Hurst exponents for Hasegawa-Wakatani turbulence

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1 Introduction

In magnetically confined plasma, it is of great interest to understand the statistics of the turbulent fluxes, which may be understood as stochastic. From a practical viewpoint, the turbulent fluxes may damage confinement by carrying strong transport; more theoretically, the turbulent fluxes fully characterize the dynamical evolution of the profiles. In particular, the *tail* of the distribution of fluxes contains information about intermittency: a fat tail is symptomatic of large transport events such as avalanches. Avalanches are known to carry a large fraction of the overall transport [1].

Besides the distribution itself, another useful measure of intermittency is a quantity known as the *Hurst exponent* H, named for hydrologist Harold Hurst [2]. The Hurst exponent, in a temporal analog of the structure functions used in fluid turbulence, gives the mean square deviation of a time series as a power of the time interval τ. Explicitly, for a time series $X(t)$, H is defined by [3]

$$E\left[(X(t+\tau)-X(t))^2\right]=\tau^{2H}. \tag{1}$$

H ranges between 0 and 1. $H=1/2$ manifestly corresponds to Brownian diffusion or white noise. The ranges $0\leq H<1/2$ and $1/2<H\leq 1$ are dubbed, respectively, *sub-*

diffusive and *superdiffusive*. The former is indicative of 'sticky,' oscillatory motion with temporal anticorrelations. The latter is indicative of strong memory effects and is characterized by burstiness, intermittency, and sustained events such as avalanches. Note that $H = 1$ corresponds to ballistic motion.

Indeed, analysis of tokamak experiments generally indicate $H \sim 0.7$ [4], suggestive of avalanching. Given that physical tokamaks include a multitude of complex physics which is difficult to model, one wonders how generic such a result is. Can simple reduced models, which can be simulated quickly on a supercomputer, reproduce such a result?

In this work, we answer this question in the affirmative. We study the simplest interesting model for drift-wave turbulence, the 2D Hasegawa-Wakatani system [5]

$$\partial_t n + \mathbf{v} \cdot \nabla n = \alpha(\tilde{\phi} - \tilde{n}) + D\nabla^2 n - \eta\nabla^4 n \tag{2}$$

$$\partial_t \nabla^2\phi + \mathbf{v} \cdot \nabla\nabla^2\phi = \alpha(\tilde{\phi} - \tilde{n}) - \mu\nabla^2\phi - \eta\nabla^6\phi. \tag{3}$$

Here $\mathbf{v} = \hat{z} \times \nabla\phi$ and the tildes indicate fluctuations from zonal averages. The system has been normalized according to $x/\rho_x \to x, \omega_{ci}t \to t, e\phi/T_e \to \phi, \log n \to n$.

These equations describe $E \times B$ convection of particles and vorticity, in the presence of instability driven by parallel electron resistivity. The tildes on the RHS are necessary to generate zonal flows in a 2D system; physically, this represents the fact that zonal components of the fluctuations do not contribute to the parallel current [6]. We have also included (classical) density diffusion D, drag μ which damps large-scale flows, and small hyperdiffusion/hyperviscosity η which removes energy from fine scales.

Determining the evolution of the (zonally averaged) vorticity and density profiles is of high interest. The Hurst exponents for these profiles provides information about the intermittency of the turbulent particle and vorticity fluxes. In what follows, we compute the Hurst exponents from simulation, in both the cases where the turbulence is driven by a fixed background gradient and where it is instead driven by fixed fluxes at the boundaries. We find that the result $H \sim 0.7$ is robust even in this dramatically simplified system, regardless of the drive. This suggests that intermittency is a generic feature of drift wave turbulence.

2 Methods

To simulate Eqs. 2–3, we use the BOUT++ framework [7] on a 512×512 grid. The grid step is $\Delta x = \Delta y = 0.1$, so ρ_* for this system is ~ 0.02. We fix the system to be weakly adiabatic, setting $\alpha = 2$. We take $\mu = D = 10^{-2}$, $\eta = 10^{-4}$. The equations are integrated using the Karniadakis method with an integration timestep 10^{-3}.[1] Simulation data are outputted every $\Delta t = 0.5$, for a total of 2000 timesteps. We use homogeneous Dirichlet boundary conditions for both ϕ and $\nabla^2\phi$ and homogenous Neumann conditions for n.

