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Chapter 12

Noninertial Reference Frames

12.1 Accelerated Coordinate Systems

A reference frame which is fixed with respect to a rotating rigid body is not inertial. The parade example of this is an observer fixed on the surface of the earth. Due to the rotation of the earth, such an observer is in a noninertial frame, and there are corresponding corrections to Newton's laws of motion which must be accounted for in order to correctly describe mechanical motion in the observer's frame. As is well known, these corrections involve fictitious centrifugal and Coriolis forces.

Consider an inertial frame with a fixed set of coordinate axes $\hat{\mathbf{e}}_\mu$, where μ runs from 1 to d, the dimension of space, and a noninertial frame with axes $\hat{\mathbf{e}}'_\mu$. Any vector $\boldsymbol{A}$ may be written in either basis:

$$\boldsymbol{A} = \sum_\mu A_\mu \, \hat{\mathbf{e}}_\mu = \sum_\mu A'_\mu \, \hat{\mathbf{e}}'_\mu \ , \tag{12.1}$$

where $A_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}_\mu$ and $A'_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}'_\mu$ are projections onto the different coordinate axes. We may now write

$$\begin{aligned} \left(\frac{d\boldsymbol{A}}{dt}\right)_{\rm inertial} &= \sum_\mu \frac{dA_\mu}{dt} \, \hat{\mathbf{e}}_\mu \\ &= \sum_i \frac{dA'_\mu}{dt} \, \hat{\mathbf{e}}'_\mu + \sum_\mu A'_\mu \, \frac{d\hat{\mathbf{e}}'_\mu}{dt} \ . \end{aligned}$$

The first term on the RHS is $(d\boldsymbol{A}/dt)_{\rm body}$, the time derivative of $\boldsymbol{A}$ along body-fixed axes, *i.e.* as seen by an observer rotating with the body. But what is $d\hat{\mathbf{e}}'_i/dt$? Well, we can always expand it in the $\{\hat{\mathbf{e}}'_i\}$ basis:

$$d\hat{\mathbf{e}}'_\mu = \sum_\nu d\Omega_{\mu\nu} \, \hat{\mathbf{e}}'_\nu \quad \Longleftrightarrow \quad d\Omega_{\mu\nu} \equiv d\hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu \ . \tag{12.2}$$

Note that $d\Omega_{\mu\nu} = -d\Omega_{\nu\mu}$ is antisymmetric, because

$$0 = d\big(\hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu\big) = d\Omega_{\nu\mu} + d\Omega_{\mu\nu} \ , \tag{12.3}$$

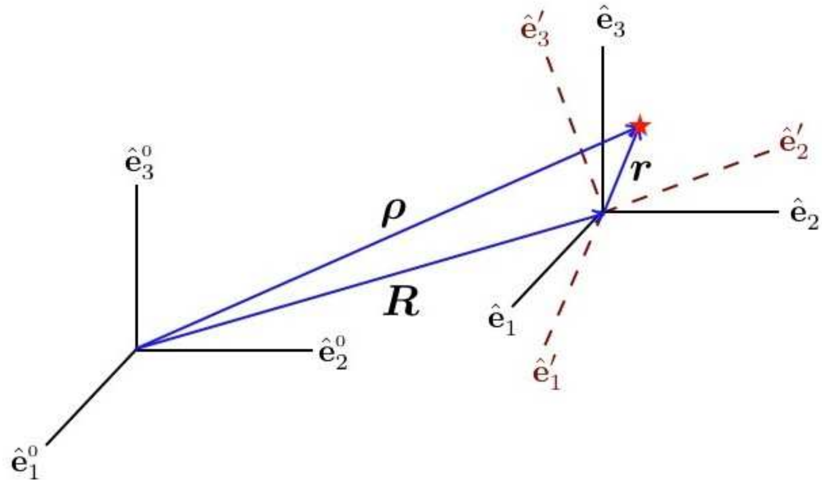

Figure 12.1: Reference frames related by both translation and rotation. Note $\hat{e}_\mu = \hat{e}_\mu^0$.

because $\hat{e}_\mu' \cdot \hat{e}_\nu' = \delta_{\mu\nu}$ is a constant. Now we may define $d\Omega_{12} \equiv d\Omega_3$, *et cyc.*, so that

$$d\Omega_{\mu\nu} = \sum_\sigma \epsilon_{\mu\nu\sigma}\, d\Omega_\sigma \quad , \quad \omega_\sigma \equiv \frac{d\Omega_\sigma}{dt} \ , \tag{12.4}$$

which yields

$$\frac{d\hat{e}_\mu'}{dt} = \boldsymbol{\omega} \times \hat{e}_\mu' \ . \tag{12.5}$$

Finally, we obtain the important result

$$\boxed{\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{A}} \tag{12.6}$$

which is valid for any vector $\boldsymbol{A}$.

Applying this result to the position vector $\boldsymbol{r}$, we have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \ . \tag{12.7}$$

Applying it twice,

$$\begin{aligned}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega} \times\right)\left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega} \times\right)\boldsymbol{r} \\ &= \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + 2\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \ .\end{aligned}$$

Note that $d\boldsymbol{\omega}/dt$ appears with no "inertial" or "body" label. This is because, upon invoking eq. 12.6,

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{\omega} \ , \tag{12.8}$$

and since $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$, inertial and body-fixed observers will agree on the value of $\dot{\boldsymbol{\omega}}_{\text{inertial}} = \dot{\boldsymbol{\omega}}_{\text{body}} \equiv \dot{\boldsymbol{\omega}}$.

