Drag Reduction by Polymers

→ These two lectures explore the physics of drag reduction in (mainly) turbulent pipe flow by small amounts of polymer additives.

→ Caveat Emptor:

- remains an open problem — no clean solution
- some facts, ideas, and many speculations abound
- cannot be balanced, inclusive or up to date in short discussion
- reflects my personal view/perspective

but:

- the subject is interesting, fun, and sometimes amazing
- has captivated several "Gents"
- you will learn something!
Outline

1. Drag and Drag Reduction in Pipe Flow

2. A Primer on Polymers in Dilute Solutions
   cf. P. F. de Gennes "Scaling Concepts in Polymer Physics"
   ... on a short list of great physics books
   also M. Doi, S. Edwards, "Theory of Polymer Dynamics"
   M. Doi, "Soft Matter Physics"
   H. B. Ried et al., "Dynamics of Polymeric Liquids, Vol. I, II."

3. Some Ideas from the Greats

4. A Continuum Model (which looks familiar)


6. Related Observations / Thoughts.
Drag and Drag Reduction in Pipe Flow

Recall Drag \( \propto \text{Re}^{1/2} \text{Pr}^{1/2} \text{P}_{\text{core}} \)

\[ \Delta p \rightarrow L - 0 - W \]

\[ \Delta p \pi a^2 = \pi \left( \frac{\partial \bar{u}}{\partial z} \right) \]

\[ \rho \bar{u}_t^2 = - \frac{\partial}{\partial x} \left( \bar{u}_t \right) \]

\[ \Pi = \rho \bar{u}_t^2 \]

\[ \bar{v}_t = u_t \frac{L}{x} \rightarrow \text{mixing length} \]

Key:
- Momentum transport \( \leftrightarrow \) turbulent stress
- No-slip B.C.
- Scale covariance between
\[ x_{SL} = \frac{\sqrt{\bar{u}_t^2}}{\bar{u}_t} \text{ and } \rho. \]

Result:
\[ \langle u_t \rangle = u_t \ln \left( \frac{X}{x_{SL}} \right) \]

and
\[ C = \frac{u_t^3}{X} \rightarrow \text{turbulent dissipation strongest} \]
\[ \text{at } x = x_{SL} + \]

\[ \rightarrow \text{strongest shear, etc.} \]
Score: \[ \chi \text{ vs } Re \text{ plot} \]
\[ \chi = \frac{2\alpha \Delta p/L}{\frac{1}{2} \rho U^2} \]

But:

→ Drag Reduction Happens with Polymers!!
- c.f. → W/wo comparison by NYFD circa 1969
  → nb: unclear whether injection or lining of hoses.

→ Compare jet w/wo:

---
- observe absence with of droplets
- small scale structure
- Filaments with

N.B. Structure of turbulent jet with polymer additive is strikingly different from w/o.
more rapidly than in turbulent flow.

Figure 32 shows a logarithmic graph of \( \lambda \) as a function of \( R \). The steep straight line corresponds to laminar flow (formula (43.6)), and the less steep curve (which is almost a straight line also) to turbulent flow. The transition from the first line to the second occurs, as the Reynolds number increases, at the point where the flow becomes turbulent; this may occur for various Reynolds numbers, depending on the actual conditions (the intensity of the perturbations). The resistance coefficient increases abruptly at the transition point.

\[
\lambda \sim \frac{\Delta P}{\rho u'^2}
\]

![Graph showing laminar and turbulent flow](image)

**Fig. 32**
Fire hose demonstration.
New York City, 1969

w/wo polymers.
jet comparison

from Bird et al.
FIGURE 2.7.4. High-speed photographs taken 2 m from the office of the same two jets shown in Fig. 2.7.3. (Reproduced from photographs courtesy of J. J. Taylor, Independent Consultant, Santa Barbara, CA, and J. W. Hoyt, Department of Mechanical Engineering, San Diego State University.)
Drag Reduction in Pope Flow

the 'marquee problem'.

Now:

- Few ppm of polymers sufficient to significantly reduce drag
- No impact on laminar drag curve
- Suggests only turbulence can access beneficial aspects of polymer solution
- Drag reduction increases with concentration (though occurs for 5 ppm)
- Some evidence of a "maximal" drag reduction i.e. parallel to but displaced from laminar curve.

N.B.: P.S. Varsh A.I.Ch.E. Journal Vol. 21, #4 625 (1975)

It's an old but excellent review!

Good job on basis of drag reduction phenomenology.
Figure 1-1. Plot of the "friction factor" defined in the text as a function of Reynolds number for turbulent water flowing in a pipe with various concentrations of PEO. Points labelled "N" are with no polymer additives, while points labelled "P" have polymers in varying concentrations; from Ref. [2]

- Few ppm sufficient
- Laminar friction factor unchanged.
- Drag reduction increases with concentration
- Maximal drag reduction
shown in Figure 7, despite differences among their trajectories at the lower Ref. Perusal of the experimental details associated with the data of Figure 7 will show that, besides being independent of pipe diameter, the asymptotic maximum drag reduction is strikingly insensitive to polymer species, molecular weight, and concentration, which contrasts with the pronounced dependence of drag reduction on these polymeric parameters witnessed in the preceding polymeric regime. The maximum drag reduction asymptote has been widely observed and seems to be a fundamental feature of the drag reduction phenomenon. Some recent results are presented in Figure 8, P-K coordinates, in which the solid line labeled mdr asymptote corresponds with Equation (4) of the text. The magnitudes of maximum drag reduction are worth noting. At Ref = (300, 1000, 3000), the respective fractional flow enhancements relative to Newtonian are Sp = (0.54, 1.13, 1.49); at the corresponding Re = (4.4 \times 10^4, 2.5 \times 10^4, 10^5), the respective fractional drag reductions relative to Newtonian are Rp = (0.59, 0.73, 0.80).

1.1.6 Transition. The transition from laminar to turbulent flow with dilute polymer solutions can be appreciated by observing first some of the gross flow trajectories depicted in Figures 1 and 7. In Figure 1, the polymer solution shows the laminar, Newtonian, and polymeric regimes, an LNF trajectory, while in Figure 7 we observe two LPM trajectories, for the 38 wppm PAM solution (hollow squares) and for the 1000 wppm PEO solution (hollow diamonds), and two Lm trajectories, for the 300 wppm PEO solution (triangles) and the 100 wppm PAM solution (solid squares). Evidently, the existence of one laminar and three turbulent flow regimes makes for three possible laminar-to-turbulent transitions, which we designate L \rightarrow N, L \rightarrow P, L \rightarrow M, respectively. These seem to have the following characteristics:

1. L \rightarrow N. The transition with polymer solution is pre-

![Fig. 6. Relationship between intrinsic slope increment and number of backbone chain links for some linear, random-coiling macromolecules. Solid line corresponds to a slope modulus \( \kappa = 70 \times 10^{-6} \). Data from Table 2.](image)

![Fig. 7. Gross flow trajectories exhibiting maximum drag reduction. In all cases solvent was distilled water, temperature 25 C. (Polymer, \( M \times 10^{-6}, c \text{ wppm}, d \text{ mm} \)) as follows: (PEO, 0.76, 1000, 2.92), (PEO, 5.2, 30, 4.57), (PEO, 6.1, 450, 32.1), (PAM, 12.5, 40, 9.53), (PAM, 12.5, 110, 8.46).](image)

![Fig. 8. The maximum drag reduction asymptote in pipe flow of polymer solutions. See Table 4 for experimental details.](image)
Note: Pipe diameter and other details irrelevant.