As previously mentioned, we employ two different turbulence drives. For the first set of simulations, we add a background term $\kappa\partial_y\phi$ to Eq. 2 for $\kappa = 1$ and $\kappa = 3$. Note that the linear instability boundary for this set of parameters is around $\kappa = 0.5$, though the Dimits shift pushes the effective boundary to $\kappa \simeq 0.75$.

The second set of simulations is driven by sources and sinks $S(x)$ localized near the boundaries. In particular, we fix

$$S(x) = \frac{f}{2\sqrt{\pi}\ell}\left(\exp(-x^2/\ell^2) - \exp(-(L-x)^2/\ell^2)\right), \tag{4}$$

with f a variable parameter representing the flux from each source/sink, L is the system size, and the lengthscale ℓ is set to $L/5$. See Fig. 1. We choose $f = 1$ and $f = 3$ for our simulations.

This latter system is arguably more physical than the gradient-driven system, which is more commonly studied. In a real tokamak, the driving gradients are set up by fluxes from the particle/heat sources and the divertor. From the intuition about avalanches from

[1] 10^{-4} was originally intended, but tragically the simulations needed to be reran with much less available time.

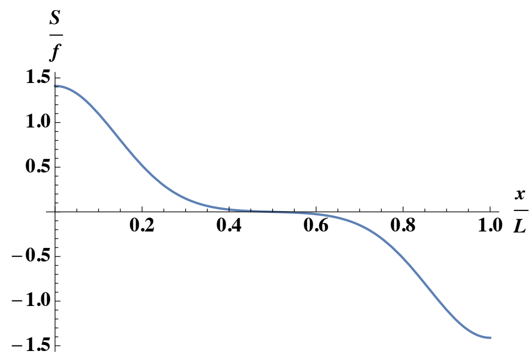


Figure 1: Plot of normalized source $S(x)/f$ for the flux-driven simulations.

sandpile models, one might a flux-driven system to exhibit a higher degree of intermittency. However, it is worth noting that the gradient-driven system helps simulate the profile stiffness exhibited in a flux-driven system via the Neumann BCs. We will see this, too, apparently is good enough to produce avalanches.

To compute the Hurst parameter, we employ "R/S analysis." For each zonally averaged field $\langle n \rangle$ and $\langle \nabla^2 \phi \rangle$, we break the time series into subdomains of 256 timesteps each, which we assume to have independent Hurst exponents (since the exponent may be dynamic). At each gridpoint in x, we compute, for a given interval of n time steps, the expectation $E[R(n)/S(n)]$, where $R(n)$ is the range of the time series over the interval, and $S(n)$ is the standard deviation. Then H may be obtained by fitting

$$E[R(s)/S(n)] = cn^H$$

We evaluate the LHS for $n = 4, 8, 16, 32, 64, 128, 256$ and take the expectations over the $256/n$ subintervals.

	H_n	$H_{\nabla^2\phi}$
$\kappa = 1$	0.74	0.69
$\kappa = 3$	0.66	0.75

Table 1: Hurst exponents for gradient-driven turbulence.

	H_n	$H_{\nabla^2\phi}$
$f = 1$	0.82	0.75
$f = 3$	0.83	0.74

Table 2: Hurst exponents for flux-driven turbulence.

