Aside - Intermittency

- Why bother studying percolation?
  What is relevance of connections profile?

- Intermittency
  - Turbulence not space filling
  - Homogeneous but non-uniform excitation

- Simple Model?
  - B model of H41 cascade

- Fractal dimension
- Anomalous exponents
Dissipation and dissipative structures?

N41 phenomenology suggests a uniform distribution of dissipation.

In reality,

distribution of dissipation is variable in intensity.

Not space filling/ intermittent.

I.e.,

Some departure from N41 spectrum concomitant.

How characterize?? Need a phenomenology, first.
Fractal Intermittency Models

- Fractal? Why?
- What does "dimension" mean?

n.b. Fractal concepts enable geometric phenomenology

Dimension

How define dimension?

Consider structure

Covering: \( N \)-dimensional cubes (cubes - Cartesian)

of size \( \varepsilon \)

(Covering set by space structure & embedded in)

If \( \tilde{N}(\varepsilon) = \# \text{ cubes to cover} \)

\[
D_0 = \lim_{\varepsilon \to 0} \frac{\ln \tilde{N}(\varepsilon)}{\ln (1/\varepsilon)}
\]
Box-Counting Dimension:

\[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln (1/\varepsilon)} \]  

**Check:**

1. **Finite # Points**
   
   \[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln \rho}{\ln (1/\varepsilon)} = 0 \]

2. **Line = length \( f \)**
   
   \[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln \tilde{N}(\varepsilon)}{\ln (1/\varepsilon)} \]
\[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln \frac{\varepsilon}{2} + \ln \frac{\varepsilon}{4}}{\ln \left( \frac{\varepsilon}{2} \right)} \rightarrow 1 \]

\[ D_0 = 1 \quad \checkmark \]

(c) Closed Curve - Area \( A \)

\[ D_0 = \lim_{\varepsilon \to 0} \frac{\ln A/\varepsilon^2}{\ln (\varepsilon/2)} \]

\[ = \lim_{\varepsilon \to 0} \frac{\ln A + 2 \ln (\varepsilon/2)}{\ln (\varepsilon/2)} \]

\[ = 2 \quad \checkmark \]

Now something juicier.

- **Middle Third Cantor Set**

  Define odd set:

  \[ \overline{01} \]

  Chop out middle \( 1/3 \):

  \[ \overline{1/3 \ 2/3} \]

  Elements:

  \[ 1/7, 3/13, 4/7, 7/13, 1 \]
For each \( n \), cover with \( 2^n \) pieces, \( (1/3)^n \) length:

\[
D_0 = \lim_{n \to \infty} \frac{\ln 2^n}{\ln (1/3)^n} = \lim_{n \to \infty} \frac{n \ln 2}{n \ln 3}
\]

\[
D_0 = \frac{\ln 2}{\ln 3}
\]

- Fractal dimension
- \( 0 < D_0 < 1 \)
- Embedded in \( 0 = 1 \) space \( A < D_{embed} \)
- Note \( \Rightarrow N \to Power Law \)

\[
N(\varepsilon) \sim \varepsilon^{-D_0}
\]

box counting dimension

Fractals are self-similar.
Could also encounter Koch curve (i.e., every 4 \rightarrow 16 \rightarrow \ldots \rightarrow \frac{1}{3}) upon iteration.

\[ D_0 = \lim_{n \to \infty} \frac{N(n)}{\ln(n)} \]

\[ = \lim_{n \to \infty} \frac{4^n}{n \ln(\frac{\ln(4)}{\ln(3)})n} \]

\[ = \lim_{n \to \infty} \frac{2n \ln 2}{n \ln 3} = \frac{2 \ln 2}{\ln 3} \]

\[ \theta = \frac{2 \ln 2}{\ln 3} \]

\[ D_0 \sim 1.26 \ldots \]

Here example of a fractal which thickens, i.e., \( D > 1 \). Needed elsewhere.

- akin "coast-of-Scotland" problem (Richardson & al., Mandelbrot...
The increased resolution reveals longer, more convoluted coastline.

\[ N(e) \sim e^{-D} \]

Rougher on smaller scale... (N increases with e^D).

**Why Fractals?**

- Self-similar structures with dimension \( D \) different from embedding space (i.e., 3).