12.1.1 Translations

Suppose that frame K moves with respect to an inertial frame K^0, such that the origin of K lies at $\boldsymbol{R}(t)$. Suppose further that frame K' rotates with respect to K, but shares the same origin (see Fig. 12.1). Consider the motion of an object lying at position $\boldsymbol{\rho}$ relative to the origin of K^0, and $\boldsymbol{r}$ relative to the origin of K/K'. Thus,

$$\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r} \, , \tag{12.9}$$

and

$$\begin{aligned} \left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\text{inertial}} &= \left(\frac{d\boldsymbol{R}}{dt}\right)_{\text{inertial}} + \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \\ \left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + 2\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \, . \end{aligned} \tag{12.10}$$

Here, $\boldsymbol{\omega}$ is the angular velocity in the frame K or K'.

12.1.2 Motion on the surface of the earth

The earth both rotates about its axis and orbits the Sun. If we add the infinitesimal effects of the two rotations,

$$\begin{aligned} d\boldsymbol{r}_1 &= \boldsymbol{\omega}_1 \times \boldsymbol{r}\, dt \\ d\boldsymbol{r}_2 &= \boldsymbol{\omega}_2 \times (\boldsymbol{r} + d\boldsymbol{r}_1)\, dt \\ d\boldsymbol{r} &= d\boldsymbol{r}_1 + d\boldsymbol{r}_2 = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\, dt \times \boldsymbol{r} + \mathcal{O}\big((dt)^2\big) \, . \end{aligned} \tag{12.11}$$

Thus, *infinitesimal rotations add.* Dividing by dt, this means that

$$\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}_i \, , \tag{12.12}$$

where the sum is over all the rotations. For the earth, $\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rot}} + \boldsymbol{\omega}_{\text{orb}}$.

- The rotation about earth's axis, $\boldsymbol{\omega}_{\text{rot}}$ has magnitude $\omega_{\text{rot}} = 2\pi/(1\,\text{day}) = 7.29 \times 10^{-5}\,\text{s}^{-1}$. The radius of the earth is $R_{\text{e}} = 6.37 \times 10^3$ km.
- The orbital rotation about the Sun, $\boldsymbol{\omega}_{\text{orb}}$ has magnitude $\omega_{\text{orb}} = 2\pi/(1\,\text{yr}) = 1.99 \times 10^{-7}\,\text{s}^{-1}$. The radius of the earth's orbit is $a_{\text{e}} = 1.50 \times 10^8$ km.

Thus, $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} = 365.25$, which is of course the number of days (*i.e.* rotational periods) in a year (*i.e.* orbital period). There is also a very slow precession of the earth's axis of rotation, the period of which is about 25,000 years, which we will ignore. Note $\dot{\boldsymbol{\omega}} = 0$ for the earth. Thus, applying Newton's second law and then invoking eq. 12.10, we arrive at

$$m\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\rm earth} = \boldsymbol{F}^{(\rm tot)} - m\left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\rm Sun} - \ 2m\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\rm earth} - \ m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\ , \tag{12.13}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm rot} + \boldsymbol{\omega}_{\rm orb}$, and where $\ddot{\boldsymbol{R}}_{\rm Sun}$ is the acceleration of the center of the earth around the Sun, assuming the Sun-fixed frame to be inertial. The force $\boldsymbol{F}_{(\rm tot)}$ is the total force on the object, and arises from three parts: (i) gravitational pull of the Sun, (ii) gravitational pull of the earth, and (iii) other earthly forces, such as springs, rods, surfaces, electric fields, *etc.*

On the earth's surface, the ratio of the Sun's gravity to the earth's is

$$\frac{F_\odot}{F_{\rm e}} = \frac{GM_\odot m}{a_{\rm e}^2}\bigg/\frac{GM_{\rm e} m}{R_{\rm e}^2} = \frac{M_\odot}{M_{\rm e}}\left(\frac{R_{\rm e}}{a_{\rm e}}\right)^2 \approx 6.02\times 10^{-4}\ . \tag{12.14}$$

In fact, it is clear that the Sun's field precisely cancels with the term $m\,\ddot{\boldsymbol{R}}_{\rm Sun}$ at the earth's center, leaving only gradient contributions of even lower order, *i.e.* multiplied by another factor of $R_{\rm e}/a_{\rm e} \approx 4.25\times 10^{-5}$. Thus, to an excellent approximation, we may neglect the Sun entirely and write

$$\boxed{\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}'}{m} + \boldsymbol{g} - 2\,\boldsymbol{\omega}\times\frac{d\boldsymbol{r}}{dt} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})} \tag{12.15}$$

Note that we've dropped the 'earth' label here and henceforth. We define $\boldsymbol{g} = -GM_{\rm e}\,\hat{\boldsymbol{r}}/r^2$, the acceleration due to gravity; $\boldsymbol{F}'$ is the sum of all earthly forces other than the earth's gravity. The last two terms on the RHS are corrections to $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ due to the noninertial frame of the earth, and are recognized as the Coriolis and centrifugal acceleration terms, respectively.