**Significant:** Critical stress!

There is a critical wall stress ($\mu \tau^*_{cr}$) for onset of drag reduction.

Wall stress monotonically related to radius of gyration of polymers.

\[ \mu \tau^*_{cr} \]

*Structure of Profiles:

- Drag reduction does not change viscous sublayer structure.
  
  i.e. viscous flow does not 'activate' the polymer.

- Drag reduction leaves core of inertial sublayer and core unchanged.

**BUT:**
Solution viscosities are generally so close to solvent that shear thinning, which almost all polymer solutions exhibit to some extent at high enough shear rates, has little discernible effect on the gross flow. In fully turbulent pipe flow, dilute polymer solutions exhibit three distinct regimes which are, in order of increasing flow rate:

1. A regime without drag reduction in which the friction factor relation is the same as for solvent, that is, the usual Prandtl-Karman (or Blasius) law for Newtonian turbulent flow

\[ f^{-rac{1}{4}} = 4.0 \log_{10} Re^{1/4} - 0.4 \]  

(2)

2. A regime with drag reduction in which the friction factor relation depends upon the nature of the polymer solution. Anticipating later results, an approximate relation for this regime is

\[ f^{-rac{1}{4}} = (4.0 + \delta) \log_{10} Re^{1/4} - 0.4 - 8 \log_{10} \sqrt{2} W^* \]  

(3)

where \( \delta, W^* \) are polymer solution parameters.

3. An asymptotic regime of maximum possible drag reduction in which the friction factor relation is insensitive to the polymer solution employed, being, universally,

\[ f^{-rac{1}{4}} = 19.0 \log_{10} Re^{1/4} - 32.4 \]  

(4)

The foregoing will respectively be termed the Newtonian (N), polymeric (P), and maximum drag reduction (M) regimes, the laminar (L) regime having been noted first. Laminar to turbulent transition is not well detected by gross flow studies and therefore only a brief description will be given in Section 1.1.6 after the turbulent flow regimes have been considered.

1.1.2 Polymeric Regime. Aspects of the polymeric regime are illustrated in Figures 2a to 2d using F-K coordinates with abscissae Re_\(f^{1/4} \), all based on solvent viscosity.

The effect of pipe diameter is shown in Figure 2a which contains data for solvent (hollow points) and the same polymer solution in each of three pipes, of inside diameters respectively 2.92, 8.46, and 32.1 mm. Two features merit attention. First, the onset of drag reduction in the three cases occurs at Re_\(f^{1/4} \) \approx 400, 1100, and 4000, the ratio of which, 0.36:1.0:3.6, closely approximates the ratio of pipe diameters, 0.35:1.0:3.8. Thus, regardless of pipe size, the given polymer solution reduces drag only after a certain wall shear stress \( T_w^* \approx 7.0 \) N/m² has been exceeded. Second, it will be noticed that, after onset, the polymer solution data describe approximately straight lines which have much the same slope in all pipes. The difference between polymer solution and solvent slopes, called the slope increment, is in this case \( \delta = 11 \pm 2 \).

The effect of polymer concentration is shown by Figure 2b which displays data taken in the same pipe for solutions of the same polymer ranging in concentration from 50 to 1000 wppm. For clarity, solvent (distilled water) data have been omitted, being replaced by solid lines representing the Prandtl-Karman law (2) to which they closely adhered; also, ordinates have been shifted upward in an obvious way for each successive set of data, and straight lines have been drawn through the polymer solution points after onset. In each case it can be seen that the onset of drag reduction occurs at a rather well defined Re_\(f^{1/4} \). Further, in all four solutions, spanning a 20-fold concentration range, onset occurs at much the same Re_\(f^{1/4} \) \approx 1150 \pm 100. Thus, the onset wall shear stress is essentially independent of polymer concentration. After onset the polymer solution data are quite well approximated by the indicated straight lines, the slopes increasing with increasing concentration such that the slope increment is \( \delta = (4.3, 6.9, 11.7, 18.5) \) for \( c = (44, 100, 300, 940) \) wppm, respectively. Scrutiny of these results will reveal that \( \delta \) varies approximately as the square root of polymer concentration with \( 8/c^0.5 = 0.65 \pm 0.05 \).

The effect of polymer molecular weight is illustrated in Figure 2c, using data taken in the same pipe with distilled water solutions of various homologous PEO polymers ranging in molecular weight from \( 0.1 \times 10^6 \) to \( 8 \times 10^6 \). It is apparent that, as molecular weight is increased, the...
The effect of solvent viscosity on drag reduction is still uncertain, fragmentary evidence (Section 2) suggesting that $T_w^*$ is independent of viscosity.

1.1.3 Onset. In summary, the onset of drag reduction occurs at a rather well-defined onset wall shear stress $T_w^*$. For a given polymer solution, $T_w^*$ is essentially the same in pipes of different diameters. For solutions of a given polymer-solvent combination, $T_w^*$ is approximately independent of polymer concentration. And onset depends on the polymer random-coil size in solution, with $T_w^*$ increasing as $R_G$ decreases. The onset of drag reduction implies incipient interaction between the turbulent flow and the polymer molecule in solution and, among experimentally accessible quantities, $T_w^*$ and $R_G$ are respectively the flow and polymeric parameters most relevant to onset. The relationship observed between $T_w^*$ and $R_G$ in the pipe flow of PEO solutions is indicated in Figure 3, log-log coordinates. Amid scatter, $T_w^*$ varies as an inverse 2 to 3 power of $R_G$. The solid line in the figure has been drawn with slope $-3$ and represents the data within a factor of 2 over a 500-fold onset of drag reduction occurs at lower $Re_f^{4/5}$, and strikingly lower concentrations are required to yield a given slope increment. Molecular weight thus strongly influences the effectiveness of a polymer in reducing drag and, speaking qualitatively, $T_w^*$ varies inversely and $8$ directly as $M$.

The effect of polymer random coil size in solution is shown in Figure 2d, using data derived from Pruitt, Rosen, and Crawford (1966), who studied drag reduction by the same PEO polymer in water, a good solvent, and in 0.6m K$_2$SO$_4$, a poorer solvent. From their reported intrinsic viscosities of the polymer in these two solvents, $R_G$ is estimated to be 96 nm in water and 65 nm in 0.6m K$_2$SO$_4$. In Figure 2d, all of the data shown were obtained in the same pipe; the bottom-most solid line represents the Prandtl-Karman law, polymer concentrations are distinguished by symbols, water solution data (hollow) are fitted through by broken lines, and 0.6m K$_2$SO$_4$ solution data (solid) by light continuous lines. It is evident that the change in solvent from water to 0.6m K$_2$SO$_4$ shifts the onset of drag reduction from $Re_f^{4/5} \approx 500$ to $Re_f^{4/5} \approx 800$ but does not appreciably alter the slopes exhibited by the polymer solutions after onset. Thus, a decrease in the dimensions of a macromolecular random-coil in solution increases the onset wall shear stress but does not much affect the slope increment.

