Notes 6 → Percolation Basics

Why Percolation?

- The problem of large $k_u$...

recall: $k_u \sim \frac{\text{bac} \Delta B}{\Delta_L B}$

$\sim \sqrt{\text{bac} / \Delta_L}$ etc.

⇒

Limiting case, well defined paradigms:
(Toy models!?)

$k_u \ll 1$ → diffusion, extensively studied
$\text{bac} \to 0$

$k_u \gg 1 \Rightarrow k_u \to \infty \Rightarrow \text{bac} \to 0$

i.e. no variation of $\Delta B$ in $\mathbb{Z}$ (confine).

→ random (by assumption) array of magnetic cells ($\mathbb{Z} \cdot B = 0$!)

Static, disordered 2D medium.
Akin array of randomly placed static vertices, as in Taylor-McNamara problem (see notes).

Ignoring Do Pe etc. need macroscopic connection — a percolation - to transport (hills and lakes as before).

\[ \text{n.b. need not have connection if } D_0 = 0 \]

though expect

\[ \text{Dec} = D_0 \times D_{\text{cell}} \]

\( x + \beta = 1 \)

explicit dependence on colliding diffusivity.

The problem is percolation of static random function of 2 variables akin to problem of effective conductivity of medium with random mixture of conducting and insulating elements — identical.
Some properties of system/problem sticking to magnetic example:

- Spectrum of $A$,
  \[ b = DA \chi \]

- Random phases

- $\langle A \rangle = 0$

- $\langle A^2 \rangle = 1$

and small auto-correlation at large distances,

\[ \rho_{\text{Lac}} \ll L \]

System size

\[ \langle A^2 \rangle_n \approx e^{-k^2/k_0^2}, \quad k > k_0 \]

\[ \langle A^2 \rangle_0 \approx \left( \frac{k}{k_0} \right)^{2n}, \quad n > 0 \]

\[ \sigma = \langle A^2 \rangle_n \]
A > E \rightarrow \{ \text{island} \}\, \{ \text{magnetic vortex} \}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{island_vortex.png}
\caption{Island and magnetic vortex diagram}
\end{figure}

\rightarrow A \leq E, \quad E > 0 \rightarrow \text{percolation}

E = 0, \text{ critical phase}

i.e. recall topographic map

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{topographic_map.png}
\caption{Topographic map}
\end{figure}

\text{percolation/connection for } A \leq E.

l(E) = \text{length island or isoline surrounding island}

l \uparrow \Rightarrow E \downarrow \Rightarrow \text{percolation}
in such 2D topology with $D \to \infty$ (i.e. nothing kicking particle off field line)

$D_{\infty} = 0$

i.e. turbulent diffusion approximation not applicable (i.e. all particles trapped or near closed cells)

either no transport or "burst" along macroscopic connection.

Interesting to consider:

2D system $\leftrightarrow \nu u \gg 1$

$\langle B \rangle \neq 0$

$B \gg \langle B \rangle$

akin to solar tachocline

Aside: References
- "The Almighty Chance"
  - Zeldovich, Ruzmaikin, Sokolov
  - serious CV of statistical dynamics
  - dated, highly recommended
  - percolation chapter postal

See also:
- Ya. B. Zeldovich, 1983 (postal) (magnetic problem)
- Stauffer, 1973 (review) (postal) (review of percolation theory)
- Stauffer and Honary, monograph

Now view as

\[ \langle B \rangle \rightarrow B_0 \]

unperturbed large scale field
\[ \text{BL} \quad \text{confined to narrow channels.} \]

i.e. real field pattern is ensemble of cells + strings of deformed \( \text{BL} \).

Question: does \( \text{BL} \) percolate? i.e. extend to \( a \), system size as \( a \to \infty \).

Physics: will AlPicnic excitation propagate?

More generally, what is response of such a system to external excitation?

