PHYS235: Final Project
Nonlocal Transports and Avalanches

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1 Introduction

In our lecture, we mentioned that the Lévy alpha-stable distribution and its "fat-tail effect" that illustrates rare events beyond the garden-variety normal distribution. To see the motivation, recall the Lévy alpha-stable distribution $\mathcal{L}_\alpha(a, x)$:

$$\begin{aligned} P_{(x_0,t_0;x_n,t_N)} = P_{a,\alpha(x)} = \mathcal{L}_\alpha(a, x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{P}_{a,\alpha(k)}dk \\ &= \frac{1}{|x|^{1+\alpha}}\Big[c^\alpha(1 + sgn(x)\beta)sin(\frac{\pi\alpha}{2})\frac{\Gamma(\alpha+1)}{\pi}\Big], \end{aligned} \tag{1}$$

where α is the **stability parameter** that characterizes the distribution. This is a generalized equation of Central Limit Theory (CLT). We knew from class that when $\alpha = 2$, the Lévy stable distribution will reduce back to the Gaussian Distribution. Clearly, we can related diffusivity D to a such that:

$$\begin{cases} a = Dt \\ \\ \mathcal{L}_{\alpha=2}(a, x) \propto e^{-x^2/a}, \quad \text{Gaussian distribution } (\alpha = 2). \end{cases} \tag{2}$$

The parameter a carries a physical meaning of the *local transport* diffusivity D that corresponds to the Gaussian-Markov process. In this line of thought, it's natural for one to think of the properties and physical meanings of a when $\alpha \neq 2$, for all cases in Lévy stable distributions.

It's difficult, however, to analyze the general form of a given that not all the Fourier component $\hat{P}_{a,\alpha(k)} = e^{-a|k|^\alpha}$ of Lévy stable distributions are trivially transformable. But we can still consider one of a paradigmatic example of Lévy stable distribution, the Cauchy (Lorenzian) Distribution, where $\alpha = 1$. For the Cauchy distribution, the transport is non-local, associating with anomalous diffusivities. The flux of Possible tools to deal with this are Continuous Time Random Walk (CTRW) and Fractional Kinetics (FK). Our goal here is to find out the physical meaning of a in the Cauchy distribution, which relates to avalanches in high Knudsen number system. We'll begin with Dif-Pradalier et al. (2010a) research and find the physical meaning of a generalized diffusivity by dimensional analyses. A research done by Luciani et al. (1983) also presented experiments of the non-local heat transport.

2 Cauchy Distributions

Dif-Pradalier et al. (2010a) use the *GYSELA* and *XGC1* codes, where heat fluxes are **self-consistent**, to run the simulations and analyze the physical scale of the a in Cauchy distribution.

Mathematically, they generalized the turbulent heat Q from local formalism $Q_{(r)} = -n(r)\chi(r)\nabla T(r)$ to the generalized heat transfer integral. Here, $n(r)$ is the density, χ is the turbulent diffusivity, and T is the temperature. The generalized heat transfer equation is

$$Q(r) = -\int \mathcal{K}_r(r, r')\nabla T(r')dr', \tag{3}$$

where the kernel $\mathcal{K}_r$ is the **generalized diffusivity**. The equation (3) is able to describe the non-local transport – the heat flux at a position r depends on the whole temperature profile in the region around r. They found that the kernel (generalized diffusivity) to be of Lévy type appeared self-consistency. For the Cauchy type kernel is

$$\mathcal{K}_r(r, r') = \frac{\Lambda}{\pi}\frac{a/2}{(a/2)^2 + |r - r'|^2}, \tag{4}$$

where Λ is the **strength parameter** which is dimensionless, and a is the **stability parameter**. In this Cauchy distribution case, one can effortless find out the dimension of a is a length. Thus, a can be further interpreted as a radial **influence length**– a transport event at location r can drive a flux up to a distance a from the position r (see figure 1). As a result, the larger a, the stronger