AICHe Journal (Vol. 21, No. 4)
ciscely the same as in the usual Newtonian case (Rotta, 1956). There is a critical Reynolds number $Re_{cr}$, which marks the beginning of a regime of intermittent laminar and turbulent slugs that finally terminate in fully turbulent flow.

2. $L \rightarrow P$. This transition is very similar to Newtonian, with much the same critical Reynolds number, turbulent slug lengths, and growth rates. However, the turbulent slug formation frequency seems to be greater than Newtonian and increases with the drag-reducing ability of the polymer solution.

3. $L \rightarrow M$. In this transition, exhibited by very strongly drag-reducing solutions, velocity fluctuations are apparent at $Re \approx 1500$, decidedly lower than the Newtonian $Re_{cr} = 2000$, but intermittency cannot be readily discerned.

The kind of transition observed apparently depends upon whether or not the onset of drag reduction occurs in the turbulent slugs which appear for $Re > Re_{cr}$. Also, $Re_{cr}$ in polymer solutions appears always to be equal to or less than $Re_{cr}$ in solvent; in no case do the polymer solutions delay transition. It should be pointed out that the preceding summarize our data for triggered (that is, high inlet disturbance) transition in pipes (Ohara, 1968) and agree substantially with the conclusions of Paterson and Abernathy (1972), but are contrary to those of Castro and Squire (1967), Giles and Pettit (1967), and White and MoEliott (1970).

1.2.2 Velocity Defects. A portion of the mean velocity data considered in Figure 9 are shown again in a velocity-defect plot, Figure 10, on arithmetic coordinates of the velocity defect $U^* = U - U^*$ versus radius-normalized distance from the wall $\xi$. This is the conventional, and sensitive, method of displaying mean velocity profiles relative to the mean velocity on the pipe axis, the arithmetic abscissa emphasizing the central region of the pipe. The solid line represents the Newtonian velocity defect law including a small, but significant, wake correction as given by Hinze (1959); solvent data reported by the investigators quoted in Figure 10 were all in close agreement with this line but are omitted for clarity. Of the results with polymer solutions, those at the lower, but nevertheless considerable, drag reduction (Virk $Sp = 0.29$, Goren $Sp = 0.64$, Rollin $Sp = 0.70$) show velocity defect profiles essentially the same as Newtonian for $1.0 > \xi > 0.05$. As drag reduction is increased (Goren $Sp = 0.94$), the velocity defect remains Newtonian toward the axis $1.0 > \xi > 0.2$ but is greater than Newtonian toward the wall $\xi < 0.2$. At maximum drag reduction (Rollin $Sp = 1.59$), the velocity defect exceeds Newtonian over the entire pipe cross section. Evidently, the mean velocity profiles retain a Newtonian velocity defect structure over a region, $1 > \xi > \xi_c$, which we will call the Newtonian plug, the inner (wallward) boundary of which $\xi_c$ moves progressively toward the pipe axis $\xi = 1$ with increasing drag reduction.

2.3 Velocity Profile Zones. The characteristic feature of mean velocity profiles during drag reduction seems to be the appearance of a region, lying somewhere between the viscous sublayer and the outer Newtonian plug, in which the mean velocity increases to above Newtonian by an amount $S^*$. While such a region, which we will call the elastic sublayer, can actually be seen in the data of Rudd (1969), roughly $15 < y^* < 60$ in Figure 9, other profiles at lower drag reduction do not extend close enough to the wall to make it visible. However, from the pipe axis inward, these profiles are observed to be parallel-shifted upward from Newtonian by $S^*$, and from the wall outward we have the conviction that, in polymer solutions as in Newtonian fluids, the profiles must start out along a viscous sublayer and, hence, between the viscous sublayer and the Newtonian plug must be a region, characteristic of drag reduction, across which the effective slip $S^*$ occurs. Mean velocity profiles during drag reduction have thus three zones from the wall outward: (1) a viscous sublayer, (2) an elastic sublayer, characteristic of drag reduction, across which the effective slip $S^*$ occurs, and (3) a Newtonian plug which retains the Newtonian defect structure though absolute velocities

At zero drag reduction (Virk $Sp = 0$), the polymer solution profile adheres to the Newtonian wall law. At low drag reduction (Patterson $Sp = 0.27$, Virk $Sp = 0.29$), the polymer solution profiles are shifted upward from but parallel to the Newtonian wall law. At the parallel upward shift, called (after Oldroyd, 1948) the effective slip $S^*$, increases with increasing drag reduction (Goren $Sp = 0.84$, Rollin $Sp = 0.70$, Elata $Sp = 0.4$). At still higher drag reductions (Goren $Sp = 0.94$, Rudd $Sp = 1.0$), and shifted region of roughly Newtonian slope is still discernible, but at the lower $y^*$, toward the wall, the profiles exhibit a slope significantly greater than Newtonian. Finally, at maximum drag reduction conditions (Rollin $Sp = 1.59$), the entire profile is approximately semilogarithmic with a slope $\approx 13$, about five times that of the Newtonian wall law.

1.2.2 Velocity Defects. A portion of the mean velocity data considered in Figure 9 are shown again in a velocity-defect plot, Figure 10, on arithmetic coordinates of the velocity defect $U^* = U - U^*$ versus radius-normalized distance from the wall $\xi$. This is the conventional, and sensitive, method of displaying mean velocity profiles relative to the mean velocity on the pipe axis, the arithmetic abscissa emphasizing the central region of the pipe. The solid line represents the Newtonian velocity defect law including a small, but significant, wake correction as given by Hinze (1959); solvent data reported by the investigators quoted in Figure 10 were all in close agreement with this line but are omitted for clarity. Of the results with polymer solutions, those at the lower, but nevertheless considerable, drag reduction (Virk $Sp = 0.29$, Goren $Sp = 0.64$, Rollin $Sp = 0.70$) show velocity defect profiles essentially the same as Newtonian for $1.0 > \xi > 0.05$. As drag reduction is increased (Goren $Sp = 0.94$), the velocity defect remains Newtonian toward the axis $1.0 > \xi > 0.2$ but is greater than Newtonian toward the wall $\xi < 0.2$. At maximum drag reduction (Rollin $Sp = 1.59$), the velocity defect exceeds Newtonian over the entire pipe cross section. Evidently, the mean velocity profiles retain a Newtonian velocity defect structure over a region, $1 > \xi > \xi_c$, which we will call the Newtonian plug, the inner (wallward) boundary of which $\xi_c$ moves progressively toward the pipe axis $\xi = 1$ with increasing drag reduction. Too, the Newtonian defect structure is retained despite absolute velocities significantly, and by various amounts, higher than Newtonian as already seen in the parallel upward shifts exhibited by these profiles in Figure 9.