12.2 Spherical Polar Coordinates

The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ varies with position. In terms of the body-fixed triad $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we have

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}
\end{aligned} \tag{12.16}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the *colatitude* (*i.e.* $\lambda \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ is the latitude). Inverting the relation between the triads $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we obtain

$$\begin{aligned}
\hat{\boldsymbol{x}} &= \sin\theta\,\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{y}} &= \sin\theta\,\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{z}} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ .
\end{aligned} \tag{12.17}$$

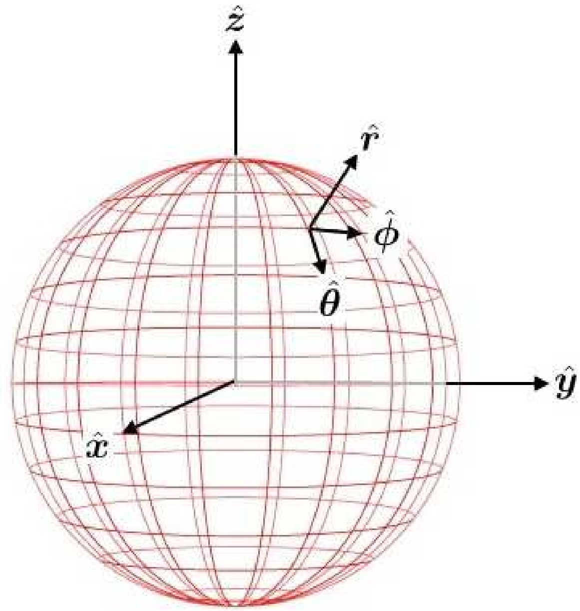

Figure 12.2: The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$.

The differentials of these unit vectors are

$$\begin{aligned} d\hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}\, d\theta + \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \\ d\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{r}}\, d\theta + \cos\theta\, \hat{\boldsymbol{\phi}}\, d\phi \\ d\hat{\boldsymbol{\phi}} &= -\sin\theta\, \hat{\boldsymbol{r}}\, d\phi - \cos\theta\, \hat{\boldsymbol{\theta}}\, d\phi\ . \end{aligned} \tag{12.18}$$

Thus,

$$\begin{aligned} \dot{\boldsymbol{r}} &= \frac{d}{dt}\big(r\,\hat{\boldsymbol{r}}\big) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}} \\ &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\ . \end{aligned} \tag{12.19}$$

If we differentiate a second time, we find, after some tedious accounting,

$$\begin{aligned} \ddot{\boldsymbol{r}} = \big(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\!\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} + \big(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{\theta}} \\ + \big(2\,\dot{r}\,\dot{\phi}\,\sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi}\,\cos\theta + r\,\sin\theta\,\ddot{\phi}\,\big)\,\hat{\boldsymbol{\phi}}\ . \end{aligned} \tag{12.20}$$

12.3 Centrifugal Force

One major distinction between the Coriolis and centrifugal forces is that the Coriolis force acts only on moving particles, whereas the centrifugal force is present even when $\dot{\boldsymbol{r}} = 0$. Thus, the equation for stationary equilibrium on the earth's surface is

$$m\boldsymbol{g} + \boldsymbol{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = 0\ , \tag{12.21}$$

involves the centrifugal term. We can write this as $\boldsymbol{F}' + m\widetilde{\boldsymbol{g}} = 0$, where

$$\begin{aligned}\widetilde{\boldsymbol{g}} &= -\frac{GM_\mathrm{e}\,\hat{\boldsymbol{r}}}{r^2} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \\ &= -\big(g_0 - \omega^2 R_\mathrm{e}\,\sin^2\theta\big)\,\hat{\boldsymbol{r}} + \omega^2\,R_\mathrm{e}\,\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\ ,\end{aligned} \tag{12.22}$$

where $g_0 = GM_\mathrm{e}/R_\mathrm{e}^2 = 980\,\mathrm{cm/s^2}$. Thus, on the equator, where $\theta = \frac{\pi}{2}$, we have $\tilde{\boldsymbol{g}} = -\big(g_0 - \omega^2 R_\mathrm{e}\big)\,\hat{\boldsymbol{r}}$, with $\omega^2 R_\mathrm{e} \approx 3.39\,\mathrm{cm/s^2}$, a small but significant correction. You therefore weigh less on the equator. Note also that $\widetilde{\boldsymbol{g}}$ has a component along $\hat{\boldsymbol{\theta}}$. This means that a plumb bob suspended from a general point above the earth's surface won't point exactly toward the earth's center. Moreover, if the earth were replaced by an equivalent mass of fluid, the fluid would rearrange itself so as to make its surface locally perpendicular to $\widetilde{\boldsymbol{g}}$. Indeed, the earth (and Sun) do exhibit quadrupolar distortions in their mass distributions – both are oblate spheroids. In fact, the observed difference $\tilde{g}(\theta=0) - \tilde{g}(\theta=\frac{\pi}{2}) \approx 5.2\,\mathrm{cm/s^2}$, which is 53% greater than the naïvely expected value of $3.39\,\mathrm{cm/s^2}$. The earth's oblateness enhances the effect.

