Problem Set 3
Due December 2

1) a) Consider a weakly damped linear harmonic oscillator driven by white noise.

i) Derive the fluctuation spectrum at thermal equilibrium.

ii) What value of forcing is required to achieve stationarity at temperature $T$?

b) Now consider a forced nonlinear oscillator

$$
\ddot{x} + \gamma x + \omega_0^2 x + \alpha x^3 = \tilde{f}.
$$

Again, assume $\tilde{f}$ is white noise. Characterize the equilibrium fluctuation spectrum. Hint: You may find it useful to review Section 29 of "Mechanics", by Landau and Lifshitz.

2) a) Derive the Fokker–Planck Equation for $f(x, v, t)$ for sedimentary particles in a fluid. Discuss the physics of all terms.

b) Now derive the Schmoluchowski equation for the above; solve it.

c) How might one get from $a \to b$ directly?
3) Consider an elastic dumbbell of Stokesian particles in a fluid flow $v(x, t)$, at temperature $T$.

\[ \text{\includegraphics[width=0.3\textwidth]{dumbbell.png}} \]

a) Derive the Fokker–Planck equation for the length $l$.

b) What is the mean square length $l$?

Assume the dumbbell has spring constant $k$. The fluid has viscosity $\nu$.

c) Now take the flow as turbulent, so $v(x, t) = \langle v(x, t) \rangle + \tilde{v}(x, t)$, where $\tilde{v}$ is random. Repeat a) and b), above.

4) Give a general, but purely classical, derivation of the Fluctuation–Dissipation Theorem.

5) Prove the Fluctuation–Dissipation Theorem for a multi-field system. (You may find Landau and Lifshitz’s “Statistical Physics” useful, here.)
6) a) Derive the Fokker–Planck equation for $\langle f(p, t) \rangle$, the mean distribution function for a system for particles moving according to the Hamiltonian equations of motion for Hamiltonian $H = H(p, q)$. Assume the average is over $q$.

b) Show that in the F–P equation, the drift and diffusion partially cancel, so the F–P equation simplifies to

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial p} \left[ D_p \frac{\partial \langle f \rangle}{\partial p} \right]$$

c) What is $D_p$?

7) a) State and prove the Central Limit Theorem starting from the Chapman–Kolmogorov equation. Chandrasekhar is a good reference.

b) Give a concise summary of when the Central Limit Theorem applies — i.e. what conditions must be met?

c) What happens if the probability of step size $x$ is:

$$p(x) = \frac{1}{1 + x^4}$$
8) Consider a function \( q \) which satisfies:

\[
\tau \frac{\partial q}{\partial t} = -a(T, T_c)q - bq^3 + \tilde{f}
\]

Here \( \langle \tilde{f}^2 \rangle = |\tilde{f}_0|^2 \tau_c \delta(t_1 - t_2) \).

(a) Derive the Fokker–Planck equation for \( P(q, t) \). Solve and discuss the stationary solution for \( T > T_c \), \( T < T_c \), \( T = T_c \).

(b) How does \( P(q, t) \) evolve if \( T \) passes adiabatically thru \( T_c \)? Here “adiabatically” means \( \tau_c \left( \frac{\partial T}{\partial t} \right) T \ll 1 \).

(c) Discuss the behavior when

\[
a = a_0 + \bar{a}
\]

\( \langle \bar{a}^2 \rangle = \bar{a}^2 \tau_0 \delta(t_1 - t_2) \)

9) (a) Generalize the calculation of the current between stable states \( A, B \) in the Kramers problem to the case where \( n_B \neq 0 \).

(b) For what ratio \( n_A/n_B \) does \( j = 0 \)?

(c) Calculate \( j \) for \( n_B \) finite and \( n_A \to 0 \). Compare this to the value of \( j \) calculated in class.