Self-Organized Criticality II
Hydrodynamic Models

Recall

→ SOC idea
→ Sandpile Model (CA)

Now, natural to ask:

→ Is there a continuum model as avalanche $\rightarrow \Delta$?

Can one think in terms of avalanche turbulence?

→ Can one exploit symmetry in deriving SOC model, much as symmetry exploited in Ginzburg-Landau model

These bring us to the hydrodynamic model of SOC

→ Continuum model

→ Valid for large scales, long time scales.
Begin

Consider:

- box with ejecting boundary on RHS
  accumulating boundary on LHS

- SOC profile, TBD

- noise

[Diagram of a box with a line going up and to the right, labeled "SOC profile", and another line going down and to the right, labeled "drops"]

Now consider deviations from SOC profile, i.e.

- bumps, blobs

- voids, holes
- Also, assume conservation of "stuff" in the profile, up to boundary losses and noise source, Call stuff $\mathbf{p}$.

- Idea is to describe dynamics of deviation from SOC state

\[ \mathbf{p} = \mathbf{p}_{SOC} + \delta \mathbf{p} \]

Formally pointed, not calculated.

\[ \partial_t \mathbf{p} + \partial_x \left[ \Gamma'(\delta \mathbf{p}) - \mathbf{D}_0 \partial_x \delta \mathbf{p} \right] = \mathbf{S} \]

- $\Gamma'(\delta \mathbf{p})$ is flux induced by deviation from SOC state

- Obviously, $\mathbf{p}$ and noise so $\delta \mathbf{p}$ evolve via $\mathbf{D} \Gamma'$ only

- Background diffusion posited.
- $\Gamma(\delta p) \to a \to \delta p \to a$

- $\delta p \to a \to \delta \to a$

- How constrain $\Gamma(\delta p) \to a$ Symmetry

- In spirit of Ginzburg/Landau prescription.

Now consider

Blob spreads out, conserving area.

Likewise void.
left-right

Now if symmetry broken by

\( T \) = 0

up-slope

down-slope

dump \( \rightarrow \) greater (extent in steepness) on down-slope

\( \Rightarrow \) bumps / local excesses propagate down gradient, to right

Necessarily,

\( \Rightarrow \) void \( \rightarrow \) greater (extent on up-slope (steepness) than down-slope

\( \Rightarrow \) voids / local deficits propagate up gradient, to left
→ Both criteria locally

→ Both criteria common sense.

Now, observe

1. Reflection $x \to -y$

i.e.

2. Bump hole interchange

$\rightarrow$ void up

$\rightarrow$ right $\checkmark$ right $\rightarrow$ $\checkmark$

Same flux direction $\checkmark$
This brings us to the principle of joint reflection symmetry!

\[ \Gamma' = \Gamma \]
\[ x \rightarrow -x \]
\[ \delta p \rightarrow -\delta p \]

This constrains the form of \( \Gamma' (\delta p) \)!

How?

\( \text{N.B.} \) : Full flux is complicated.

Seek flux in large scale, long time limit \( \Rightarrow \) smoothest term.

So have

\[ \delta t \delta p + \dot{x} \left[ \Gamma (\delta p) - D \delta x \delta p \right] = \tilde{S} \]

\( \Gamma (\delta p) \) must satisfy joint reflection symmetry.
Then formally:

\[ p(q, p) = \sum_0^N \left( A_n (q, p)^n + B_m (q, p)^m \right) \]

\[ + 2 (q, p)^x + 2 (q, p)^y + \int \]

\[ \text{JRS: \textit{Universit \textit{reflection symmetry}}.} \]

1. \[ n=1 \text{ violates JRS} \]

2. \[ x = 0.5 p^2 + h.o.t. \]
   \[ x > 0 \]

3. \[ m=1 \text{ OK} \]

4. \[ m=2 \text{ OK} \]

5. \[ -D_{\alpha} (q, p)^2 + h.o.t. \]
   \[ \alpha > 0 \text{ (will be shown)} \]

6. \[ x = 1 \text{ violates JRS} \]

\[ x = 2 \text{ too fine scaled} \]

\[ \text{ignore.} \]
\( z = 1, \ r = 1 \) violates JRS

so to lowest order in roughness:

\[
\frac{\partial p}{\partial t} + \nabla \left[ x \frac{\partial p^2}{\partial x} - D \frac{\partial p}{\partial x} \right] = \frac{\partial \eta}{\partial x}
\]

\( \eta, D \) are constants to be specified, as in G-L theory one.

Re-work \( \eta \) onto \( \eta' \):

\[
\frac{\partial \eta'}{\partial t} + \nabla \left[ x \frac{\partial \eta'^2}{\partial x} - D \frac{\partial \eta'}{\partial x} \right] = \frac{\partial \eta}{\partial x}
\]

- hydro model limit is noisy Burgers
- exactly solvable for \( \eta = 0 \)
- basic solution structure is shocks
- Shocks thin and weaken,...