12.3.1 Rotating tube of fluid

Consider a cylinder filled with a liquid, rotating with angular frequency ω about its symmetry axis $\hat{\boldsymbol{z}}$. In steady state, the fluid is stationary in the rotating frame, and we may write, for any given element of fluid

$$0 = \boldsymbol{f}' + \boldsymbol{g} - \omega^2\,\hat{\boldsymbol{z}}\times(\hat{\boldsymbol{z}}\times\boldsymbol{r})\ , \tag{12.23}$$

where $\boldsymbol{f}'$ is the force per unit mass on the fluid element. Now consider a fluid element on the surface. Since there is no static friction to the fluid, any component of $\boldsymbol{f}'$ parallel to the fluid's surface will cause the fluid to flow in that direction. This contradicts the steady state assumption. Therefore, we must have $\boldsymbol{f}' = f'\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the local unit normal to the fluid surface. We write the equation for the fluid's surface as $z = z(\rho)$. Thus, with $\boldsymbol{r} = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$, Newton's second law yields

$$f'\,\hat{\boldsymbol{n}} = g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}}\ , \tag{12.24}$$

where $\boldsymbol{g} = -g\,\hat{\boldsymbol{z}}$ is assumed. From this, we conclude that the unit normal to the fluid surface and the force per unit mass are given by

$$\hat{\boldsymbol{n}}(\rho) = \frac{g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}}}{\sqrt{g^2 + \omega^4\,\rho^2}} \qquad , \qquad f'(\rho) = \sqrt{g^2 + \omega^4\rho^2}\ . \tag{12.25}$$

Now suppose $\boldsymbol{r}(\rho,\phi) = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$ is a point on the surface of the fluid. We have that

$$d\boldsymbol{r} = \hat{\boldsymbol{\rho}}\,d\rho + z'(\rho)\,\hat{\boldsymbol{z}}\,d\rho + \rho\,\hat{\boldsymbol{\phi}}\,d\phi\ , \tag{12.26}$$

where $z' = dz/d\rho$, and where we have used $d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\,d\phi$, which follows from the first of eqn. 12.18 after setting $\theta = \frac{\pi}{2}$. Now $d\boldsymbol{r}$ must lie along the surface, therefore $\hat{\boldsymbol{n}}\cdot d\boldsymbol{r} = 0$, which says

$$g\,\frac{dz}{d\rho} = \omega^2\,\rho\ . \tag{12.27}$$

Integrating this equation, we obtain the shape of the surface:

$$z(\rho) = z_0 + \frac{\omega^2\rho^2}{2g}\ . \tag{12.28}$$

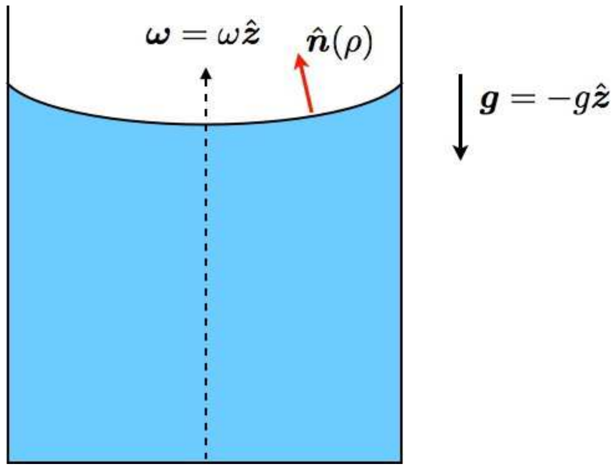

Figure 12.3: A rotating cylinder of fluid.

12.4 The Coriolis Force

12.4.1 Projectile motion

The Coriolis force is given by $\boldsymbol{F}_{\text{Cor}} = -2m\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}}$. According to (12.15), the acceleration of a free particle $(\boldsymbol{F}' = 0)$ isn't along $\widetilde{\boldsymbol{g}}$ – an orthogonal component is generated by the Coriolis force. To actually solve the coupled equations of motion is difficult because the unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ change with position, and hence with time. The following standard problem highlights some of the effects of the Coriolis and centrifugal forces.

PROBLEM: A cannonball is dropped from the top of a tower of height h located at a northerly latitude of λ. Assuming the cannonball is initially at rest with respect to the tower, and neglecting air resistance, calculate its deflection (magnitude and direction) due to (a) centrifugal and (b) Coriolis forces by the time it hits the ground. Evaluate for the case $h = 100$ m, $\lambda = 45°$. The radius of the earth is $R_{\text{e}} = 6.4 \times 10^6$ m.