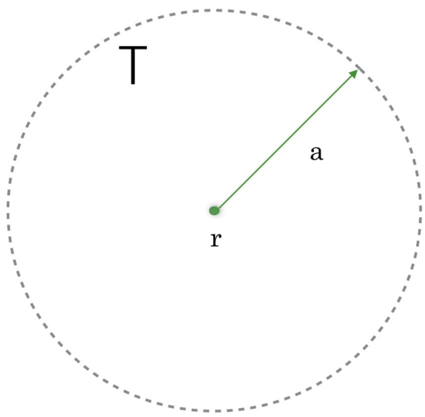

Figure 1: Non-local transport.

the non-locality. The values of a are mesoscale scale– the scale is bellow the system size (S) but larger than the turbulent autocorrelation length (l_c): $l_c \ll a \ll S$.

Since the anomalous transport can be easily affected by the turbulence intensity, one could imagine the Λ is depend on the turbulent intensity $I_{turb} = I_{turb}[\frac{L^2}{T^2}] \propto \delta v^2$. The velocity can be written as

$$v = \bar{v} + \delta v,$$

where $\bar{v}$ is the mean velocity, and δv is denoted as the fluctuation of velocity.

To broaden the physical meaning of the influence length a from $\alpha = 2$ to $\alpha \neq 2$, one can start a dimension analysis from the strength parameter Λ. It can be predicted as:

$$\Lambda \propto I^{\beta}\tau^{\gamma}L^{\delta}, \tag{5}$$

where τ is the timescale, L is the length scale. As the strength parameter $\mathcal{K}$ is the generalized diffusivity, we have

$$\Lambda \cdot L = L^2 D$$
$$\Lambda = LD,$$

which lead us to $\Lambda = \Lambda[\frac{L^3}{T}]$. We have the following by the dimension analysis:

$$\begin{cases} -2\beta + \gamma = -1, \\ \\ 2\beta + \delta = 3. \end{cases} \tag{6}$$

We choose $\beta = \gamma = 1$ and $\delta = 1$ under the assumption that the heat transport is linearly proportional to the turbulence intensity ($\beta = 1$). Thus we have

$$\Lambda \propto I\tau L. \tag{7}$$

The coefficient of Λ can be determine by the dimensionless turbulent fluctuation

$$(\frac{R}{L_T} - \frac{R}{L_{T,c}})^p,$$

where p is the critical exponent with value ~ 0.5, the $L_{T,c}$ is the critical temperature gradient length, and R is the chosen scale of region. The $\frac{R}{L_{T,c}}$ is the critical gradient. The concept of this is originated in the mixing length formalism from Spiegel (1963).

They indicate the flux as:

$$Q \propto (\frac{dT}{dz} - \frac{d\bar{T}}{dz}) \propto (\frac{1}{L} - \frac{1}{L_{crit}}).$$

These fluctuations follow from the intrinsically global character of the simulation as well as the existence of avalanche-like large-scale (L_T) events. More detail discussion can be found in the research of Dif-Pradalier et al. (2009). The candidates of the length are: mean free path (MFP, λ_0), influence length (a), autocorrelation length (l_{ac}), gyroradius (ρ_i), temperature gradient length (L_T), and the avalanche size (L_{avl}). On the other hand, candidates of the timescale are: autocorrelation time (τ_{ac}), avalanche time (τ_{avl}), electron mean collision time ($\tau_{e,coll}$), and the hydrodynamic timescale (τ_{hyd}). An analogy of the non-local transport is the avalanche– the spreading behavior of the temperature profile in the space in magnetic fusion energy (MFE) confinement is like a avalanching of a sand pile. The sand pile will redistribute its shape by avalanching triggered by the gradient of gravitational potential $\nabla\phi_g$ (see figure 3) that will determine a scale L_{crit}. And L_{crit} can be related to the temperature difference $\frac{1}{T_{crit}}\nabla T_{crit}$ in MFE confinement. The wind blows through the surface of the sand pile will trigger the avalanche, and resembles the sheared electric fields which triggers the avalanche in the MFE confinement. Based on these analogies, the heat/particle flux has its counterpart– a sand/grain flux. This indicates that the avalanches of a sand pile can be equivalent to the heat/particle flux transport in MFE confinement. Consequently, back to the dimension analysis, the turbulent autocorrelation length (l_{ac})

