Physics 235

Lecture II

Stochastic Fields:

Kuba # Collisionless and Collisional Heat Transport

Recall:

- Chirikov criteria for onset Hamiltonian chaos

\[ \kappa u = \frac{S_0}{\Delta n} \sim \frac{\rho_e B_0}{\Delta n} \]

\[ \kappa u < \frac{\pi}{\Delta} \quad (\Rightarrow \sqrt{\frac{\rho_e B_0}{\Delta}}) \]

\[ D_n = \sum \frac{p_{B_n}^2}{B_0} \int d^2 \delta(\kappa u) \]

\[ = \left( \sum \frac{1}{B_0} \right)^2 \rho_e u \]
More on kubo #

\[ \frac{dr}{dz} = \frac{Br}{Bo} \]

\[ \Rightarrow dr = \int (\frac{Br}{Bo}) dz \]

Now line trajectory decoheres from perturbation for \( L > \text{len} \).

What do the symbols mean?

\[ \text{lauc} = \frac{1}{\langle A(\text{len}) \rangle} \]

\( \text{lauc} \) is the spatial correlation length.

\( \Delta r = \text{lauc} \frac{Br}{Bo} \)

\[ \Rightarrow \text{ spatial excursion of } \frac{Bo}{lauc} \]

Can identify \( \Delta r \equiv \text{scatterer radial correlation length (i.e. spatial spatial width)} \)

then:

\[ ku = \frac{dr}{\Delta r} = \frac{\text{lauc} \frac{Br}{Bo}}{\Delta r} \rightarrow \text{kubo #} \]

and can then post:

\[ ku < 1 \Rightarrow \text{many kicks of coherence length} \]

\[ \Rightarrow \text{diffusion process} \]
\[ \nu_n \approx I \rightarrow \text{R.H. "natural state of EM turbulence}\]

\[ \nu_n \approx \text{critical balance} \]

\[ \nu_n > I \rightarrow \text{more than one } \Delta \text{ on } \nu \]

\[ \rightarrow \text{ strong scattering } \rightarrow \text{ percolation} \]

\[ \rho \text{LT} \]

Here \( \nu_n \ll I \), at first. So proceed via Quasi/linear theory.

\[ \Gamma_n = \left( \frac{\mathbf{E} \cdot \mathbf{F}}{B_0} \right) \]

\[ = \sum_n \left( \frac{\mathbf{B}_n - \mathbf{B}_0}{B_0} \right) \tilde{F}_n \]

\[ = - (c \mathbf{E}_z - E_0 \mathbf{B}_0) \tilde{F}_n = - \mathbf{B}_0 \cdot \frac{\partial \tilde{F}}{\partial n} \]

So

\[ \Gamma_n = - \Delta m \frac{\partial \tilde{F}}{\partial n} \]

\[ \Delta m = \sum_n \left| \mathbf{B}_n \right|^2 \pi D \left( c \mathbf{E}_z - \mathbf{E}_0 \mathbf{B}_0 \right) \]

Magnetic diffusivity

\[ = \sum_n \left| \mathbf{B}_n \right|^2 \pi \Omega \left( c \mathbf{E}_z \right) \]

\[ \sim \left( \frac{c \mathbf{B}_n}{B_0} \right)^2 \Delta \epsilon \]

\[ \text{What is } \Delta \epsilon \text{?} \]
N.B.: \( \sum = \sum \) \( \text{Spatial spread} \)
\( n, m, n \) \( \text{scattered} \) \( \text{collimated} \)

\[ n = \frac{m}{Z} \quad \text{d}n = \frac{1}{Z} \frac{Z}{m} \frac{Z}{d} \]

\( \Rightarrow \) spatial scale of spectral width (\( \Delta r \))
\( \text{sets} \) \( k_{\text{hi}} \approx \frac{|k_{\text{lo}} A_r|}{L_o} \)

\( \boxed{\text{Lanczos} \sim L_o / \text{Keto} \ A_r} \)

Lines then diffuse as:

\( \langle \Delta r^2 \rangle \sim \Delta m \quad \text{Broaden}
\)

N.B. Line Liouville eqn can be obtained by reducing/simplifying of OKE

\[ \frac{\partial}{\partial t} + \left( \mathbf{u} \cdot \nabla \right) \mathbf{F} = -\frac{1}{\rho} \nabla \cdot \mathbf{F} + \frac{v_{11} \nabla \cdot \mathbf{F} - \nabla \cdot (v_{12} \mathbf{F})}{B_0} \]