SOLUTION: The equation of motion for a particle near the earth's surface is

$$\ddot{\boldsymbol{r}} = -2\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}} - g_0\,\hat{\boldsymbol{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\ , \tag{12.29}$$

where $\boldsymbol{\omega} = \omega\,\hat{\boldsymbol{z}}$, with $\omega = 2\pi/(24\,\text{hrs}) = 7.3 \times 10^{-5}\,\text{rad/s}$. Here, $g_0 = GM_{\text{e}}/R_{\text{e}}^2 = 980\,\text{cm/s}^2$. We use a locally orthonormal coordinate system $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + (R_{\text{e}} + z)\,\hat{\boldsymbol{r}}\ , \tag{12.30}$$

where $R_{\text{e}} = 6.4 \times 10^6$ m is the radius of the earth. Expressing $\hat{\boldsymbol{z}}$ in terms of our chosen orthonormal triad,

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ , \tag{12.31}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the polar angle, or 'colatitude'. Since the height of the tower and the deflections are all very small on the scale of R_{e}, we may regard the orthonormal triad as fixed and time-independent, although, in general, these unit vectors change as a function of $\boldsymbol{r}$. Thus, we have $\dot{\boldsymbol{r}} \simeq \dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}$, and we find

$$\begin{aligned} \hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} &= (\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times (\dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}) \\ &= -\dot{y}\,\cos\theta\,\hat{\boldsymbol{\theta}} + (\dot{x}\,\cos\theta + \dot{z}\,\sin\theta)\,\hat{\boldsymbol{\phi}} - \dot{y}\sin\theta\,\hat{\boldsymbol{r}} \end{aligned} \tag{12.32}$$

and

$$\begin{aligned} \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) &= \omega^2(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times \Big((\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times (R_\text{e}\,\hat{\boldsymbol{r}} + \overbrace{x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}})}^{\text{negligible}}\Big) \\ &\approx \omega^2(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times R_\text{e}\sin\theta\,\hat{\boldsymbol{\phi}} \\ &= -\omega^2 R_\text{e}\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} - \omega^2 R_\text{e}\sin^2\!\theta\,\hat{\boldsymbol{r}}\ . \end{aligned} \tag{12.33}$$

Note that the distances x, y, and z are all extremely small in magnitude compared with R_e.

The equations of motion, written in components, are then

$$\begin{aligned} \dot{v}_x &= g_1\sin\theta\cos\theta + 2\omega\cos\theta\,v_y \\ \dot{v}_y &= -2\omega\cos\theta\,v_x - 2\omega\sin\theta\,v_z \\ \dot{v}_z &= -g_0 + g_1\sin^2\!\theta + 2\omega\sin\theta\,v_y\ , \end{aligned} \tag{12.34}$$

with $g_1 \equiv \omega^2 R_\text{e}$. While these (inhomogeneous) equations are linear, they also are coupled, so an exact analytical solution is not trivial to obtain (but see below). Fortunately, the deflections are small, so we can solve this perturbatively. To do so, let us write $\boldsymbol{v}(t)$ as a power series in t. For each component, we write

$$v_\alpha(t) = \sum_{n=0}^{\infty} v_{\alpha,n}\,t^n \quad , \tag{12.35}$$

with $v_{\alpha,0} = v_\alpha(t=0) \equiv v^0_\alpha$. Eqns. 12.34 then may be written as the coupled hierarchy

$$\begin{aligned} n v_{x,n} &= g_1\sin\theta\cos\theta\,\delta_{n,1} + 2\omega\cos\theta\,v_{y,n-1} \\ n v_{y,n} &= -2\omega\cos\theta\,v_{x,n-1} - 2\omega\sin\theta\,v_{z,n-1} \\ n v_{z,n} &= -(g_0 - g_1\sin^2\!\theta)\,\delta_{n,1} + 2\omega\sin\theta\,v_{y,n-1} \quad . \end{aligned} \tag{12.36}$$

Integrating $\boldsymbol{v}(t)$, we obtain the displacements,

$$x_\alpha(t) = x^0_\alpha + \sum_{n=0}^{\infty} \frac{v_{\alpha,n}}{n+1}\,t^{n+1} \quad . \tag{12.37}$$

Now let's roll up our sleeves and solve for the coefficients $v_{\alpha,n}$ for $n = 0, 1, 2$. This will give us the displacements up to terms of order t^3. For $n = 0$ we already have $v_{\alpha,0} = v^0_\alpha$. For $n = 1$, we use Eqns. 12.36 with $n = 1$ to obtain

$$\begin{aligned} v_{x,1} &= 2\omega\cos\theta\,v^0_y + g_1\sin\theta\cos\theta \\ v_{y,1} &= -2\omega\cos\theta\,v^0_x - 2\omega\sin\theta\,v^0_z \\ v_{z,1} &= 2\omega\sin\theta\,v^0_y - g_0 + g_1\sin^2\!\theta \quad . \end{aligned} \tag{12.38}$$

Finally, at level $n = 2$, we have

$$\begin{aligned}
v_{x,2} &= \omega\cos\theta\, v_{y,1} = -2\omega^2\cos\theta\,(\cos\theta\, v_x^0 + \sin\theta\, v_z^0) \\
v_{y,2} &= -2\omega\cos\theta\, v_{x,1} - 2\omega\sin\theta\, v_{z,1} = -2\omega^2 v_y^0 + \omega\sin\theta\,(g_0 - g_1) \\
v_{z,2} &= \omega\sin\theta\, v_{y,1} = -2\omega^2\sin\theta\,(\cos\theta\, v_x^0 + \sin\theta\, v_z^0) \quad .
\end{aligned} \tag{12.39}$$