Now, seek long wavelength approximation to nonlinear flux

\[ \left[ \frac{dx \log p^2}{dy} \right]_n \to \frac{\partial y}{\partial n} \log p_n \]

\[ = \frac{\chi k^2 \partial p_n}{\nu} \]

Turbulent viscosity

N.B. \[ \times \partial p^2 \to -Q(C_d p) \partial p \]

\[ \rightarrow -D(\partial p \partial \log p) \partial p \]

clear correspondence to expected CL expression for flux, with threshold.
Now,

\[ N_{k,w} = \int [\hat{d}k \, dP^2] \int_{\eta,\omega} \rightarrow v k^2 \, dP \]

\[ = \int \frac{\omega^2}{k^2 w} \, dP \]

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where:

The nonlinear screening of the coupling time

\[ -i (\omega + \omega') + (k + k')^2 \Delta + (k + k')^2 \delta \]

\[ = -i \omega (k + k') \, dP \]

and substituting gives:

\[ N_{k,w} = v k^2 \, dP \]

where
For \( k, \omega \to 0 \) 

Long smooth 
slow limit

\[
\frac{\rho}{T} = \sum_p \frac{\hbar \nu_p}{2} \frac{1}{\omega_p} \left[ \frac{\omega^2 + (\hbar \nu_p)^2}{\omega^2 + (\hbar \nu_p)^2} \right] 
\]

where neglected \( \rho \) relative to \( \nu^0 \).

Now need related \( \delta \rho \) to noise
(i.e. \( k, \omega \to \) high freq. short wavelength modes excited). This must also include nonlinear response self-consistently

\[
(-i \omega + \hbar^2 \nu) \delta \rho_{\nu} = \delta \rho_{\nu}
\]
\[ n = \alpha^2 \sum \frac{15 \ell^2}{(\ell^2 + r)^3} \frac{1}{1 + \left( \frac{\ell}{\sqrt{\alpha k^2}} \right)^2} \]

\[ \sum_{\ell \neq 0} = \int_{\text{kin}} d\ell \int_{\text{kin}} d\omega \]

and

\[ B_{\omega, \ell} \rho^2 = \delta^2 \rightarrow \text{white noise} \]

\[ \rho = \frac{\delta^2}{C_1 \alpha^2} \sum_{\ell \neq 0} \int_{\text{kin}} \frac{d\ell}{\ell^{14}} \]

\[ \alpha \rightarrow \text{infrared divergence} \]

\[ \text{conserved order parameter} \]

\[ \text{why?} \] (flux term) [8.17]

- slow modes
Slow modes $\rightarrow$ damping drops

$$\gamma \sim -k^2 v$$

$$\rightarrow \gamma_0 \ll \gamma$$

Weak noise + tiny decay $\Rightarrow$

Strong intensity.

$\Rightarrow$ General point: weakly damped

Modes dangerous if any excitation available.

$$V_t = \left( C_1 x^2 S_0^2 \int_{0}^{\infty} \frac{dk}{k^4} \right)^{1/3}$$

$$= \left( C_1 x^2 S_0^2 \right)^{1/3} k_{min}^{-1}$$

$V_t$ depends explicitly on cut-off scale.
Now meaning? 

\[ k^{-1} \equiv \frac{\ell}{l} \rightarrow \text{scale being observed} \]

\[ d\ell' < d\ell \rightarrow \text{scatter} \]

\[ \Rightarrow \]

\[ \sqrt{t} \sim \sqrt{t_0} \Delta \ell \]

\[ \left\{ \begin{array}{l}
\sqrt{t} \text{ grows with scale of interest} \\
\sqrt{t} \text{ is differing} \Rightarrow \\
\end{array} \right. \]

\[ \frac{d\langle \Delta \ell^2 \rangle}{dt} \sim \sqrt{t} , \quad \text{but} \]

\[ \Delta \ell^2 \sim \sqrt{t_0} \Delta \ell \]

\[ \Rightarrow \left\{ \begin{array}{l}
\Delta \ell \sim \sqrt{t_0} \Delta t \\
\text{Pulse prepates ballistically not diffusively.} \\
\end{array} \right. \]
infused divergence ultimately identifies ballastic propagation
supported by scaling analysis

if 2D anisotropic pile:

$$\frac{d\mathbf{r}}{dp} + \mathbf{a}_0 \left\{ \mathbf{x} \cdot dp^2 - \mathbf{D}_{ij} dp^3 \right\} = \mathbf{r}_0 dp$$

$$\mathbf{a}_0 = \frac{\mathbf{V} \cdot \mathbf{D}}{|\mathbf{V}|} \rightarrow \text{derivative parallel to pile gradient on surface.}$$

see refs for more.