Natural candidates to describe:

- Intermittent dissipation events
- Geometry of dissipative structures in intermittent turbulence/cascade

Cascade is due hierarchical, embedded process; dissipative structure does not fill space.

\[ - \text{ intermittency correction to } k^4 l \text{ spectrum} \]
The idea:

- Fractal structure is picture/phenomenology of observed departure from 1/f spectrum.

- Trends of scalings → plausible (i.e. fit)

but

- Theory based on NSE, does not predict $A_0\,!$

N.B. Geometrically symmetrical motivated phenomenology is extremely useful, i.e. Landau-Ginzburg etc.
which brings us to:

→ \( \varepsilon \) - model  \( \text{(Frisch, Sulem, Nelkin)} \)

→ basic ideas  \( \text{(Mandelbrot)} \)
  
  active region
   - cascade is self-similar fractal structure with \( D < 3 \)
   - dissipation events are patchy
   - forces correction to \( \eta^4 \)

→ Analysis

→ why intermittency? \( \Rightarrow \) \text{cascade}

→ vortex stretching is \text{very} nonlinear

\[
\begin{align*}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla P + \nu \nabla^2 \mathbf{v} \\
\partial_t \mathbf{V} &= -\nabla (\mathbf{w} + \mathbf{V}^2) + \mathbf{V} \times \mathbf{W} + \nu \nabla^2 \mathbf{V} \\
\mathbf{W} &= \nabla \times \mathbf{V} \quad \Rightarrow \text{vorticity (key physics)}
\end{align*}
\]
\[ \nabla \times (\nabla \times \mathbf{u}) + \nabla \cdot \mathbf{u} = \nabla \times \mathbf{u} \]  

\[ \frac{d}{dt} \mathbf{u} = \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} \]  

\[ \text{Kelvin Thm} \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \cdot \mathbf{u} = \text{const.} \]

Vorticity, enstrophy

\[ \nabla \times \mathbf{u} = 0 \]

\[ \mathbf{u} = \mathbf{0} \]

\[ \frac{d}{dt} \mathbf{u} = \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} \]

\[ \text{Kelvin Thm} \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \cdot \mathbf{u} = \text{const.} \]

Vortex stretching

Heuristic only

Fast (nearly explosive) growth of vorticity

\[ \langle \omega^2 \rangle \rightarrow \text{produced to disson} \]

Bursts, etc.

Vortex stretching feeds on self

\[ \text{localized process} \]

\[ \text{embedded} \]

\[ \text{patchy cascade} \]

\[ \text{occupation factor} \]

\[ E \sim R_n \frac{V_n^3}{l_n} \]

\[ \text{mean dissonance rate} \]

\[ R_n \equiv \text{Fraction of space active in n-th step} \]

\[ \text{cascade} \]
N.B.

\[ - \oint \mathbf{v} \cdot d\mathbf{l} = \int \mathbf{w} \cdot d\mathbf{q} = \text{const} \]

\[ \omega_1 r_1^2 - \omega_2 r_2^2 \Rightarrow \text{vorticity increase on small scale} \]

N.B. analogy

\[ \mathbf{F} + \mathbf{v} \times \mathbf{B} = m \frac{\mathbf{v}}{c} \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \Rightarrow \partial_t \mathbf{B} = \nabla \times \left( \frac{\mathbf{v} \times \mathbf{B}}{c} \right) + \eta \nabla^2 \mathbf{B} \]
\( \beta_n \) → Fraction of volume active in \( n \)th step of cascade

N.B. - if each eddy scale \( l \rightarrow l/2 \) per step

then # children to fill space per step i.e. \( 2^3 = 8 \)

\[ -8 = \beta = \frac{N}{2^3} \leq 1 \]

# off-spring

occupation reduction factor

\[ \beta_n = (\beta)^n = \left(\frac{N}{2^3}\right)^n \rightarrow n \text{ steps} \]

→ new interpretation only

\[ N \equiv 20 \quad 0 < 3 \]

(simplify an interpretation)

box counting dimension

\[ \beta_n = (2^{d-3})^n \]
So, taking mean energy balance

\[ \overline{E} = \frac{3n V_n^2}{\ell n} \]

\[ \beta_n = \frac{2^{n+1} (n-3)}{(n-1) (n-2)} \]

\[ \beta_n = \left( \frac{\ell_0}{\ell n} \right)^{3-n} \]

\[ V(\ell n) \sim (\overline{E} \ell n)^{1/3} \left( \frac{\ell n}{\ell_0} \right)^{-\frac{1}{3} (3-n)} \]