3 Results

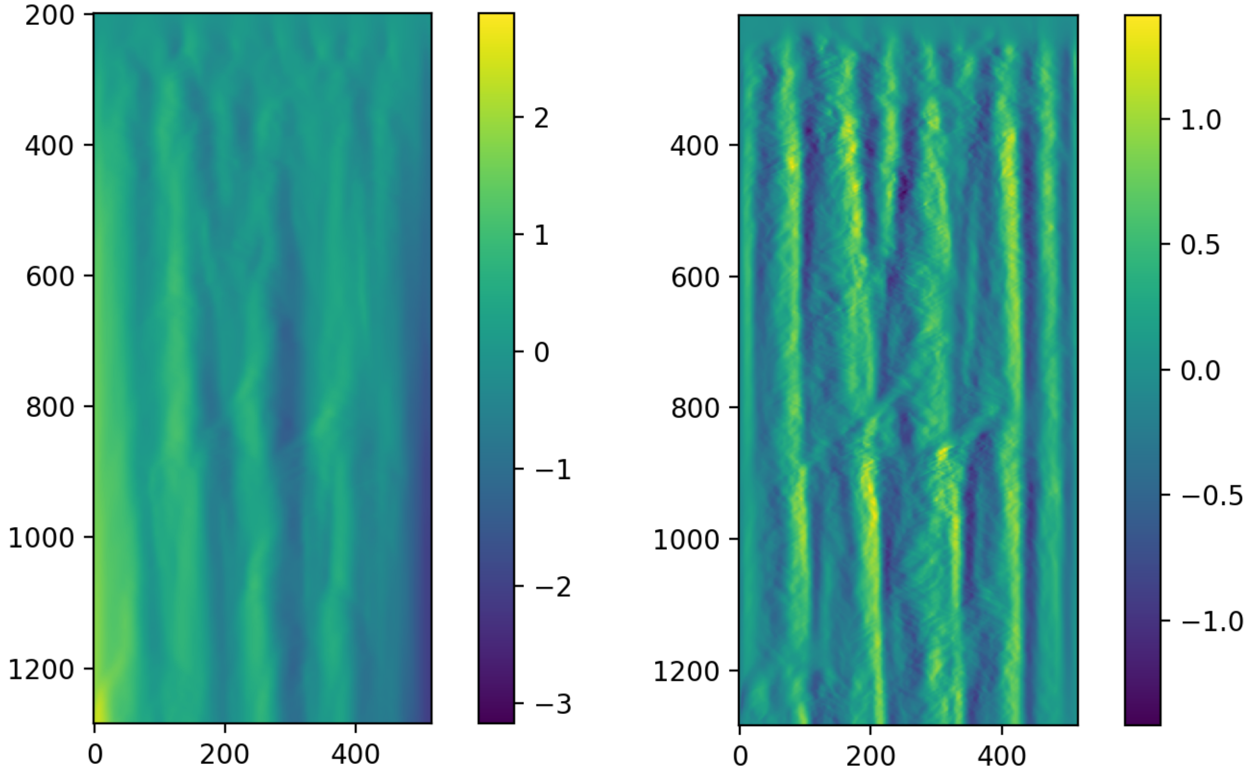
3.1 Gradient-driven turbulence

For gradient-driven turbulence, plots of the time evolution of the mean profiles are shown in Figs. 2–3. Note that $\kappa = 1$ features strong shear layer mergers and large-range avalanches (appearing as diagonal streaks), near $N_t = 800$ for example, whereas the profiles are fairly rigid for most of the simulation in the case $\kappa = 3$. Avalanches still appear in the larger gradient case, but these appear to be on a significantly smaller scale.

The Hurst exponents are summarized in Table 1. Note that while we computed the Hurst exponent dynamically, it was found to vary by not more than 10%, so the exponents in the table are averaged over time intervals. It is worth noting that the positional variance in the exponent was also found to be insignificant, $\lesssim 10\%$. In all cases, we recover $H \sim 0.7$.

3.2 Flux-driven turbulence

We now present the corresponding results for flux-driven turbulence. Again, we see in Figs. 4–5 that the avalanches appear to be on a larger scale for a weaker drive. In Table 2 we show the Hurst exponents, which again are ~ -0.7.