Thus, the displacements are given by

$$\begin{aligned}
x(t) &= x(0) + v_x^0\, t + \tfrac{1}{2}\big(2\omega\cos\theta\, v_y^0 + g_1\sin\theta\cos\theta\big)\, t^2 - \tfrac{2}{3}\,\omega^2\cos\theta\,(\cos\theta\, v_x^0 + \sin\theta\, v_z^0)\, t^3 + \mathcal{O}(t^4) \\
y(t) &= y(0) + v_y^0\, t - \omega\,(\cos\theta\, v_x^0 + \sin\theta\, v_z^0)\, t^2 - \tfrac{2}{3}\,\omega^2 v_y^0\, t^3 + \tfrac{1}{3}\,\omega\sin\theta\,(g_0 - g_1)\, t^3 + \mathcal{O}(t^4) \\
z(t) &= z(0) + v_z^0\, t + \tfrac{1}{2}\big(2\omega\sin\theta\, v_y^0 - g_0 + g_1\sin^2\!\theta\big)\, t^2 - \tfrac{2}{3}\,\omega^2\sin\theta\,(\cos\theta\, v_x^0 + \sin\theta\, v_z^0)\, t^3 + \mathcal{O}(t^4) \quad .
\end{aligned} \tag{12.40}$$

When dropped from rest, with $x(0) = y(0) = 0$ and $z(0) = h_0$, we have

$$\begin{aligned}
x(t) &= \tfrac{1}{2} g_1 \sin\theta\cos\theta\, t^2 + \mathcal{O}(t^4) \\
y(t) &= \tfrac{1}{3}\,\omega\sin\theta\,(g_0 - g_1)\, t^3 + \mathcal{O}(t^4) \\
z(t) &= h_0 - \tfrac{1}{2}\,(g_0 - g_1\sin^2\!\theta)\, t^2 + \mathcal{O}(t^4) \quad .
\end{aligned} \tag{12.41}$$

Recall $g_1 = \omega^2 R_e$, so if we neglect the rotation of the earth and set $\omega = 0$, we have $\omega = g_1 = 0$, and $z(t) = h_0 - \frac{1}{2} g_0 t^2$ with $x(t) = y(t) = 0$. This is the familiar high school physics result. As we see, in the noninertial reference frame of the rotating earth, there are deflections along $\hat{\boldsymbol{\theta}}$ given by $x(t)$, along $\hat{\boldsymbol{\phi}}$ given by $y(t)$, and also a correction $\Delta z(t) = \frac{1}{2} g_1 \sin^2\theta\, t^2 + \mathcal{O}(t^4)$ to the motion along $\hat{\boldsymbol{r}}$. To find the deflection of an object dropped from a height h_0, solve $z(t^*) = 0$ to obtain $t^* = \sqrt{2h/(g_0 - g_1\sin^2\!\theta)}$ for the drop time, and substitute. For $h_0 = 100\,\mathrm{m}$ and $\lambda = \frac{\pi}{2}$, find $\delta x(t^*) = 17\,\mathrm{cm}$ south (centrifugal) and $\delta y(t^*) = 1.6\,\mathrm{cm}$ east (Coriolis). Note that the centrifugal term dominates the deflection in this example. Why is the Coriolis deflection always to the east? The earth rotates eastward, and an object starting from rest in the earth's frame has initial angular velocity equal to that of the earth. To conserve angular momentum, the object must speed up as it falls.

Exact solution for velocities

In fact, an exact solution to (12.34) is readily obtained, via the following analysis. The equations of motion may be written $\dot{\boldsymbol{v}} = 2i\omega\mathcal{J}\boldsymbol{v} + \boldsymbol{b}$, or

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_x \end{pmatrix} = 2i\,\omega \overbrace{\begin{pmatrix} 0 & -i\cos\theta & 0 \\ i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 0 \end{pmatrix}}^{\mathcal{J}} \begin{pmatrix} v_x \\ v_y \\ v_x \end{pmatrix} + \overbrace{\begin{pmatrix} g_1\sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\!\theta \end{pmatrix}}^{\boldsymbol{b}} \quad . \tag{12.42}$$

Note that $\mathcal{J}^\dagger = \mathcal{J}$, *i.e.* $\mathcal{J}$ is a Hermitian matrix. The formal solution is

$$\boldsymbol{v}(t) = e^{2i\omega\mathcal{J}t}\,\boldsymbol{v}(0) + \left(\frac{e^{2i\omega\mathcal{J}t} - 1}{2i\omega}\right)\mathcal{J}^{-1}\,\boldsymbol{b}\,. \tag{12.43}$$

When working with matrices, it is convenient to work in an eigenbasis. The characteristic polynomial for $\mathcal{J}$ is $P(\lambda) = \det(\lambda \cdot 1 - \mathcal{J}) = \lambda\,(\lambda^2 - 1)$, hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = +1$, and $\lambda_3 = -1$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ i \\ \sin\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ -i \\ \sin\theta \end{pmatrix} \quad . \tag{12.44}$$

Note that $\boldsymbol{\psi}_a^\dagger \cdot \boldsymbol{\psi}_{a'} = \delta_{aa'}$.