\( \mathbf{F} = \frac{\mathbf{F}}{\mathbf{F}} = \nabla \phi \)
\[ \nabla \cdot \mathbf{F} + \mathbf{J} \cdot \mathbf{F} = 0 \]

**Scales**

Now, scales:

\[ \rho \rightarrow \text{ (scatters)} \]

- Field line memory length
- Self-coherence of scattered field.

\[ l_0 \rightarrow \text{ line decorrelation length} \]

\[ l_0 \rightarrow \text{ length over which line scattered from its unp} \]

\[ \frac{\mathrm{d} \Theta}{\mathrm{d} z} = \frac{\mathbf{B}_0(x)}{\mathbf{B}_0} \]

\[ \text{but } \mathbf{r} \text{ scattered, } \mathbf{r} \]

\[ \frac{\partial y}{\partial z} = \mathbf{B}_0(\mathbf{r}) + \frac{\mathbf{B}'(\mathbf{r})}{\mathbf{B}_0} \]

\[ \mathbf{B}_0 \]

\[ \frac{\partial y}{\partial z} = \frac{\mathbf{B}_0}{\mathbf{B}_0} \]

\[ \langle y^2 \rangle = \left( \frac{\mathbf{B}_0}{\mathbf{B}_0} \right)^2 \]
\[ \langle dy^2 \rangle \sim \frac{B_0^2}{B^2} \left( \langle dx^2 \rangle \right)^2 \]
\[ \sim \frac{B_0^2}{3B^2} D_m Z^3 \]

**analogous to shear dispersion**

\[ \langle k^2 \rangle \sim \frac{B_0^2}{3B^2} \]

\[ \sim \left( \frac{k}{\sqrt{3}} \right)^{1/3} \]

**orbit expectation length**

\[ \langle k \sigma \rangle \sim \left( \frac{B_0 D_m}{3} \right)^{1/3} \]

\[ \sim \left( \frac{k}{\sqrt{3}} \right)^{1/3} \]

Also: **stretching**

\[ \langle x_1 x_2 \rangle \sim \langle dx^2 \rangle \]
For all regime validity:

\[ \lambda \alpha < \lambda c \]

\[ \text{ limit } \]

and another (particle) length: \( \lambda_{mp} \)

\[ \lambda < \lambda c < \lambda_{mp} \rightarrow \text{ so called } \]

\[ \text{ collisionless regime } \]

\[ \lambda_{mp} < \lambda < \lambda c \rightarrow \text{ collisional } \]

which brings us to: something physical

- (Electron) Heat Transport

\[ \text{ Theme: interference processes} \]

\[ \text{ N.B. nobody cares about long diffusion} \]

- people (i.e. experiments) do care about:
  - heat
  - particle transport

\[ \text{ Rochester + Rosenwald} \]

PRL '78

a MUST
Let's begin with heat transport.

Consider the heat transport equation.

- Linear wave behavior

Recall the thought problem. Is it so simple?

But, let's assume parallel collisions (only) happen. (Particle stays on line.)

Motion along line is diffusive:

\[ \frac{\partial^2 u}{\partial x^2} = D_{ll} \frac{\partial u}{\partial t} + \frac{1}{V_{ll}} \text{parallel thermal diffusion} \]

So, for slug heat:

\[ \text{Law} \sim D_{ll} \frac{1}{2} \sim D_{ll} \text{(Kirt)} \]

So, had scutter:

\[ X \approx 1 \text{ Law} \frac{1}{2} \sim D_{ll} \text{(Kirt)} + \frac{1}{2} \]
Point: the may wander
but
- particle kicked back along line
- even though field
  no net radial wander as particle kicked back

Less any:
- collisions control irreversibility

Need:
- need get kicked off field line

Coarse graining:
- FLR \Rightarrow \theta e
- $k_L$
- drifts

Smear particle location over a resolution cell.
1. Consider the following argument:

Consider a disk of radius $r$. 

2. Move the disk forward, noting that $D \cdot B = 0$.

$\Rightarrow$ map is area-preserving.

\[ \begin{align*} 
1 & \quad h_1 > 0 \\
0 & \quad h_0 < 0 \\
\text{(Ch: Lyapunov Exn).} 
\end{align*} \]
but coarse graining occurs at

\[ A_{\alpha} f = A_{\delta} f \text{ (for } \delta \text{)} \]

coarse graining of structure from previous

and can continue...