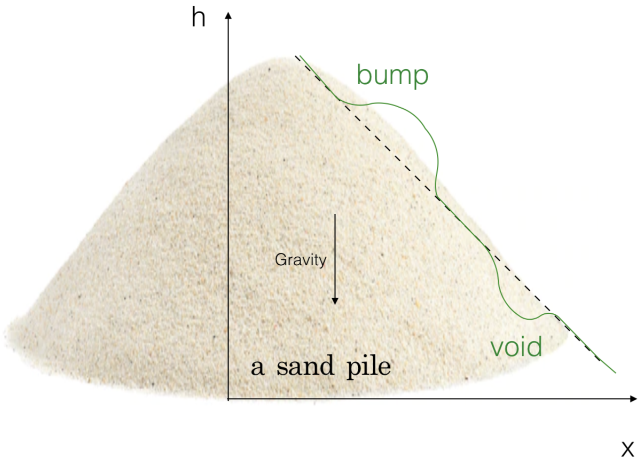

Figure 2: The avalanche of a sand pile.

resembles the cell size of a sand pile (L_{cell}), for the analogy between the local turbulence mechanism in MFE confinement and the topping rules of a sand pile. The mesoscale influence length (a) resembles the avalanche size L_{avl} such that $L_{cell} \ll L_{avl} \ll S$.

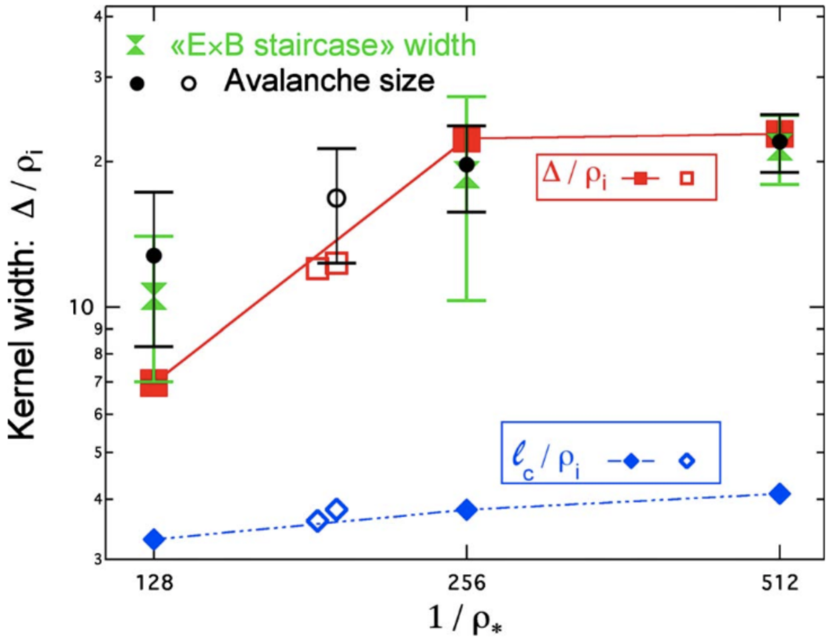

Figure 3: The avalanche size L_{avl} and the influence size a (here denoted as Δ) (Dif-Pradalier et al., 2010a).

Dif-Pradalier et al. (2010b) did simulations with *GYSELA* and *XGC1* and showed that the avalanche size L_{avl} is comparable the influence length a (see figure 2). They have a mesoscale a that is larger than the ion Larmor gyroradius (ρ_i) and the auto-correlation length (l_{ac}), but smaller than the temperature gradient (L_T) and the system size:

$$\rho_i < l_{ac} \ll a \sim L_{avl} \ll L_T < S.$$

Let's have a deeper discussion about the strength parameter $\Lambda(\propto I\tau L)$ based on this order of sclaes. The turbulence intensity $I = I[\frac{L^2}{T^2}]$ which indicates that the stronger turbulent the system is, the larger heat flux transport will the system have. Second, in terms of the candidates of the length scale, we have:

$$\begin{cases} L \propto \rho_i^{\delta} \cdot l_{ac}^{\gamma} \cdot a^{\beta} \cdot L_T^{\sigma} \\ 1 = \delta + \gamma + \beta + \sigma \end{cases}$$

One of a possible length scale is that $L = \sqrt[4]{\rho_i \cdot l_{ac} \cdot a \cdot L_T}$. Finally, we discuss the possible candidate for the timescale. By observing all possible timescale, one can find out a possible candidate $\tau = \tau_{avl}$. This indicates that the larger the avalanching time, the bigger the MFP will be. This leads to a larger influence length a and hence enhance the heat flux transport.