Expanding $\boldsymbol{v}$ and $\boldsymbol{b}$ in this eigenbasis, we have $\dot{u}_a = 2i\omega\lambda_a u_a + b_a$, where $u_a = \psi^*_{ia}\, v_i$ and $b_a = \psi^*_{ia}\, b_i$. The solution is

$$u_a(t) = u_a(0)\, e^{2i\lambda_a \omega t} + \left(\frac{e^{2i\lambda_a\omega t} - 1}{2i\lambda_a\omega}\right) b_a \quad . \tag{12.45}$$

Since the eigenvectors of $\mathcal{J}$ are orthonormal, $u_a = \psi^*_{ia}\, v_i$ entails $v_i = \psi_{ia} u_a$, hence

$$v_i(t) = \sum_j \left(\sum_a \psi_{ia}\, e^{2i\lambda_a\omega t}\, \psi^*_{ja}\right) v_j(0) + \sum_j \left(\sum_a \psi_{ia} \left(\frac{e^{2i\lambda_a\omega t} - 1}{2i\lambda_a\omega}\right) \psi^*_{ja}\right) b_j \quad . \tag{12.46}$$

Doing the requisite matrix multiplications, and assuming $\boldsymbol{v}(0) = 0$, we obtain

$$\begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} t\sin^2\!\theta + \frac{\sin 2\omega t}{2\omega}\cos^2\!\theta & \frac{\sin^2\!\omega t}{\omega}\cos\theta & -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta \\ -\frac{\sin^2\!\omega t}{\omega}\cos\theta & \frac{\sin 2\omega t}{2\omega} & -\frac{\sin^2\!\omega t}{\omega}\sin\theta \\ -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta & \frac{\sin^2\!\omega t}{\omega}\sin\theta & t\cos^2\!\theta + \frac{\sin 2\omega t}{2\omega}\sin^2\!\theta \end{pmatrix} \begin{pmatrix} g_1 \sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\!\theta \end{pmatrix} , \tag{12.47}$$

which says

$$\begin{aligned} v_x(t) &= \left(\tfrac{\sin 2\omega t}{2\omega t} - 1\right) g_0 t \sin\theta\cos\theta + \tfrac{\sin 2\omega t}{2\omega t}\, g_1 t \sin\theta\cos\theta \\ v_y(t) &= \tfrac{\sin^2\!\omega t}{\omega t}\,(g_0 - g_1)\, t\sin\theta \\ v_z(t) &= -\left(\cos^2\!\theta + \tfrac{\sin 2\omega t}{2\omega t}\sin^2\!\theta\right) g_0 t + \tfrac{\sin^2\!\omega t}{2\omega t}\, g_1 t \sin^2\!\theta \, . \end{aligned} \tag{12.48}$$

One can check that by expanding in a power series in t we recover the results of the previous section.

12.4.2 Foucault's pendulum

A pendulum swinging over one of the poles moves in a fixed inertial plane while the earth rotates underneath. Relative to the earth, the plane of motion of the pendulum makes one revolution every day. What happens at a general latitude? Assume the pendulum is located at colatitude θ and longitude ϕ. Assuming the length scale of the pendulum is small compared to $R_{\rm e}$, we can regard the local triad $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{r}}\}$ as fixed. The situation is depicted in Fig. 12.4. We write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}} \, , \tag{12.49}$$

with

$$x = \ell\,\sin\psi\,\cos\alpha \quad , \quad y = \ell\,\sin\psi\,\sin\alpha \quad , \quad z = \ell\,(1 - \cos\psi) \, . \tag{12.50}$$

In our analysis we will ignore centrifugal effects, which are of higher order in ω, and we take $\boldsymbol{g} = -g\,\hat{\boldsymbol{r}}$. We also idealize the pendulum, and consider the suspension rod to be of negligible mass.

The total force on the mass m is due to gravity and tension:

$$\begin{aligned}\boldsymbol{F} &= m\boldsymbol{g} + \boldsymbol{T} \\ &= \big(-T\sin\psi\,\cos\alpha,\ -T\sin\psi\,\sin\alpha,\ T\cos\psi - mg\big) \\ &= \big(-Tx/\ell,\ -Ty/\ell,\ T - Mg - Tz/\ell\big)\ .\end{aligned} \tag{12.51}$$

The Coriolis term is

$$\begin{aligned}\boldsymbol{F}_{\rm Cor} &= -2m\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}} \\ &= -2m\,\omega\big(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\big)\times\big(\dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}\big) \\ &= 2m\omega\,\big(\dot{y}\cos\theta,\ -\dot{x}\cos\theta - \dot{z}\sin\theta,\ \dot{y}\sin\theta\big)\ .\end{aligned} \tag{12.52}$$

The equations of motion are $m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{F}_{\rm Cor}$:

$$\begin{aligned} m\ddot{x} &= -Tx/\ell + 2m\omega\cos\theta\,\dot{y} \\ m\ddot{y} &= -Ty/\ell - 2m\omega\cos\theta\,\dot{x} - 2m\omega\sin\theta\,\dot{z} \\ m\ddot{z} &= T - mg - Tz/\ell + 2m\omega\sin\theta\,\dot{y}\ .\end{aligned} \tag{12.53}$$