Ludwig Boltzmann asserts we no memory between steps (1 bump)
collision time)

so initial spot expands with random walk, as

\[ \langle \Delta n^2 \rangle \sim \text{D} \Delta \text{time}. \]
c.e. coarse graining (interval size $\langle \Delta v^2 \rangle$) step ! \\

6) then, for $x_1$ (on-line): $\frac{1}{v_c} \sim v_c \\
\frac{x_1}{v_c} \sim \frac{\langle \Delta v^2 \rangle}{v_c} \sim \frac{\Delta x}{v_c} \sim \frac{x_1}{v_c} \sim v_{in} \cdot 10\, M.$ \\

$\Rightarrow x_1 \sim v_{in} \cdot 10\, M.$ \\

$\Rightarrow$ collisionless stochastic field heat, diffusivity \\
$\Rightarrow$ manifestly independent of collisionality \\
$\Rightarrow$ yet clearly dependent on collisions and coarse graining! \\

Lesson: coarse graining essential to irreversibility \\

Collisions $\rightarrow$ arrow of time.
Stoch. Fields cont'd

Exercises (suggested):

c') Derive the magnetic diffusivity with magnetic drifts. How do these modify \( D_m \)? Explain why high energy particles (runaways) are confined longer than thermal.

c'') Formulate the theory of diffusion due stochastic fields in toroidal geometry using selfsimilar mode formulation for the fluctuations.

c''') What happens to net cross field transport in a standing spectrum of e.m. and magnetic perturbations; when might transport vanish? Why?
Collisional Regime = More challenging

Here: \( l_{cc} \approx l_{mfp} < L \) 

(Short mean free path)

Point: \( l_{mfp} \approx L \Rightarrow \) particle tends to walk its parallel and undergo many bounces in \( L \). So parallel motion is diffusive.

\( \Rightarrow \) perpendicular motion is continuous

\( \langle \Delta r^2 \rangle \sim D_T \) \( \propto \frac{\text{parallel correlation length}}{l_{mfp}} \) 

(\( \approx \) significant diffusive regime)

But also note that parallel motion is diffusive, so:

\( \langle \Delta r^2 \rangle \sim D_T \) \( \propto \frac{\text{parallel correlation length}}{l_{mfp}} \)
but time set by:

\[ \frac{\nu_1}{b_{cs}} \sim \frac{1}{t} \]

\[ \frac{\langle \Delta \nu^2 \rangle \sim \nu_1 \Delta \nu}{\nu_{cs}} \]

\[ \sim \frac{D_m \nu_1}{\nu_{cs}} \sim \frac{D_m \nu_1 / \nu_{cs}}{\nu_{cs}} \]

\[ \nu_{cs} = \frac{D_m}{\nu_{cs}} \]

\[ \nu_{cs} \]

May what is \( b_{cs} \)?

Notice \( b_{cs} \) is set by competition between 2 processes:

1. width of circle increases due diffusion (cause another)

\[ \Rightarrow \]
\[\frac{d\sigma}{d\Omega} \sim \left(\frac{\alpha \cdot \text{d}t}{\Delta \cdot \text{d}t}\right)\]
\[d\sigma \sim \left(\frac{\Delta}{\alpha \cdot \text{d}t}\right)^{\frac{1}{2}}\]

but
\[\frac{x_{11}}{b(\Delta)}^2 \sim \frac{1}{\Delta \cdot \text{d}t}\]

\[d\sigma \sim \left(\frac{\alpha \cdot \text{d}t}{\Delta \cdot x_{11}}\right)^{\frac{1}{2}}\]
\[+ \frac{x_{11}}{\Delta^2} \sim \frac{\alpha \cdot \text{d}t}{\Delta^2}\]

2. Width shrinks due to stochastic instability and area conservation:

\[\frac{d\sigma}{dL} = -\frac{1}{\lambda c}\] (exponential decay)
then homogeneously:

$$\Delta t \sim (\rho_t / \chi_t)^{1/2}$$

and

smeared

\[ \Delta t \sim (\rho_t / \chi_t)^{1/2} \]

\[ \sigma \sim \rho_t (\rho_t / \chi_t)^{1/2} \]

\[ 2 + T - \chi_t \chi_t \Delta T - \rho_t \Delta \rho - T = 0 \]

\[ \Rightarrow \frac{\chi_t}{\rho_t} \sim \frac{\rho_t}{\chi_t} \]