Now, seek \( \langle B \rangle \to \text{mean field} \). What is \( \text{if?} \)

- Define system as strip of width \( a \).

\[ k_{\text{min}} = \frac{2\pi}{a} \quad \rightarrow \quad \frac{1}{a}. \]
- take \( A_k \sim k^m \) (see before)

so

\[ B_k \sim k^{m+1} \]

\[ <B> = \left( \langle k^2 \rangle_{k < \frac{1}{a}} \right)^{\frac{1}{2}} \]

Long wavelength components define mean field

so

\[ <B> = \left( \frac{1}{a} \int \frac{dk}{2\pi} k \frac{2m}{k^2} \right)^{\frac{1}{2}} \]

\[ = (\frac{1}{a})^{m-2} \]

then, for \( <B> \neq 0 \) as \( a \to \infty \) (i.e. not decrease with increasing distance \( a \))

For \( a \to \infty \)

\[ m = -2 \n
\[ <B> \neq 0 \]

\[ <B> \text{ constant} \]
\[ M \sim -2 \]

currents

white noise

\[ j_{\Sigma} \sim k^2 A \eta \sim k^0 \sim \text{const.} \]

random currents

Random currents will result in percolating mean field structure determined by currents.

N.B. More generally:

- need \( \langle j(x) j(x+t) \rangle \geq 0 \) (correlated currents) for percolating \( \langle B \rangle \).

- no percolation for anti-correlated currents.

\[ \text{Some general features of percolation problems} \]

- explore formation of \( L \sim \]

- linked paths.

\[ L \sim (N)^{1/3} \Delta x, \quad \text{thermodynamic limit} \]
- Basic game is some D-lattice:
  
  \[
  \begin{array}{cccccc}
  x & x & x & x & x & \text{etc.} \\
  x & x & x & x & x & \text{etc.} \\
  x & x & x & x & x & \text{etc.} \\
  \end{array}
  \]

  With probability \( p \) of occupation of a site.

  - connections \( A \) with \( p \).

  - bond \( x \times x \) percolation

  site \( x \)

- Universality is key concept

  \[
  \text{for } \frac{A}{x} < 1, \quad L \rightarrow \infty
  \]

  details of specific site element irrelevant.

  i.e. classic is details of, say, conductor or insulator shape element.
- caveat: Fractal elements can complicate universality
- example: pancakes in 3D.

Universality →
- define as simple problem
- concept as for phase transition
  → \( T_c \) as \( T \to T_c \)
  → \((T - T_c)^2\) in regime interest
  → scaling!


- contrast with diffusion:
  (i.e. on lattice)
  → diffusion: randomness arises from particle trajectory dynamics
  i.e. \( r^2 \) and overlap
  → percolation: randomness encoded in site probabilities.
D matters

if $p = p_c$ \to critical occupation density
threshold for
percolation

1D: $p_c = 1$

all sites occupied.

2D: $p_c = 0.59$ triangular lattice
$p_c = 0.50$ square lattice

etc.

How describe percolation?

Scaling Theory of Percolation

Clusters - c.f. Stuuffer

percolation - lattice

random occupation $p$ yes

$1 - p$ no.

$p > p_c$ \to 1 cluster spanning lattice.

$p < p_c$ \no

\no spinners

$p_c \to \text{Layat}
a cluster of s-sites, nearest neighbors coupled, ends empty

seek describe scaling of cluster properties as \( p \to 0 \)

i.e. characterize distribution of of cluster sizes

how behave as \( p \to 0 \)

\[
N_s = \# s\text{-clusters}
\]

\[
N_s = \frac{\text{avg } \# \text{site of } s \text{ clusters}}{s} = n_s(p)
\]

population density for \( p \) of

of interest is:

\[
\sum_{n_s} \sum_{p} n_s(p) \to \text{cluster population}
\]

\[
\sum_{n_s} S n_s(p) \to \text{mass cluster probability}
\]
\[ \Sigma(p) \equiv \text{correlation length} \]

etc.

- Aim of scaling theory is \( N \).