Correction due to E0

\[ \overline{E}_n \sim \frac{\ell n}{(\ln \ell_0)^2} - \frac{\ell n}{(\ln \ell_0)^2} \left( \frac{\ell n}{\ell_0} \right)^{\frac{2}{3} (3-n)} \]

Velocity in active region

\[ \sim \overline{E} \ell n \left( \frac{\ell n}{\ell_0} \right)^{\frac{8}{3} (3-n)} \]
\[ E(k) \sim \varepsilon^{2/3} k^{-4/3} (k l_0)^{-(3-D)/3} \]

\[ E(k) \sim \varepsilon^{2/3} k^{-4/3} (k l_0)^{-(3-D)/3} \]

- connection to \( k^4 \), in proportion \( 3-D \)

- slight steepening of spectrum

\[ \text{can deduce effective dimension from fit to spectral data} \]

Finally, dissipation scale changes:

\[ \frac{c l_0}{l_0^2} = \frac{V(l_0)}{l_0} \]

\[ \frac{R_2}{l_0^2 V_0} \]

but

\[ V(l_0) \sim \varepsilon l_0^{1/3} l_0^{1/3} (k l_0)^{-(3-D)/3} \]

\[ \varepsilon \sim \frac{V_0}{l_0} \]
\[ l_d \sim l_0 \left( \text{Re} \right)^{-3/4} \]

\[ \text{Re} = \frac{l_0 V_0}{\sqrt{g}} = \frac{-1/3}{\sqrt{g}l_0}^{4/3} \]

\( D = 3 \)

\[ l_d \sim l_0 \left( \frac{-1/3}{\sqrt{g}l_0}^{4/3} \right)^{-3/4} \]

\[ \sim \frac{l_0}{\sqrt{g}l_0}^{-1/4} \cdot \sqrt{g}^{1/4} + 3/4 \]

\[ l_d \sim \sqrt{3/4} \cdot \sqrt{1/4}. \quad D = 3 \]

modified for \( D < 3 \).
More Intermittency

A)

$\Delta W$ Explanation and What do we get from $\beta$-Model?

- Higher moments are a more severe probe of small scale structure of turbulence than energy is!

Recall $H_4 \Rightarrow \langle \delta u(r) \rangle \sim E^{1/3} L^{1/3}$

$\Rightarrow \langle \delta u(r)^p \rangle \sim E^{p/3} L^{p/3}$

So normalizing:

$\frac{\langle \delta u(r)^p \rangle}{\langle \delta u(r)^2 \rangle^{p/2}} \sim 1$

Normalized moments all independent of scale. $\Rightarrow$ Testable Prediction

So what of higher moments, i.e. $p > 2$?

Special interest in:
\[ \rho = 3 - \text{Skewness} \quad (\text{Symmetry}) \]

Why? Turbulence \(\to\) statistical approach/picture

Naively a Gaussian distribution (i.e. random)

\(\Rightarrow \rho \to 0\).

But:

\[ \dot{\theta} E \sim \dot{\theta} v^2 \sim v^3 \]

Net energy transfer in cascade, and

\[ \langle v^3 \rangle \neq 0 \]

Similarly, \(\rho = 4 - \text{Kurtosis}\) \(\kappa\)

\[ \kappa = \frac{\langle \delta v^4 \rangle}{\langle \delta v^2 \rangle^2} \to 3 \quad \text{for Gaussian process, measure of compactness/weight of tails of distribution) } \]

\(\kappa \gg 1\) is indicative of strong correlations and non-Gaussian behavior and fat tails.
The data (mostly in time, i.e., \( V_i(t_1) \), etc.)

\[ < \langle V_i(t) \rangle^p > \]

\( \langle \text{model} \rangle (V \sim 2.8) \)

Reality depends \( \beta \).