Finally, need correlation length $\xi$, aka $\sigma$. Assume set by
the

$0 \rightarrow D$

$\lambda = \sigma \chi_t / \rho_t \sim \rho_t / \sigma$
\[ b_{c5} \sim b_{c5} \ln \left( \frac{1}{b_{c5}} \right) \]

\[ b_{c5} \sim b_{c5} \left( \frac{3n}{4} \right)^{1/2} \]

\[ b_{c5} \sim b_{c5} \ln \left( \frac{3n}{4} \right)^{1/2} / k_{c5} \]

\[ \Rightarrow \]

\[ n_{1} \sim \ln \left( \frac{3n}{4} \right) / b_{c5} \]

Apart from log factor:

\[ n_{1} \sim n \ln \left( \frac{3n}{4} \right) \left( \frac{\text{length}}{b_{c5}} \right) \]

\[ \Rightarrow \] reduced relative to collisionless values.
Lesson: - collisions reduce (knuff kl) reduce \( \alpha \) relative to "collisionless case"

- interplay of perp and parallel diffusion

- again critical to knock particle off field line.

Now, the above calculation requires thought. It is much more convenient to crack mindlessly.

\[ \rightarrow \text{Hydro approach: Kedemtsen and Rogutse (not mindless but systematic)} \]

Consider heat flux along wiggling fields \( \phi \)

\[ q = -x_{\parallel} \frac{\partial}{\partial \phi} b - x_{\perp} \frac{\partial}{\partial \phi} T \]

\[ \phi_{\parallel} \gg \phi_{\perp} \]
Here:  \[ b = b_0 + \vec{b} \]

\[ \vec{b} = \vec{D} \text{ Drifting}\]

\[ \vec{D} = \omega z + \vec{b} \cdot \vec{D}_f \]

piecewise wedging line

Seek mean radial heat Flux

\[ \langle q_{in} \rangle = -K_n \left( \frac{\partial T}{\partial r} \right) (r<T) \]

usual quadratic

\[ -K_n \langle \vec{b} \cdot \vec{D}_f \rangle \rightarrow \text{ cubic} \]

\[ -K_n \langle \vec{b} \cdot \vec{D}_f \rangle \rightarrow \text{ cubic} \]

Now \( \frac{\partial T}{\partial r} \rightarrow \frac{\partial T}{\partial r} \)

\( \frac{\partial T}{\partial r} \rightarrow \frac{\partial T}{\partial r} \)

\( \rightarrow b \cdot \vec{D}_f \rightarrow K_n \)}
so cubic nonlinearity dominates for \( \text{k}_\text{u} > 1 \).

\( \text{k}_\text{u} < 1 \Rightarrow \text{drop cubic} \).

To compute \( \langle \text{Zr} \rangle \), we need

- retain 1) casually and 2)
- iterate for \( \overline{\text{Z}} \) using

\[
\overline{\text{Z}} = 0 \quad \text{i.e. adiabatic limit.}
\]

Thinking (gasp!) first:

\[
\langle \text{Zr} \rangle = - \nabla \mu \left[ \langle \text{kn} \rangle \nabla T + \langle \text{Gradz} \overline{\text{T}} \rangle \right] - \rho \frac{\partial \langle \text{T} \rangle}{\partial t}
\]

\[
= - \nabla \mu \left[ \langle \text{kn} \rangle \frac{\overline{\text{b}} \cdot \nabla \text{T} \rangle \right] - \frac{\gamma}{\kappa} \rho \langle \text{T} \rangle
\]

Inheritance:

\( \partial_t \langle \text{T} \rangle + \Delta \overline{\text{T}} \)
Point: need non-zero $\mathbf{q} \cdot \partial \mathbf{r}$ fluctuation to drive heat flux

\[ \text{Temperature cannot be constant along line to drive parallel heat flux} \]

\[ \mathbf{v} \cdot \mathbf{Z} = 0 \Rightarrow \text{result must multiply } \mathbf{v}_1 \text{ dependence!} \]

\[ \mathbf{v}_1 \text{ to balance} \]

\[ \mathbf{v}_1 \mathbf{Z} = 0 \]

\[ \mathbf{D} \cdot \mathbf{Z} = 0 \]

\[ \mathbf{D}_1 \mathbf{Z}_1 + \mathbf{D}_1 \mathbf{Z}_x = -2 \mathbf{K}_1 \mathbf{D}_1 \frac{\partial T}{\partial n} \]

\[ \text{i.e.} \]
\[ z = - x_1 \left( A + b \cdot D \right) (T_0 + T) (\tilde{a} + \tilde{b}) \]

\[ -y_1 T \]