This yields more info.

i.e. \( p_\infty \equiv \text{percolation probability} \)

[= \text{Fraction of sites belonging to percolation network}]

[= \text{"strength" of infinite network}]

Now, can relate different quantities; i.e. any lattice site can be:

\( p \) empty \( \rightarrow \) \( \text{Prob} = 1 - p \)

\( p \) part of a percolation cluster \( \rightarrow \) \( \text{Prob} = p p_\infty \)

\( p \) part of a finite cluster \( \rightarrow \) \( \text{Prob} = p (1 - p_\infty) \)
of course, being part of a finite cluster

\[ \text{Prob} = p(1 - P_{\infty}) = \sum_n n \cdot P_n \]

\[ p \]

\[ 1 - p + P_{\infty} + \sum_5 S_n = 1 \]

So computing \( N_5 \) gives \( P_{\infty} \).

What we are interested in:
- Scaling exponents near \( p \).

As for phase transitions, we'll aim to (understand) the relationship between scaling exponents. So, determinism \( \neq \) from simulation/\( RG \) calculation (I'm sure) etc. gives all.
Scaling Exponents

\[ \sum_0 N_s (p) \sim |p - p_0|^{2 - \alpha} \]

\[ \sum_0 s N_s (p) \sim |p - p_0|^{\beta} \]

\[ \sum_0 s^2 N_s (p) \sim |p - p_0|^{-\gamma} \]

\[ \sum (p) \sim (|p - p_0|)^{-\nu} \]

e tc.

For 2D, \( \alpha = 0.7 \)

\[ \beta = 0.74 \]

\[ \gamma = 2.4 \]

\[ \nu = 1.35 \]

Aims is to set relations.
1D Percolation \rightarrow Trivial but Illustrative

Of course, \( p_c = 1 \).

- \( \text{Prob occ} = p \)
- Cluster \( \rightarrow 8 \text{ sites} + 2 \text{ ends} \) (empty 1)

i.e. \( P(5 \text{ cluster}) \sim p^5 (1-p)^2 \)

\( \# 5 \text{ sites} \rightarrow 2 \text{ ends} \)

\[ \langle S \rangle \sim \frac{p^5}{(1-p)^2} \]

\( \rho \rightarrow \sum_{\text{sites}} \left( \frac{1}{1-p} \right) \]

\( \langle S \rangle \rightarrow 0 \) for \( s \rightarrow \infty \) (exponentially)

\[ N_s = \rho^5 (1-p^2) \]

\[ \frac{\# 5 \text{ clusters/ site}}{\text{thermodynamic limit} (N \rightarrow \infty)} \]

\( \rightarrow \) no worry re:

\( N_s \rightarrow 0 \) for \( s \rightarrow \infty \) (exponentially)
prob. of site in cluster of size $s = s N_s$

\[ \text{Prob} = \sum_{s=0}^\infty N_s s = \rho \]

any cluster

\[ \sum_{s} \frac{\rho^s (1-\rho)^s}{s!} = (1-\rho)^2 \sum_{s} \rho^s \frac{d}{d\rho} \frac{1}{s!} \]

\[ = (1-\rho)^2 \rho \frac{d}{d\rho} \sum_{s} \rho^s \]

\[ = (1-\rho)^2 \rho \frac{d}{d\rho} \left( \rho / 1-\rho \right) \]

\[ = \rho \]

Probability occupied.

N.B. Standard trick in statistical mechanics $\rightarrow$ relate index to derivative.
Now, define

\[ W_0 = \frac{\sum s}{\sum \frac{s^2}{5}} \]

average cluster size:

\[ \bar{s} = \frac{\sum s W_0}{\sum W_0} = \frac{\sum s s^2}{\sum s} \]

(why moments scaling of some concern)

\[ \bar{s} = \frac{\sum (1-\rho)^2 p s^2}{\sum (1-\rho)^2 p s} = \frac{1+p}{1-p} \]

after some truth.
Obviously, \( S \to \infty \) \( p \to p_0 = 1 \).

if \( p = 1 - p \)

\[ S \approx \frac{2}{\sigma} 1. \]

Coming to further consideration of scaling theory.