1.2.3 Velocity Profile Zones. The characteristic feature of mean velocity profiles during drag reduction seems to be the appearance of a region, lying somewhere between the viscous sublayer and the outer Newtonian plug, in which the mean velocity increases to above Newtonian by an amount $S^*$. While such a region, which we will call the elastic sublayer, can actually be seen in the data of Rudd (1969), roughly $15 < y^* < 60$ in Figure 9, other profiles at lower drag reduction do not extend close enough to the wall to make it visible. However, from the pipe axis inward, these profiles are observed to be parallel-shifted upward from Newtonian by $S^*$, and from the wall outward we have the conviction that, in polymer solutions as in Newtonian fluids, the profiles must start out along a viscous sublayer and, hence, between the viscous sublayer and the Newtonian plug must be a region, characteristic of drag reduction, across which the effective slip $S^*$ occurs. Mean velocity profiles during drag reduction have thus three zones from the wall outward: (1) a viscous sublayer, (2) an elastic sublayer, characteristic of drag reduction, across which the effective slip $S^*$ occurs, and (3) a Newtonian plug which retains the Newtonian defect structure though absolute velocities

### Figures

- **Figure 9.** Mean velocity profiles during drag reduction. See Table 5 for experimental details.
- **Figure 10.** Velocity-defect plot, showing velocity defect profiles.
- Slip layer // Buffer layer forms with polymers
- Viscous
- Note slip/buffer layer:
  - Steepens local DU
  - Necessarily reduced local momentum transport
    ➔ Momentum transport barrier
  - Slip layer expands with C2 approaching ultimate profile.
- Rei: Momentum transport barrier,
  - Momentum flux is local <UV>
  - Stress drops with polymer additive
  - C(u,v) = <uv>/kHHV drops with polymers ➔ effect on cross-phase.
agrees with Logan's results (Figure 11b). Concerning the variation of turbulent intensity profiles with drag reduction, the data of Logan ($\delta'_{F} = 0.4$) and Rudd ($\delta'_{R} = 1.0$) respectively yield a ratio of maximum axial intensity in polymer solution to that in solvent ($\mu_{max,\delta'_{P}} / \mu_{max,\delta'_{R}}$) of 1.3 and 1.8, while in the pipe core the regions where $u'^{+}$ is the same as in solvent are $1.0 > \xi > 0.4$ and $1.0 > \xi > 0.6$, respectively. At (or near) maximum drag reduction, there is only uncertain information from streak photography (Seyer and Metzner, 1969; Arunachalam, Hummel, and Smith, 1972), which suggests that ($\mu_{max,\delta'_{P}} / \mu_{max,\delta'_{R}}$) is 2.0 to 2.5.

1.3.3 Turbulence Structure Zones. The axial intensity profile, for which the most information exists, has three zones:

1. A region $0 < y^{+} < y_{m}^{+}$, say, in which $a^{+}$ and the magnitudes of $u'^{+}$ are essentially the same as Newtonian.

2. A region $y_{m}^{+} < y^{+} < y_{*}^{+}$, say, which is characteristic of drag reduction and in which $u'^{+}$ differs from Newtonian, being higher. This region further consists of a relatively thin portion, $y_{m}^{+} < y^{+} < y_{max}^{+}$, in which $u'^{+}$ increases to a maximum value and the remaining major portion $y_{max}^{+} < y^{+} < y_{*}^{+}$ in which $u'^{+}$ decreases from the maximum to a value close to Newtonian.

3. A region $y^{+} > y_{*}^{+}$ or $1 > \xi > \xi_{e}$ in which $u'^{+}$ is the same as Newtonian.

The radial intensity, Reynolds stress, and correlation coefficient profiles do not yield information on the very near wall region (1) but do show clearly the region (2), toward the wall, which differs from Newtonian ($v'^{+}$, $\phi$, $C_{uv}$ are all relatively lower) and region (3), toward the axis, which is much the same as Newtonian.

There is evidently a striking analogy between the above turbulence structure zones and the (1) viscous sublayer, (2) elastic sublayer, (3) Newtonian plug zones of the mean velocity profiles considered earlier.

2. CORRELATION AND ANALYSIS

The object of this section is to correlate drag reduction data, using the elastic sublayer model (Virk, 1971) as a framework. The model is briefly described and then empirically related to experimental gross flow, mean velocity, and turbulence measurements with analysis and discussion of the data used for this purpose, including a section on polymer characterization. The empirical developments are, of course, substantially model independent.

2.1 The Elastic Sublayer Model

The essential physical notion is that the stimulation of polymer molecules by a turbulent shear flow creates a zone, called the elastic sublayer, which is characteristic of the drag reduction phenomenon. The elastic sublayer originates at onset, it then grows with increasing drag reduction and eventually occupies the entire pipe cross section at maximum drag reduction. By hypothesis, all drag reduction observations can be related to the properties and extent of the elastic sublayer.

The mean flow model is shown schematically in Figure 13, which defines all terminology. The general profile ABCD, consists of three zones: (1) a viscous sublayer AB, $y^{+} \equiv y_{m}^{+}$; (2) an elastic sublayer BC, $y_{m}^{+} < y^{+} < y_{*}^{+}$; (3) a Newtonian plug CD, $y_{*}^{+} \equiv 0^{+}$, each with the indicated mixing length constant and $U^{+}$-$y^{+}$ relationship. The inner edge B of the elastic sublayer stays fixed at $y_{m}^{+} = 11.6$, which is the trisection of Equations (7), (8), and (9), while its outer edge C moves along the ultimate profile BM from $y_{*}^{+}$ at zero drag reduction to $R^{+}$ at maximum drag reduction. Comparison of Figure 9 with Figure 13 shows that the model reproduces the essential features of the experimentally observed mean velocity profiles. The effective slip is related to elastic sublayer thickness by

$$S^{+} = (A_{m} - A_{n}) \ln y_{*,*}^{+}$$

Integration of the profile ABCD by segments yields the friction factor relations:

$$U_{ex}^{+} = A_{n} \ln R^{+} + B_{n} - 1.5 A_{n}$$

$$= A_{n} \ln R^{+} + B_{n} - 1.5 A_{n}$$

$$+ (A_{m} - A_{n}) \ln (y_{*}^{+} / y_{*,*}^{+}); y_{*,*}^{+} < y_{*}^{+} < R^{+}$$

$$= A_{n} \ln R^{+} + B_{n} - 1.5 A_{m}$$

$$+ y_{*,*}^{+} + R^{+}$$

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m drag reduction flows with $\langle u' v' \rangle$

$\bar{u}$ drops $\sqrt{\nu}$ increases.

cross-stream

$Re < \text{ld}$ for all cases!

2. What is a Polymer?

How does it behave in dilute solution?

(N.B. Hodges most of the really interesting questions in polymer physics we get by reptation, -----)

key: Resonances of Polymer Chain

Approach via rescaling laws -----

What is a Polymer (Chain)?

- $N > 1$ molecules in a long chain

$N \approx 10^5 - 10^6$

structure: [linear, grid]

relevant $\Rightarrow$ resist degradation
Various animals attempting to follow a scaling law.
Polyethylene

- \( \text{CH}_2 - \text{CH}_2 - \text{CH}_2 \cdots \) - plastic

Polystyrene (styrofoam)
- \( \text{CH}_2 - \text{CH} - \text{CH}_2 \text{CH}_2 \)

Polyvinyl chloride (PVC)
- \( \text{CH}_2 - \text{CH} - \text{CH}_2 - \text{CH} - \text{Cl} \)

Polyethylene dioxyde (polyox) - common drag-reduction additive

Physically:
- System is long chain of monomer, with \( N \sim 10^5 \)
- Can be pure or in solution
- Can add chemical agent to change connectivity (i.e., sulphur on vulcanization)
- Issues:
  - radius reductions (with \( N \))?
    - how big?