\[ = -x_1 \alpha \frac{\Sigma}{\beta} - x_1 \frac{\Sigma}{\gamma} = -x_1 \Xi \frac{\Sigma}{\delta} \]

\[ \frac{\Sigma}{\delta} = -x_1 \Xi \frac{\Sigma}{\gamma} \left( \frac{k_z k_0}{k_z k_0} \right) \]

\[ \frac{\Sigma}{\gamma} = -x_1 \Xi \frac{\Sigma}{\gamma} \left( \frac{k_z}{k_z} + \frac{k_0}{k_0} \right) \]

\[ = -x_1 \Xi \frac{\Sigma}{\beta} \sum_{\gamma} \left( \frac{\Xi}{\gamma} \right) \]
\[ \langle q_r \rangle_{NL} = -\chi_{rl} \frac{\partial^2}{\partial \chi^2} \sum_n \frac{x_n^2 k^2 \langle b_n b_n \rangle}{x_n^2 k^2 + \frac{\alpha}{x_n} k^2} \]

Note explicit dependence on \( x \)!

\[ \langle q_r \rangle_{NL} = -\chi_{rl} \frac{\partial^2}{\partial \chi^2} \int \frac{x_\perp^2 k^2 \langle b_\perp b_\perp \rangle}{x_\perp^2 k^2 + \left( \frac{\alpha}{x_\perp} k^2 \right)} \]

\[ = -\frac{2\pi D}{\Delta} \int \frac{x_\perp^2 \langle b_\perp b_\perp \rangle}{\left( \frac{\alpha}{x_\perp} k^2 \right) + 1} \left( \frac{x_\perp}{x_{\perp}} \right)^2 \]

\[ = -\frac{2\pi D}{\Delta} \int x_{\perp}^2 \left( \frac{x_{\perp}}{x_\perp} \right)^2 \frac{k^2}{\sqrt{\alpha^2}} \text{ auto-correlation} \]
auto correlation due to entropic virial normalization

\[
\langle z r \rangle_w = -\sqrt{\frac{n_w}{n_t}} \langle \tilde{z} \tilde{r} \rangle \text{ on } \sqrt{\frac{n_t^2 - \langle z^2 \rangle_w}{n_w}}
\]

Note: need \( D_w = -\frac{\partial \langle z r \rangle}{\partial r} \)

\((\mathbf{B} \cdot \nabla T = 0)\) for heat flux

\[-\langle \tilde{z} \rangle \text{ on } \Omega_m, \quad \sqrt{n_t^2 - 1} A_1 \]

\[
\langle 2r \rangle = -\chi_1 \langle z \tilde{r} \rangle \frac{\partial r}{\partial r} - \chi_1 \frac{d \langle \tilde{z} \rangle}{d r}
\]

\[
\chi_1 \equiv \left( \frac{2 n_t^2 \Omega_m}{A_1} \right)
\]

\[
\chi_1 \equiv \frac{\frac{\partial u_t}{\partial x}}{u_t} 0 \text{ on } \frac{A_t}{D_0}
\]
\[ \nu_{\text{eff}} \approx \frac{D_B D_M}{\Delta_I} \]

- \( \nu_{\text{eff}} \) scales with Bohm not Spitzer (\( \nu_B \))

- high line important again.

To compare \( R \& R' \):

\[ \nu_{\text{eff}} \approx \sqrt{\nu_B \nu_{\text{eff}}} < \frac{<b^2>}{k_{\text{eff}}} \]

what is \( \Delta_I \)?

Now

\[ \frac{\nu_B}{k_{\text{eff}}} \sim \frac{B}{\Delta_I^2} \quad \text{and} \quad \frac{\epsilon_{\text{diffusion}}}{\epsilon_{\text{rad}}} \sim \frac{1}{\Delta_I} \]

\[ \Delta_I \sim k_{\text{eff}} \sqrt{\nu_B / \nu_{\text{eff}}} \]

\[ \text{entire spectrum} \quad (\text{small layer}) \]
\[ X_i - \sqrt{\ln \ln \frac{L}{k_i}} \leq 2 \ln \left( \frac{24}{N_i} \right)^{\frac{1}{2}} \]

\[ Y_i \sim \frac{X_i}{Y_i} \frac{M_i}{L_i} \]

- so - modulo \( k_i \), \( A_j \) agrees with \( R \times R \) to within log. factor

\[ Y_i \sim \frac{X_i}{Y_i} \frac{M_i}{L_i} \]

- covers diffusion in \( k_i \) of \( A_j \) stochastic fields

\[ \text{Lesson: Take care re: irreversibility!} \]