These three equations are to be solved for the three unknowns x, y, and T. Note that

$$x^2 + y^2 + (\ell - z)^2 = \ell^2\ , \tag{12.54}$$

so $z = z(x,y)$ is not an independent degree of freedom. This equation may be recast in the form $z = (x^2+y^2+z^2)/2\ell$ which shows that if x and y are both small, then z is at least of second order in smallness. Therefore, we will approximate $z \simeq 0$, in which case $\dot{z}$ may be neglected from the second equation of motion. The third equation is used to solve for T:

$$T \simeq mg - 2m\omega\sin\theta\,\dot{y}\ . \tag{12.55}$$

Adding the first plus i times the second then gives the complexified equation

$$\begin{aligned}\ddot{\xi} &= -\frac{T}{m\ell}\,\xi - 2i\omega\cos\theta\,\dot{\xi} \\ &\approx -\omega_0^2\,\xi - 2i\omega\cos\theta\,\dot{\xi}\end{aligned} \tag{12.56}$$

where $\xi \equiv x + iy$, and where $\omega_0 = \sqrt{g/\ell}$. Note that we have approximated $T \approx mg$ in deriving the second line.

It is now a trivial matter to solve the homogeneous linear ODE of eq. 12.56. Writing

$$\xi = \xi_0\,e^{-i\Omega t} \tag{12.57}$$

and plugging in to find Ω, we obtain

$$\Omega^2 - 2\omega_\perp\Omega - \omega_0^2 = 0\ , \tag{12.58}$$

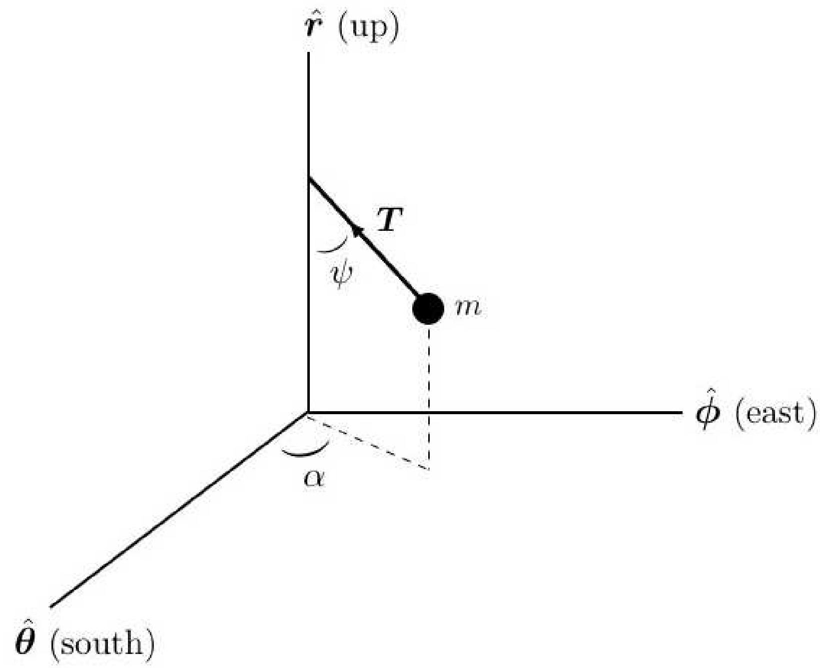

Figure 12.4: Foucault's pendulum.

with $\omega_\perp \equiv \omega \cos\theta$. The roots are

$$\Omega_\pm = \omega_\perp \pm \sqrt{\omega_0^2 + \omega_\perp^2}\ , \tag{12.59}$$

hence the most general solution is

$$\xi(t) = A_+ \, e^{-i\Omega_+ t} + A_- \, e^{-i\Omega_- t}\ . \tag{12.60}$$

Finally, if we take as initial conditions $x(0) = a$, $y(0) = 0$, $\dot{x}(0) = 0$, and $\dot{y}(0) = 0$, we obtain

$$\begin{aligned} x(t) &= \left(\frac{a}{\nu}\right) \cdot \Big\{ \omega_\perp \, \sin(\omega_\perp t) \, \sin(\nu t) + \nu \, \cos(\omega_\perp t) \, \cos(\nu t) \Big\} \\ y(t) &= \left(\frac{a}{\nu}\right) \cdot \Big\{ \omega_\perp \, \cos(\omega_\perp t) \, \sin(\nu t) - \nu \, \sin(\omega_\perp t) \, \cos(\nu t) \Big\}\ , \end{aligned} \tag{12.61}$$

with $\nu = \sqrt{\omega_0^2 + \omega_\perp^2}$. Typically $\omega_0 \gg \omega_\perp$, since $\omega = 7.3 \times 10^{-5}\,\mathrm{s}^{-1}$. In the limit $\omega_\perp \ll \omega_0$, then, we have $\nu \approx \omega_0$ and

$$x(t) \simeq a \, \cos(\omega_\perp t) \, \cos(\omega_0 t) \quad , \quad y(t) \simeq -a \, \sin(\omega_\perp t) \, \cos(\omega_0 t)\ , \tag{12.62}$$

and the plane of motion rotates with angular frequency $-\omega_\perp$, *i.e.* the period is $|\sec\theta\,|$ days. Viewed from above, the rotation is clockwise in the northern hemisphere, where $\cos\theta > 0$ and counterclockwise in the southern hemisphere, where $\cos\theta < 0$.