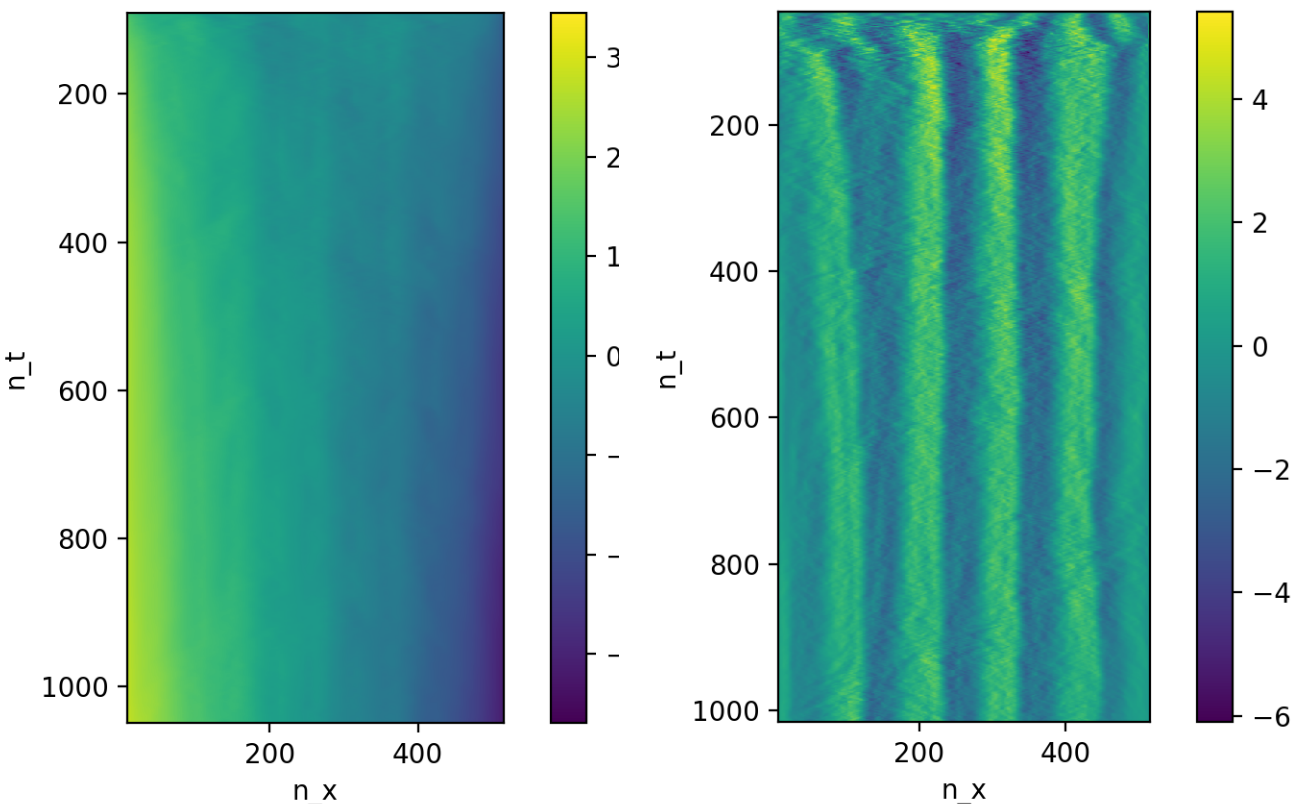


(a) (b)

Figure 2: Plot showing mean density (a) and vorticity (b) evolution for $\kappa = 1$. These snapshots show timesteps running from $N_t \sim 200$ to 1250 (vertical axis). The horizontal axis is position.

4 Conclusion

We have computed Hurst exponents for both flux- and gradient-driven turbulence in the simplest possible model of drift wave turbulence. We have recovered the result $H \sim 0.7$ which is generic to tokamak experiments and is associated with avalanches. Indeed, avalanche-like behavior is clearly visible in simulation. This is suggestive that avalanching is a quite generic feature of drift wave turbulence. Moreover, avalanching is recovered even in the absence of an explicit flux drive — a gradient drive with Neumann boundary conditions suffices. Evidently, the humble 2D Hasegawa-Wakatani system with a gradient drive may in fact be a reasonable framework for studying avalanching in confined plasma — perhaps it is not necessary to run