- "elasticity"? (drag reduction)
  - coupling to internal dents

- natural modes:
  - frequency relaxation rate
    - transport material properties

Some Basics:

- ideal chain:
  \[ \{ N \text{ monomers} \} \text{ spacing } a \]

- \# distinct conformations or walks:
  - starting from \( \ell = 0 \) and ending at \( \ell \) is \( \Omega(\ell) \) = def

- \( \ell = a_1 + a_2 + a_3 + \ldots + a_N = \sum_i a_i \)
  - each step uncorrelated
  - \( \text{CLT} \Rightarrow \)
  - \[ \langle \ell^2 \rangle \sim N(a^2) \]
    - 10 steps
    - end-to-end dist
so probability of extending to distance $r$ is 

$$P(r) = \frac{\Sigma(N)}{\Sigma_n(N)} \sim \frac{1}{N^{3/2}} \exp \left[ -\frac{3r^2}{2N\alpha^2} \right]$$

and thus can immediately write entropy at elongation $r$

$$S = \ln(P(r)) = \ln\frac{1}{P(r)}$$

$$= S(\alpha) - \frac{3}{2} \left( \frac{r^2}{R_G^2} \right)$$

$\Rightarrow$ reduces gyration

$$R_G^2 = N\alpha^2 \equiv \text{(radius)}^2 \text{ of polymer ideal chain}$$

$\Rightarrow$ radius is radius of walk of $N$ steps

but $F = E - TS \equiv \text{Helmholtz free energy}$

$$F = -k_B T \ln Z$$
\[ F(r) = F(0) + \frac{3}{2} T r^2 / R_G^2 \]

- free energy of ideal chain

- note entropy decreases but free energy increases, as chain extended

\[ \frac{F(r)}{V} \approx U(r) \quad \text{(off potential)} \]

\[ f = -\frac{\partial F}{\partial r} = -3 T r / R_G^2 \]

\[ k_{\text{eff}} = \frac{3 T}{R_G^2} = \text{entropic spring constant of ideal chain} \]

\[ \Rightarrow \text{Hookean elasticity via entropy} \]

Aside:

- \( f \) is tension constant along chain

\[ \langle r \rangle \sim N \quad \text{(linear oscill)} \]

- expect \( \langle r \rangle \) to depend on \( f, T, R_G \)

\[ \text{now } \sim f \text{ or so} \]
$\langle r \rangle \sim (Fr_0/T)^{2/3}$

$\langle r \rangle \sim R_g \left( Fr_0/T \right)^{2/3}$

and

$\langle r \rangle \sim N$ while $R_g^2 \sim N \alpha^2$

$k = 1$

and $\langle r \rangle \sim Fr_0^2/T$

But:

- Light scattering experiments supported $R_g \sim N^{0.6}$

- Polymer is SAW $\Rightarrow$ Self Avoiding Random Walk (SAW)
  + effective monomer-monomer repulsion
  + chain swells
  
  i.e. $R_g \sim N^{0.6}$ vs $N^{1/2}$

How understand simply? (Too simply?)
1. explore competition:
   a. exclusion/repulsing (swelling)
   b. contraction, due to entropic force

2. chain monomer concentration

\[ C = \frac{N}{R^d} \]

\[ F = F_{\text{elastic}} + F_{\text{repuls}} \]

\[ F_{\text{elastic}} \sim TR^2 \frac{kN^2}{2} \]

\[ F_{\text{repuls}} \sim \frac{1}{2} TV(\Gamma) C^2 \left[ \frac{R^d}{R^d + 1} \right] \]

\[ \sim \text{at-hex volume} \]

\[ C = \frac{N}{R^d} \]

\[ F = TR^2 \frac{kN^2}{2} + \frac{1}{2} TV(\Gamma) \frac{N^2}{R^d} \]

\[ \frac{\partial F}{\partial R} = TR^2 \frac{dT V(\Gamma)}{R^d} \frac{N^2}{R^{d+1}} = 0 \]

\[ \text{Heur min b} \]
\[ R \sim R_F \sim N^{3/6} \left[ \frac{e^2 V_0}{\epsilon} \right]^{1/4} \]

**Flory prediction**

\[ R_F \sim N^{3/6} \]

\[ d = 3 \]

\[ d = 4 \rightarrow RW \]

**Note:**

→ Simple but a swindle, i.e. mean field theory

1. Repulsive energy over-estimated
   - Neglects plates, \( \langle \tilde{z} \tilde{z} \rangle \)

2. Elastic energy over-estimated
   \[ \frac{1}{R^2} \rightarrow \frac{1}{R_F^2} \]

Errors cancel, still

\[ \Rightarrow \] Simple Flory argument was a major step forward in SAW understanding!

deGennes on Flory: "Simple and brilliant scheme"
How describe dynamics of chain?

Simplest approach is to view chain as one of individual beads each feeling:
- spring (centripetal) force from neighbors
- drag force on fluid

\[
\frac{\text{M}}{\text{M}_n} + \frac{\text{d}}{\text{M}} \text{v}_n = \text{F}_n
\]

assumed:
- elastic force
- drag on \( n \)th bead

\[
\begin{align*}
\text{M}^{-1} &= \sigma \pi \text{M}^2 \\
\text{F}_{\text{drag}} &= 6\pi \eta \text{M}^2 \text{v} \\
\text{v}_n &= \text{v}_n \\
\text{Stokes}
\end{align*}
\]

\[
\text{F}_n = k \left[ \text{v}_{n+1} - 2\text{v}_n + \text{v}_{n-1} \right] \quad \text{as usual.}
\]
\[ \frac{\partial n}{\partial t} = \frac{T_n}{a^2} \left[ \frac{\partial^2 n}{\partial x^2} - 2 \frac{\partial n}{\partial x} + n \right] \]

\( h a < 1 \)

usual expansion

\[ n \in \Delta = \frac{T_n}{a^2} \frac{\partial^2 n}{\partial x^2} \]

with free end: \( \frac{\partial n}{\partial x} \bigg|_0 = 0 \)

\[ \tau_p \sim \frac{p^2 T_n}{N^2 a^2} \]

relaxation time of \( p \)th mode.

\[ \tau \sim \frac{T_n}{N^2 a^2} \frac{1}{N^2 (6 \pi) a^3} \]

\( n \to \frac{1}{6 \pi M} \) graphene; \( \frac{1}{M} \) q.

\[ \tau_p \sim N^2 / p^2 \]
Now, Rouse model overlooks hydro interference.