The relation between the MFP λ_0 and the influence length a can be found in a model of delocalization (Luciani et al., 1983) . Luciani et al. (1983) derived a non-local expression for the delocalization of the collisional heat flux and made a comparison with the classical flux-limited transport (FLT). In this paper, they discuss the non-local heat transport in small Knudsen number (Kn) system where $Kn = \frac{\lambda_0}{L_T} < 2 \times 10^{-3}$. The kernel they use has a different expression but with the same concept as Dif-Pradalier et al. (2010b):

$$\mathcal{K}_r(r, r') \propto \frac{1}{2a} exp[\int_{x'}^{x} dx'' \frac{n(x'')}{a(x')n(x')}], \tag{8}$$

where a is the influence length. The influence length a can be expressed by mean free path (λ_0):

$$L_{avl} \sim a(r) = c(Z+1)^{1/2} \lambda_{0(r)}, \tag{9}$$

where c is a constant. The value of c is a constant ranging from 30 to 35. The electron MFP $\lambda_{0(r)}$ scattering by electron and ion collisions can be expressed as

$$\lambda_{0(r)} = \frac{T_e}{4\pi n_e e^4 (Z+1)}, \tag{10}$$

where Z is the atomic number. Notice that here the temperature T_e is associated with the thermal velocity (v_T) of particles:

$$T_e \propto m v_T^2.$$

If our system has magnetic fields, then the velocity will be the replaced by the rms of the $\mathbf{E} \times \mathbf{B}$ drift:

$$T_e \propto m v_{rms,\mathbf{E}\times\mathbf{B}}^2. \tag{11}$$

Luciani et al. (1983) found that the delocalization model delineates the experiments batter, compared with the results of classical linear Spitzer-Härm (SH) descriptions (see figure 4). They also figured out temperature profiles in Lagrangian coordinate (in figure 5)– one can notice that the temperature profile of the delocalized model diffuses faster and farther away from the diffusion center r. This suggests the non-local transport– more particles have longer excursions that behave like Lévy flights.

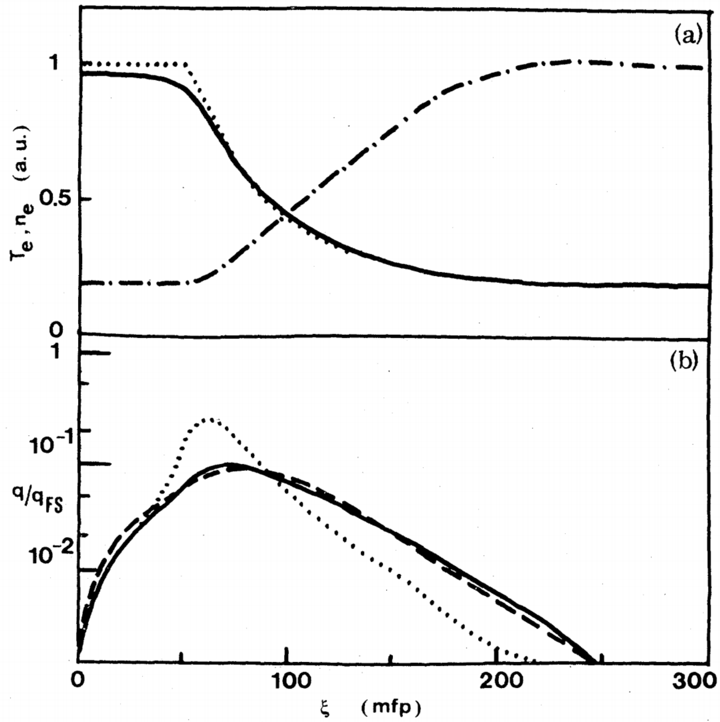

Figure 4: (a) Initial (dotted line) and the final (solid line) temperature profile, and the density profile (dash-dotted line). (b) Final heat-flux profiles, from the Fokker-Plank simulation (solid line), from SH law (dotted line), and from the delocalized model (dashed line).