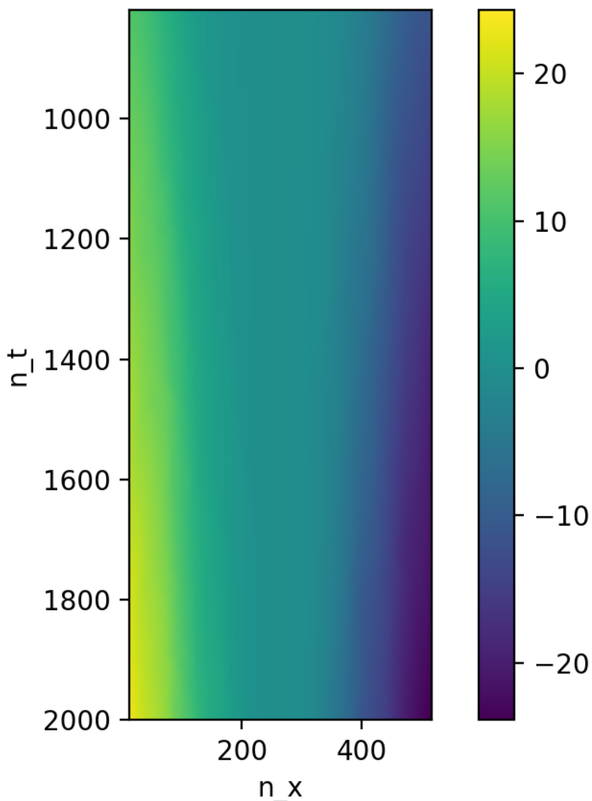
(a) (b)

Figure 3: Plot showing mean density (a) and vorticity (b) evolution for $\kappa = 3$. These snapshots show timesteps running from $N_t \sim 100$ to 1000 (vertical axis). The horizontal axis is position.

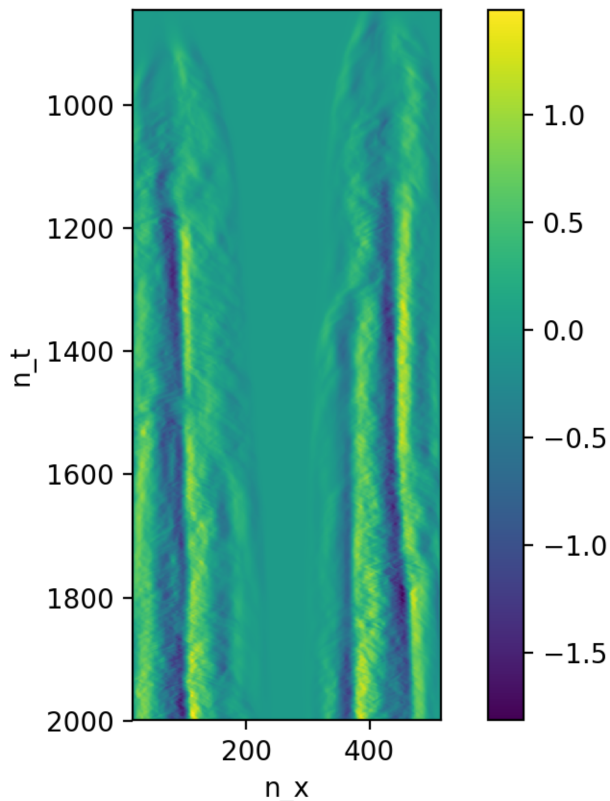
des grandes simulations gyrokinétiques,[2] or even flux-driven simulations of HW. It may be interesting to run gradient-driven simulations with different BCs which allow the profile to relax, and confirm that avalanches no longer occur and that (presumably) the Hurst parameter is no longer superdiffusive.

In simulations, it seems clear the avalanches occur on larger scales far above marginality, in both the gradient- and flux-driven cases (though it is more difficult to define marginality in the latter). This is intuitive, as avalanching is a phenomenon associated with fluctuations about a critical profile. However, this is not clearly reflected in the Hurst exponents. One should compute frequency spectra to confirm that more power is contained at small scales

[2]Apologies to Guilhem.