⇒ Better 1 elastic shape!

\[ \delta = \text{length, poly mer} \]

\[ F = F_{\text{drag}} + F_{\text{spring}} \]

\[ F_{\text{drag}} = -6\pi \eta M R F \frac{d\delta}{dt} \]

\[ F_{\text{spr}} = \frac{T \delta}{R^2_F} \]

\[ 0 = 6\pi \eta M R_F \frac{d\delta}{dt} + \frac{T\delta}{R^2_F} \]

\[ T_{\text{damp}} \sim T \sqrt{\frac{\eta M R_F}{2}} \]

\[ T_{\text{el}} \sim N^{9/5} \text{ for shape} \]

\[ \psi \sim N^2 \text{ for Rouse.} \]
\( T_d \equiv \) Zaming time

\( \Rightarrow \) characteristic relaxation time of polymer

\[ \sqrt{\gamma_z} \sim \frac{T}{6\pi \eta M R_F^2} \]

obviously, \( \frac{V(E)}{T} \) vs \( \sqrt{\gamma_z} \)

is activation criterion.

TBG ...
Some prevailing wisdom
(Lumley, DeGennaro — see posted papers)
- well stress threshold
  \[ \frac{v(E)}{E} > \frac{1}{1/2} \]
  \[(E/r)^{1/2} > \frac{1}{12} \sqrt{2} \]
  \[ c = U_{a*}/X, \text{ pipe} \]
  \[ \Rightarrow U_{a*}/a \text{ microscopically} \]

and
\[ T_w \sim C U_{a*} \]

\( \Rightarrow \) threshold vs. \( 1/R_f \sim N^{-9/5} \)

emerges easily

\( \Rightarrow \) larger polymers more easily activated
- Drag Reduction
  - difficult to see substance.
  - Lumley
    - polymer stretch, damp turbulence.
    - how??

Coil-stretch transition rests on laminar extensional flow. Dependence?

- Do Genres - see '86.
  - activation

- cascade truncation (?)
  - due to coupling to elastic ODEs dependence

- C emerges via equipartition condition for truncation?!

- why truncation?
A Model

- Continuum (Dumbbell Model)
- seek a continuum model of poly-hydro turbulence
- explicit dumbbell model - virtue of simplicity
- what are effects of (minimal) elasticity on hydro interactions, cascade, transport, etc.?
- collective dynamics? (variant of FENE-B)

\[ \frac{\partial}{\partial t} \mathbf{V}(r,t) + \nabla \cdot \mathbf{V}(r,t) = \mathbf{F}(r,t) \]

seek description in terms of distribution of \( r \rightarrow \) relative coordinate

\( R \rightarrow \) centroid coordinate

and their distribution.

Now the dumbbells (i.e. shape elements) go with the flow!
\[
\gamma = \frac{1}{\mu} \\

\gamma \left( \frac{d\mathbf{v}}{dt} - \mathbf{v} (\mathbf{v} \cdot \mathbf{v}) \right) = -\frac{\partial U}{\partial \mathbf{v}} + \frac{\mathbf{z}_1}{\eta} \quad \text{noise}
\]

Drift

\[
\gamma \left( \frac{d\mathbf{v}}{dt} - \mathbf{v} (\mathbf{v} \cdot \mathbf{v}) \right) = -\frac{3\mathbf{v}}{\eta} + \mathbf{z}_2
\]

\[
U = \frac{1}{2} \kappa (v_1 - v_2)^2
\]

Entropic force

\[
\langle z_1(t_1, z_2(t_1)) \rangle = 2\gamma k_B T \delta_{t_1, t_2} (t_1 - t_2) - \text{usual}
\]

Then:

\[
R = \frac{1}{2} (v_1 + v_2) \quad \rightarrow \text{centered}
\]

\[
z = v_1 - v_2 \quad \rightarrow \text{extension}
\]

\[
\gamma \frac{dR}{dt} = \mathbf{v} (\mathbf{v} \cdot \mathbf{v}) - \frac{3}{2} \mathbf{z}_1(t) + \mathbf{z}_2(t)
\]
expansion.

\[ \frac{dZ}{dt} = Z \cdot 1 \cdot \nabla \cdot (\nabla \theta + t) - \frac{\theta}{\delta} \frac{\partial u}{\partial y} \]

\[ + \frac{1}{\delta} \left[ \sum \theta(t) \cdot 3 \cdot 3(t) \right] \]

N.B. A hint as to where fluid equations come from.

Now to derive fluid equations for polymer DOFs, need kinetic equation

\[ F(\theta, R, V, \ldots \text{etc}) \]

distribution of polymer DOFs.

\[ \delta \text{ have Fokker-Planck equation} \]

\[ \frac{\partial F}{\partial t} = - \frac{\partial}{\partial R} \left[ \nabla (\nabla \theta) \cdot F \right] \]

\[ + \frac{\partial}{\partial R} \cdot D_0 \frac{\partial F}{\partial R} \]

\[ - \frac{\partial}{\partial z} \left[ \sum \theta \cdot D \nabla (\theta z) + \frac{\theta}{\delta} \frac{\partial u}{\partial z} \right] \]

\[ + \frac{\partial}{\partial z} \cdot D_2 \frac{\partial F}{\partial z} \]

where: \( D_0 = k_B T / 2 \delta \)

\( D_2 = 2 k_B T / \delta \)
and \( \int d^3r \int d^3z \quad F(r, z, t) = I \)

Now take moments:

\[ C(r, z, t) = \int d^3z \quad F(r, z, t) \]

\[ \Omega(r, z, t) = \int d^3z \quad \Omega(r, z, t) \]

\[ \frac{\partial C}{\partial t} = \int d^3z \quad \frac{\partial F}{\partial t} \quad \text{and} \quad \Omega \text{ (nearly) the natural order parameter.} \]

\[ \frac{\partial C}{\partial t} = -\nabla \cdot v C + D_\alpha \nabla^2 C \]

\[ \frac{\partial C}{\partial t} + \nabla \cdot v C = D_\alpha \nabla^2 C \]

As \( \nabla \cdot v = 0 \).

In homogeneous turbulence:

\( C(R, t) \to C_0 \propto \text{const.} \)
Now consider elongation moment:

\[ \varphi(x, t) = \int \frac{d^3 z}{z} z F(\varphi, z, t) \]

\[ \int \frac{d^3 z}{z} \left[ \frac{\partial F}{\partial z} + \frac{\partial}{\partial \varphi} \left( \nu(x, t) \varphi \right) \right] \]

\[ = \int \frac{d^3 z}{z} \left[ \frac{-2}{\nu} \left( \varphi - \psi \sqrt{\varphi} \right) - 2 \frac{3}{2} \frac{\partial}{\partial \varphi} \frac{\partial S}{\partial \varphi} \right] \]

\[ + \int \frac{d^3 z}{z} \left[ \frac{\partial}{\partial \varphi} \frac{\partial}{\partial z} \frac{2F}{\nu} \right] \]

\[ \text{n.b.: Closure problem for non-Hookean spring} \]

\[ \frac{\partial}{\partial z} \varphi + \nu \cdot \frac{\partial F}{\partial \varphi} = 2 \frac{\partial^2 \varphi}{\partial z^2} + \varphi \cdot \frac{\partial^2 \nu}{\partial \varphi} \]

\[ \frac{-2\nu}{\sqrt{\varphi}} + \sqrt{\varphi} \]

\[ \text{Note} \quad \frac{L}{\nu} = \omega_2 \rightarrow \text{Zimm frequency/relaxation rate} \]

\[ L = \frac{T}{R_G^2} \quad \nu = 6\pi \eta_0 \frac{R_G}{M} \]

\[ \omega_2 = \frac{T}{6\pi \eta_0 R_G^3} \quad R_G = R_F. \]
thus, have:

\[ \frac{\partial \varphi}{\partial t} + \mathbf{V} \cdot \nabla \varphi = \varphi \frac{\partial \mathbf{V}}{\partial t} - 2 \frac{\partial}{\partial t} \mathbf{V} \varphi + \rho \frac{\partial^2 \varphi}{\partial t^2} \]

