1.78. Hole in a shell

FIRST SOLUTION: We can solve this exercise by direct integration. Let's slice up the spherical shell (minus the hole) into rings parameterized by the angle θ shown in Fig. 29. The width of a ring is $R\,d\theta$, and its circumferential length is $2\pi(R\sin\theta)$. So its area is $2\pi R^2 \sin\theta\,d\theta$. All points on the ring are a distance $2R\sin(\theta/2)$ from the center of the hole. Only the vertical component of the field survives, and this brings in a factor of $\sin(\theta/2)$, as you can check. If the edge of the hole is at the small angle $\theta = \epsilon$, the total field at the middle of the hole is (writing $\sin\theta$ as $2\sin(\theta/2)\cos(\theta/2)$)

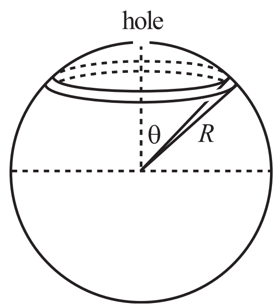

Figure 29

$$\begin{aligned} \frac{1}{4\pi\epsilon_0}\int_\epsilon^\pi \frac{\sigma 2\pi R^2 \sin\theta\,d\theta}{\big(2R\sin(\theta/2)\big)^2}\sin(\theta/2) &= \frac{\sigma}{4\epsilon_0}\int_\epsilon^\pi \cos(\theta/2)\,d\theta \\ &= \frac{\sigma}{2\epsilon_0}\sin(\theta/2)\bigg|_\epsilon^\pi \approx \frac{\sigma}{2\epsilon_0}, \end{aligned} \tag{87}$$

in the limit where $\epsilon \to 0$.

SECOND SOLUTION: The given setup with the hole is the superposition of a complete spherical shell with density σ plus a small disk with density $-\sigma$. And very close to the center of the disk, the disk looks essentially like an infinite plane. The fields due to these two objects, at the point in question, are shown in Fig. 30. The sum of the fields at the center of the hole is therefore a field with magnitude $\sigma/2\epsilon_0$, directed radially outward. Note that we obtain an outward $\sigma/2\epsilon_0$ field independent of whether we look at a point just inside or just outside the shell; these points yield $0 + \sigma/2\epsilon_0$ or $\sigma/\epsilon_0 - \sigma/2\epsilon_0$, respectively. In other words, even though the field isn't continuous across the original complete shell or across the disk, it *is* continuous across the hole. It must be continuous, of course, because there is nothing but vacuum in the hole.

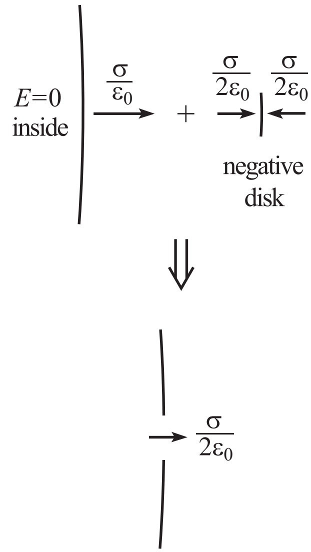

Figure 30

Since the field inside the complete sphere is zero, the field inside the sphere-plus-hole is exactly the same as the field due to the negative disk. The field lines due to a disk are shown in Fig. 2.12. Near the edge of the hole, the tangential component of the field diverges. But at points in the hole exactly on the (removed) surface of the sphere, the *radial* component of the field is exactly $\sigma/2\epsilon_0$, over the entire area of the hole.

THIRD SOLUTION: We can also solve this exercise by considering the force on the little disk, while it is still in place in the shell. If A is the area of the disk, then we know from Eq. (1.49) that the force on it is

$$A\sigma\frac{E_1 + E_2}{2} = A\sigma\frac{0 + \sigma/\epsilon_0}{2} = A\sigma \cdot \frac{\sigma}{2\epsilon_0}. \tag{88}$$

But the force on the disk equals the charge on the disk times the field at the location of the disk, due to all the *other* charge in the system (that is, the shell with the disk removed). Equation (88) therefore tells us that the (radial) field of the shell-minus-disk must be $\sigma/2\epsilon_0$, as desired.