**What does \( \beta \)-model predict?**

**Volume factor**

\[ < \langle V_i(t) \rangle^p > \sim \eta_{\beta} \left( \frac{\text{ln}(\text{var})}{\text{ln}(\text{var})_0} \right)^P \]

\( \eta_{\beta} \) set by coincidence

\[ \sim \eta_{\beta}(P) \left( \frac{\text{ln}(\text{var})}{\text{ln}(\text{var})_0} \right)^P \]

**Wt factor**

\[ y_{\alpha} = \frac{1}{3} (3 - \alpha) (3 - \rho) \]

**Exponent of intermittency correction**

\[ y_{\alpha} \]
So, normalization \( \Rightarrow \)

\[ a_p(\ell_n) \sim \frac{1}{\langle \delta V(\ell_n)^2 \rangle} \]

plugging in \( \Rightarrow \)

\[ a_p(\ell_n) \sim \frac{\ell_n}{\ell_0} \]

\[ E_p = \frac{1}{2} (3-\alpha) (2-\beta) \]

\[ \frac{1}{2} > 2 \]

\[ E_p < 0 \]

In particular:

\[ 5 = \frac{\langle \chi^2 \rangle}{\langle \psi^2 \rangle} \]

\[ \gamma \sim DV \]

\[ \sim Re \frac{\beta (3-\alpha)}{2(1+\beta)} \]

taking \( \beta \) small as effect maximal

\[ \kappa \sim s^2 \]

Note:

- departure from \( k41 \) strongest
- at smallest scales
  \( \Rightarrow \) "force" of cascade strongest
- $\theta$ model $\rightarrow$ stages in cascade have "memory" of initial scale $\Rightarrow$
  $b$ explicit beyond $E$.

- $D = 2.8$ is reasonable data fit
  $\Rightarrow$ dissipative structure is highly convoluted sheets.

- $T_0$ depends $\theta$-model or $p$?
  $\Rightarrow$ multi-fractal model $\Rightarrow$
  $\theta$-model $\rightarrow$ single dissipative structure $\rightarrow$ dimension $D$

  multi-fractal $\rightarrow$ multiple dissipative structures different

$\Rightarrow$ connection to Navier-Stokes equation and dynamics is increasingly obscure $\Rightarrow$
Natural question:

- have argued that intermittency
  - departure from simple, self-similarity scaling
  - manifested as 'fractal memory'
  - structure function
- have also stressed analogy between
  - self-similarity in space (Blast wave)
  - self-similarity in scale (K41)

So, what is analogue of intermittency for space-time similarity? c.f.

K41 $\rightarrow$ $\beta$-model

as

Spatio-temporal self-similarity $\rightarrow$ ?
Memory of initial condition!

e.g. \( F \rightarrow F \left( r / \rho(t) \right) \)

self-same variable

Note how Sedov-Taylor effectively ignored initial radius of blast.

See Sedov blast, "Scaling"

Chapter 3

Now one can go further and calculate:

\( \langle \varepsilon^2 \rangle \rightarrow \text{mean square fluctuating dissipation} \)

but \( \varepsilon \sim \nu \langle (\nabla \varepsilon)^2 \rangle \)

\( \langle \varepsilon^2 \rangle \sim \nu^2 \langle (\nabla \varepsilon)^2 (\nabla \varepsilon)^2 \rangle \sim \nu^2 \frac{\kappa}{\nu} \frac{\kappa}{\nu} \frac{\kappa}{\nu} \)

\( \langle \varepsilon^2 \rangle / \langle \varepsilon \rangle^2 \sim \frac{1}{\Re} \frac{1}{(3 - \nu)(1 + \nu)} \rightarrow \text{heuristic}! \)
Can also address:

\[ \langle \varepsilon(r) \varepsilon(r+t) \rangle \rightarrow \text{dissipation correlation} \]

Now,

\[ \langle \varepsilon(r) \varepsilon(r+t) \rangle \sim \langle \varepsilon \rangle^2 \text{Prob.} \left( r, r+t \text{ belong to m-odd} \right) \]

\[ \langle \varepsilon \rangle \sim \frac{V_m}{\ell_m} \]

\[ \sim \frac{V(\ell_m)^3}{\ell_m} \]

To allow for:

- Packing

- If correlated by \( \ell_m \) then correlated by all larger odd \( \ell \)s

\[ \langle \varepsilon(r) \varepsilon(r+t) \rangle \sim \sum_{m \geq 2} \left( \frac{V_m}{\ell_m} \right)^2 \ell_m \]

\[ \sim \varepsilon^2 \left( \frac{\ell}{\ell_0} \right)^{(2-D)} \]
In particular,

$$\langle \xi(r) \xi(r+\delta r) \rangle \sim \tilde{\xi}^2 \left( \frac{\delta r}{\lambda} \right)^{(D-3)}$$

a strong correlation in dissipation at dissipation scale.