(2) \rightarrow \text{advection}

(3) \rightarrow \text{stretching}

(4) \rightarrow \text{Zimm relaxation damping} \\
\Rightarrow \text{scale covariant.}

\text{n.b.: Obvious similarity to induction equations:}

\[ \frac{\partial \omega}{\partial t} = \text{\Omega} \times \text{\nabla} \times \omega + \nu \nabla^2 \omega \]

\[ \frac{\partial B}{\partial t} = \text{\Omega} \times \text{\nabla} \times B + \nu \nabla^2 B \]

(3) \rightarrow \text{stretching} \\
e.g. \text{flow streaming stretch the polymer}

(4) \rightarrow \text{Zimm damping \rightarrow resists stretching} \\
e.g. \text{scale covariant relaxation rate.}
are key terms.

N.B. As anticipated

\[ |\mathbf{v}| > \omega \] \text{ required for "polymer activity"}

Is this correct?

→ No

→ Direction of stretching field not established.

Points 1, 2 arbitrary!

→ I, Q equally likely.

Polymer tension better thought of as a dyad tensor

not \( \mathbf{Q}_{\mathbf{B}} \) but \( \mathbf{Q}_{\mathbf{u}} \)

has direction and tension

(\( \mathbf{F} \cdot \mathbf{T} = 0 \))
Akin to order parameter for nematic liquid crystals

\[ \phi_0 (r, t) = \sum_i \phi_i \delta (r - \mathbf{r}_i(t)) \]

and elastic energy field (elastic energy tensor)

N.B.: Elastic energy tensor is natural for weak field MITD or small scale dynamics, etc.

\[ \delta_{ij} \nabla_i \nabla_j \phi_0 + \nabla_i \phi_0 \nabla^i \phi_0 = -4 \lambda \nabla_i \phi_0 \nabla^i \phi_0 + D_0 \nabla^2 \phi_0 + \frac{4 \kappa B T}{\tau} \delta_{ij} \]

Physics picture essentially unchanged
1. \( \rightarrow \) Reduction
2. \( \rightarrow \) Stretching, need symmetrize
3. \( \rightarrow \) Relaxation damping
Noise → thermal levels of excitation.

For equilibrium (De feeble)

\[-4\omega z Q_{ij} + 4k_B T \delta_{ij} = 0\]

\[Q_{ij} = \delta_{ij} \frac{k_B T}{\gamma} = \frac{R_0 \delta_{ij}}{\gamma}\]

**Back Reaction on Fluid**

Observe that $Q_{ij}$ as elastic energy field naturally suggests writing back-reaction as $\nabla \cdot \nabla \psi$ on

RHS of N-S:

C.R.

\[c \left[ \frac{\partial U}{\partial t} + U \cdot \nabla U \right] = -\nabla p + c_p k T \nabla \phi + \nabla^2 U\]

concentration field

more precise:

\[\tau = c_p k \phi\]

stress tensor
In component form:

\[ \rho \left[ \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_j \cdot \nabla \mathbf{v}_i \right] = -\nabla P + \nabla \cdot \mathbf{T}_y + \mu \nabla^2 \mathbf{v}_i \]

- obvious analogy

**elastic stress**: \( \mathbf{T}_y = E \frac{\partial \mathbf{E}}{\partial t} \)

\( \Rightarrow \) elastic waves \( \partial^2 \mathbf{E} \)

Polymer stress \( \mathbf{T}_{ij} = \mathbf{c}_{ikj} \mathbf{Q}_{ij} \)

\( \Rightarrow \) elastic waves (thin Alfvén)

so, with polymer fluid is both viscous and elastic

i.e.

\[ \mathbf{T}_{ij} = \mathbf{c}_{ikj} \mathbf{Q}_{ij} \quad \Rightarrow \quad \mathbf{M} \mathbf{D} \mathbf{V} \]

(choose different sign conventions)

and note if linearize \( \mathbf{Q}_{ij} \)
\[ a_{ij} \cdot \tilde{\sigma}_{ij} = \rho \frac{\partial \tilde{v}}{\partial t} + \]
\[ \rho \frac{\partial \tilde{v}}{\partial t} = \nabla \cdot \vec{E} + \]
\[ \rho \frac{\partial^2 \tilde{v}}{\partial t^2} = \nabla \cdot \vec{F} + \]
\[ = \nabla \cdot \vec{E} = \nabla \cdot \left( \nabla \cdot \vec{E} \right) + \tilde{v}_j \]

suggestive of elastic wave response due polymer.

Of course:
\[ a_{ij} \cdot \tilde{v} = n \nabla \tilde{v} \]
viscous.

Welcome to "viscoelasticity" a property of polymeric fluids.

N.B. A close correspondence exists between:
- Classes of polymer hydro models
  - MHD (evident)
- Classes of viscoelastic systems
  - Oldroyd- \( B \) \( \leftrightarrow \) MHD
  see Ogilvie, Proctor
Physics of back-reaction:

Fluid element
dumbells
piercing surface of the dumbell.

the Poly-Hydra system:

\[ \partial_t Q_{ij} + \left( \mathbf{v} \cdot \nabla \right) Q_{ij} = Q_{ij} \partial_x \mathbf{v}_x^2
+ Q_{ij} \partial_x \mathbf{v}_y^2 - 4 \omega_2 Q_{ij} + D \nabla^2 Q_{ij}
+ \frac{4 \pi \rho_0}{\gamma} S_{ij} \]

and

\[ \left[ \partial_t \mathbf{v}_i + \mathbf{v} \times \mathbf{B} \right] = -\nabla \mathbf{v}_i 
+ \partial_j \epsilon_{ij} k Q_{ij}
+ n \nabla^2 \mathbf{v}_i + \mathbf{f}_i \]

Regarding energetics:
\[ E = \frac{1}{2} \int d^3r \left[ ev^2 + cpk \cdot \Phi_{\text{kin}} \right] \]

converged to:

\[ \frac{dE_{\text{tot}}}{dt} = \int d^3r \left[ cpk \cdot \left( \frac{1}{2} \rho \right) \right]^2 + 4 \mu \epsilon \cdot \Phi_{\text{kin}} \]

Do irrelevant.