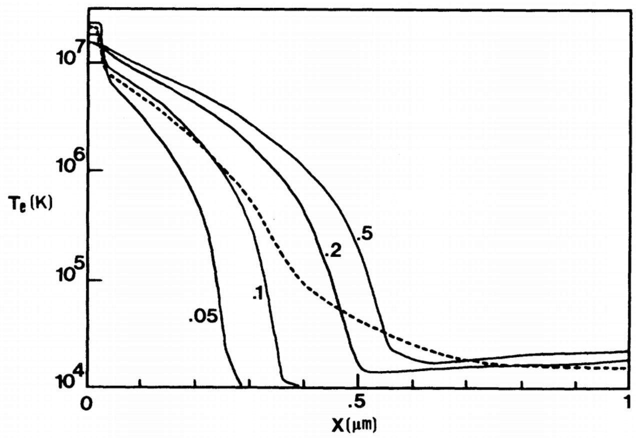

Figure 5: This is a snapshot of emperature profiles after 50 psec after the peak od the laser pulse. The solid lines are the classical flux-limited transport with different flux-limited parameter f. The dashed line is the result of the delocalized model which describes better of the final heat-flux profile (Luciani et al., 1983) .

Now we have all the elements to figure out our kernel Cauchy type kernel:

$$\mathcal{K}_r(r, r') = \frac{\Lambda}{\pi} \frac{a/2}{(a/2)^2 + |r - r'|^2}$$

$$\Lambda = \left(\frac{R}{L_T} - \frac{R}{L_{T,c}} \right)^{p=\frac{1}{2}} I \cdot \tau \cdot L$$

From all above discussions, we have:

$$\mathcal{K}_r(r,r') = \left(\frac{R}{L_T} - \frac{R}{L_{T,c}}\right)^{p=\frac{1}{2}} I \cdot \tau_{avl} \cdot \left(\rho_i \cdot l_{ac} \cdot a \cdot L_T\right)^{\frac{1}{4}} \frac{1}{\pi} \frac{a/2}{(a/2)^2 + |r-r'|^2},$$

$$\begin{aligned} &I = v_{rms,\mathbf{E}\times\mathbf{B}}^2, \\ &\tau_{avl} = \frac{L_{avl}}{v_{rms,\mathbf{E}\times\mathbf{B}}}, \\ &L_{avl} \sim a. \end{aligned} \tag{12}$$

Thus, we'll have the kernel

$$\boxed{\mathcal{K}_r(r,r') = \left(\frac{R}{L_T} - \frac{R}{L_{T,c}}\right)^{p=\frac{1}{2}} v_{rms,\mathbf{E}\times\mathbf{B}} \left(\rho_i \cdot l_{ac} \cdot L_T\right)^{\frac{1}{4}} \frac{1}{\pi} \frac{a^{\frac{9}{4}}/2}{(a/2)^2 + |r-r'|^2}.} \tag{13}$$

We've checked the dimension of $\mathcal{K}$ and it matches that of the generalized diffusivity $\mathcal{K} = \mathcal{K}[\frac{L^2}{T}]$.

In investigation of equation (2), we designate the fact that the dominated parameter of the generalized diffusivity $\mathcal{K}$ is the influence length a, with an exponent factor $\frac{9}{4}$ if $|r - r'|$ is large enough. And that the larger temperature gradient scale and rms of the $\mathbf{E} \times \mathbf{B}$ drift, the larger heat transport a system can have. Last but not least, even though r' is close to the diffusion center r, we still have non-zero generalized diffusivity, and this is the "Joseph effect" in diffusivity.

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