(a)



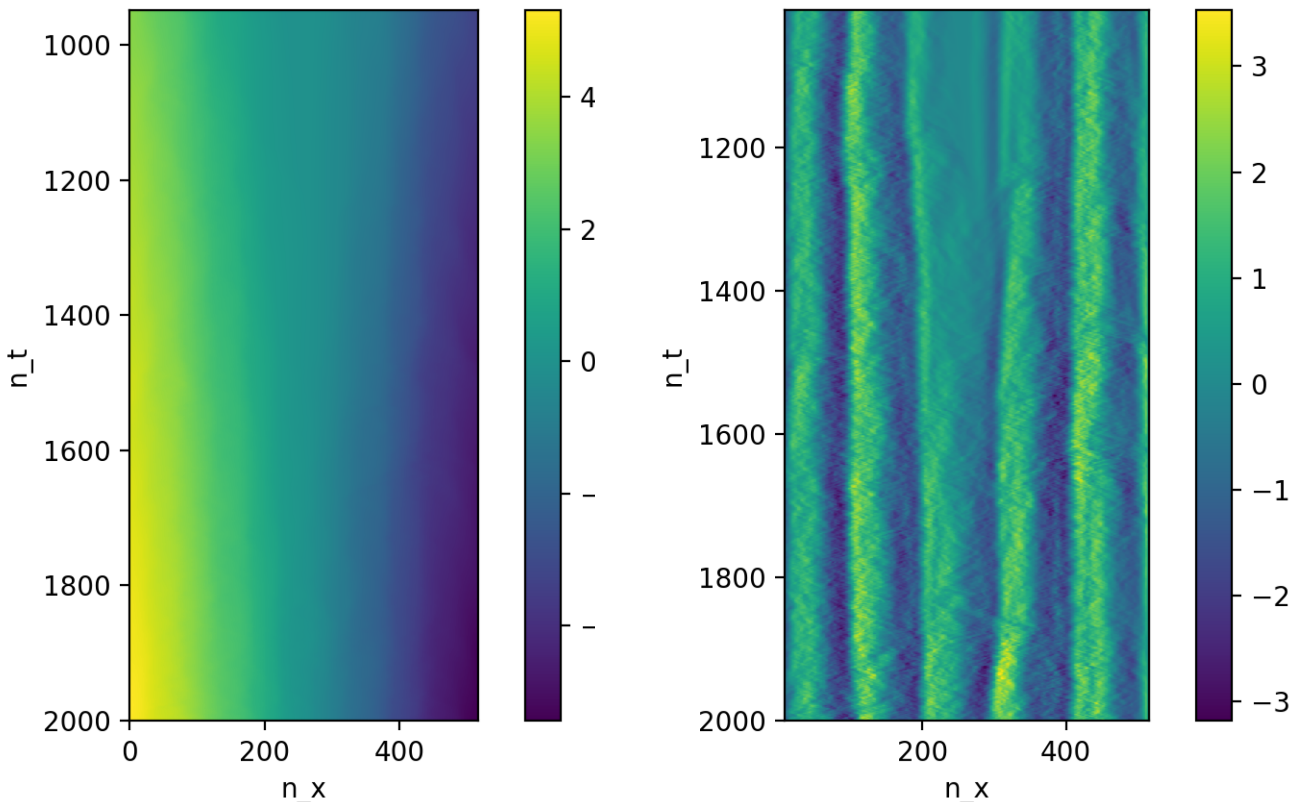
(b)

Figure 4: Plot showing mean density (a) and vorticity (b) evolution for $f = 1$. These snapshots show timesteps running from $N_t \sim 8000$ to 2000 (vertical axis). The horizontal axis is position.

for larger drives.

An interesting observation is that the Hurst exponent remained relatively consistent as the profiles involved dynamically. In particular, this conflicts with the intuition that the development of the zonal flow profile should limit avalanching. More work should be done determining the relationship of ZF to the Hurst exponent, say by running simulations with an imposed background shear flow.

These results are fairly preliminary/rushed, and there are several more directions for future work.It would be interesting to look at the case of very far above marginality, where one expects vortex interactions to drive strong intermittency. However, it is unclear how physically relevant this is for the tokamak. One may also consider alternate source/sink



(a) (b)

Figure 5: Plot showing mean density (a) and vorticity (b) evolution for $f = 3$. These snapshots show timesteps running from $N_t \sim 1000$ to 2000 (vertical axis). The horizontal axis is position.

setups, such as a source at one boundary and a fixed Dirichlet boundary condition at the other. More work could definitely be done exploring the parameter space in the flux-driven case, as time did not permit more than a perfunctory search for reasonable parameters. Finally, it is worth determining the effect of α and (dare I say) going to 3D.

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