N.B., Obvious closer analogy with MHD

\[ \frac{d}{dt} B = \nabla \times (u \times B) + \eta \nabla^2 B \]

\[ \frac{d}{dt} B + \nabla \cdot \nabla \cdot B = B \cdot \nabla u + \eta \nabla^2 B \]

Obviously no counterpart of Ziman damping in MHD

and

\[ \frac{\partial u}{\partial t} = -\nabla P + \nabla \times B + \rho \nabla^2 u \]

\[ \nabla \times B + \text{Ampere} = 0 \]

\[ \text{Ohm's} \]

\[ \nabla \cdot E = 0 \]
\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\rho \nabla p + \frac{\nabla \cdot \mathbf{B} \mathbf{B}}{4\pi} \] 

D \cdot \mathbf{B} = 0 \quad \text{can write as} \quad \nabla \cdot \left[ \frac{\mathbf{B} \mathbf{B}}{4\pi} \right]

W.B. A key element in incompressible MHD is the Alfven wave. Fermi: "Magnetic-electric wave"

\[ \nabla \cdot \mathbf{V} = 0 \quad \mathbf{B} = B_0 \mathbf{z} \]

\[ \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla p + \frac{B_0}{4\pi} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = B_0 \frac{\partial \mathbf{V}}{\partial z} \]

\[ \omega^2 = \left( \mathbf{k} \cdot \mathbf{b}_0 \right)^2 B_0^2 / 4\pi \rho \]

\[ = \nu_m^2 \nu_a^2 \]

i.e. For incompressible MHD with static field, linear wave is excitation above state

Contrast: \( D \cdot \mathbf{V} = 0 \) hydro. 

i.e. this correct
What of Poly-Hydro?  

So  

\[ \omega_{ij} = (\omega_{ij} + \omega_{ij}) (\omega_{ij} + \omega_{ij}) \]  

\[ = \omega_{ij} \omega_{ij} + \omega_{ij} \omega_{ij} + \omega_{ij} \omega_{ij} \]

usual:  

\[ \omega_{ij}^2 = \frac{1}{c_\|} \left( \nabla \cdot \mathbf{q}_\| \right)^2 \]

local field

→ Alfvén-like elastic waves present

→ More significantly:

with Alfvénic/elastic wave coupling turbulence character changes fundamentally. As a reference to MHD turbulence (weak magnetization)

→ wave-like, or 'weak'

→ wave interactions control cascade, not eddy

→ inertial range spectrum power transfer fundamentally different from physics.
Mini-Primer on MHD Cascade

- E. Galbric "Modern MHD"
- M. Kruer "Plasma Physics For Astrophysics"

- \( Z_{\pm} = V \pm b \) are exact NL solutions
  - Interact only via \( \pm - \) collisions

- Basic physics is wave interaction instead:

\[
E = \frac{\text{v}^2}{\text{a}} \rightarrow \frac{\text{v}^2}{\text{b} \text{a}} = \frac{\text{v}^2}{\text{b}}
\]

\[
\rightarrow \rightarrow
\]

- Have:

\[
E = (c_0^2 \text{z}^2) \frac{\text{T_c}}{\text{c}}
\]

\( \text{coulomb} / \text{coeff} \rightarrow \text{wave interaction} \)

\( \text{energy} / \text{coherence} \)

\[ \frac{\text{dE}}{\text{dt}} = \text{T} \rightarrow \text{transition} \]
\[ T = \sum_{i} \left[ \text{strain field of scattering wave packet} \right] \]

\[
T \sim \left( \frac{Z(l)}{l} \right)^2 \frac{1}{Tc} \]

\[
T_c \rightarrow \text{interception time}
\]

\[ (b_0/l)^{-1} \]

\[ \left( \frac{V_{\text{rms}}}{l} \right)^{-1} \rightarrow \text{wave packet passes time scale } l \]

\[ \text{Alfvén speed in RMS field} \rightarrow \text{local, produced by stretching} \]

\[
\xi = \frac{1}{l^2} \left( \frac{Z(l)}{l} \right)^2 \frac{1}{V_{\text{rms}}} \]

\[ Z(l) \sim \left( \frac{V_A}{E_c} \right)^{1/4} \]

\[ 1/4 \]
\[ E_k (e) \sim E_{\text{m}} (k) \sim (E \nu_{\text{ao}})^{1/2} \nu_{\text{ao}}^{1/2} \]

\[ E_k (h) \sim E_{\text{m}} (h) \sim (E \nu_{\text{ao}})^{1/2} \nu_{\text{ao}}^{-1/2} \]

\[ \frac{1}{T_A} \sim \frac{1}{\sqrt{l^2}} \]

\[ \frac{2(E^2 - \nu^2)}{\nu^2} T_A \]

\[ T_A = \frac{l}{V_{\text{ao}}} \]

- change in dissipation scale \( l \)
- but no truncation!

- Can argue on basis of analogy with MHD

- in activated region (in scale),
  - hydro cascade will convert to hydro/elastic cascade.
  - where \( \ell^* \sim l^* \)
- spectrum will be modified
- turbulence converted from hydro/low to elastic wave
  turbulence, with
  \[ E_q \sim E_k \to \] equilibrium of kinetic, elastic \( \partial \Phi \)

IV. Yet More
- Conclusions on Drag Reduction
- recall essence of drag is momentum transport to (no slip) wall
- polymer activated if for \( l > l_d \)
  \[ \frac{V(l)}{\bar{E}} > \frac{1}{\sqrt{l}} \] defines \( l_d \)
  \[ \Rightarrow E^{1/3} \left( \frac{V^{3/4}}{\bar{E}^{1/4}} \right) > \frac{1}{\sqrt{l}} \]
  \[ E^{1/3} \leq \frac{1}{16} = \left( \frac{V}{\bar{E}} \right)^{1/2} > \frac{1}{\sqrt{l}} \]
Hydro

Poly active

--- Hydro-elastic ---

\[ \ell^{**} \]

not clear scale chain
anisotropy, slow deG

in hydro-elastic range polymer
Dof's can/may be stretched
to equipartition with kinetic
energy.

for mean flow evolution of dilaturl layer.

- recall:

\[ \frac{\partial \langle \mathbf{V}_1 \mathbf{V}_1 \rangle}{\partial t} = - \nabla \cdot \left( \frac{3}{2} \langle \mathbf{V}_1 \mathbf{V}_1 \rangle - \frac{C_{pq}}{2\theta} \langle \mathbf{Q} \mathbf{Q} \rangle \right) \]

mean flow

\[ \text{note: total stress on } \]

\[ \text{activated region} \]

now if turbulence 'elasticized' (i.e. akin
A1 renormalized) \( \rightarrow \) converted to elastic wave

then

\[ \langle \mathbf{V}_1 \mathbf{V}_1 \rangle = \frac{C_{pq}}{2\theta} \langle \mathbf{Q} \mathbf{Q} \rangle \]
can expect reduction in net stress due to kinetic-elastic competition.

→ degree reduction will require/depend on approach to equi-partition.

→ obviously:

- higher Cp beneficial?

- glimmer of concentration dependence appearing.

→ quantitative?

- hydro-elastic layer ⇒ buffer layer, defined by transport reduction.

→ problem now closely tracks small scale dynamo problem at large Pr.

→ how close will hydro-elastic turbulence come to "equi-partition"?
on elastic_range?

Other Issues:

- elastic effect on $<u_1 u_2>$ cross_phase $\Rightarrow C[u_1 u_2]$ ?

- detailed differences between Hydro-Elasticity, MHD

but $\Rightarrow$ local $<\Phi>$ $\Rightarrow$ others seems compelling.

Subject open for further study and contradiction?