Intensity in Single-Slit Diffraction

The intensity of the diffraction pattern at any given angle ϑ is

$$I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2,$$

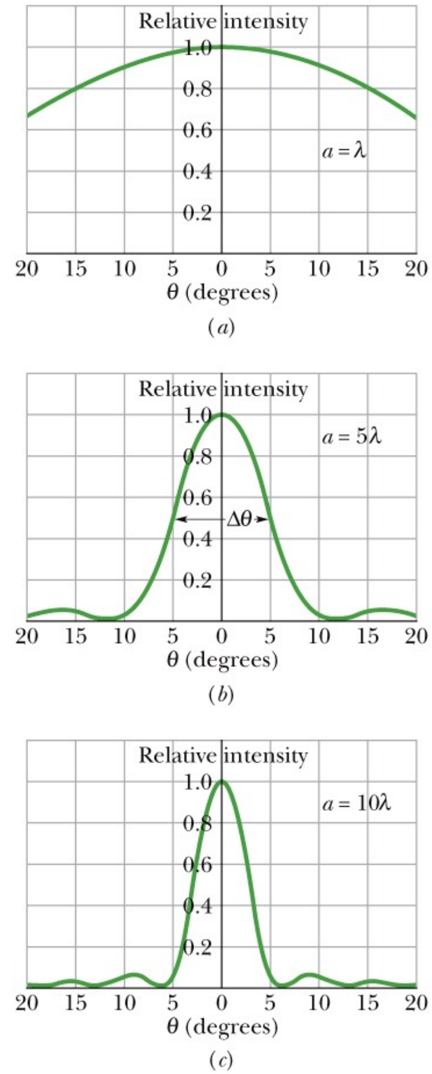
where, I_m is the intensity at the center of the pattern and

$I(\alpha) = 0$ for
$\alpha = m\pi$, $m = 1, 2, 3, ..$

$$\alpha = \tfrac{1}{2}\phi = \frac{\pi a}{\lambda}\sin\theta.$$

$\frac{\cancel{\pi}}{\lambda} a \sin\theta = m\cancel{\pi}$

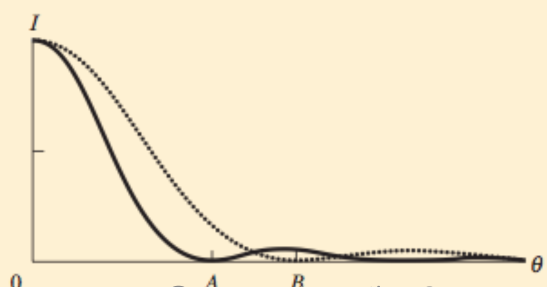
The plots show the relative intensity in single-slit diffraction for three values of the ratio a/λ. The wider the slit is, the narrower is the central diffraction maximum.



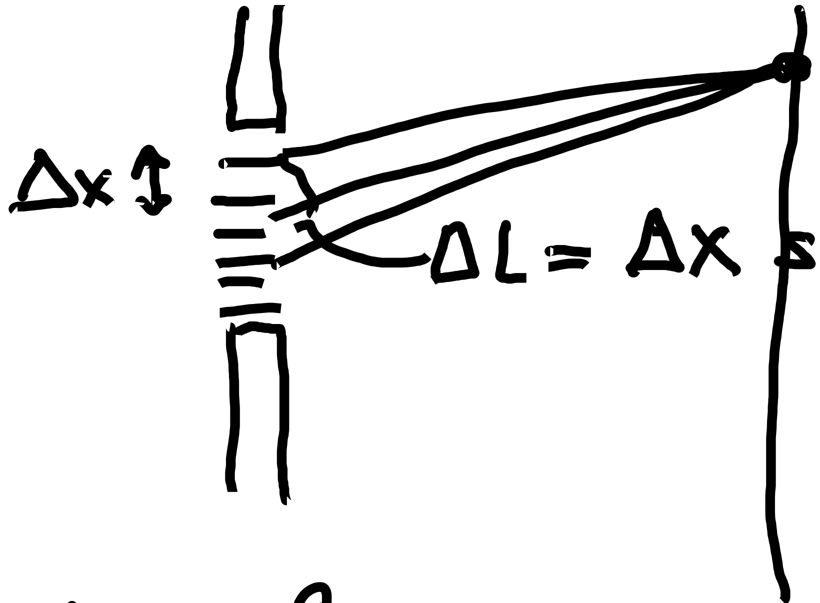
(a)
(b)
(c)

Checkpoint 3

Two wavelengths, 650 and 430 nm, are used separately in a single-slit diffraction experiment. The figure shows the results as graphs of intensity I versus angle θ for the two diffraction patterns. If both wavelengths are then used simultaneously, what color will be seen in the combined diffraction pattern at (a) angle A and (b) angle B?



Answer
(a) 650 nm
(b) 430 nm



$$\Delta L = \Delta x \sin\theta$$

$$\Delta\phi = 2\pi \frac{\Delta L}{\lambda}$$

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x \sin\theta$$

$$\Delta x = \frac{a}{N}$$

$$E = E_0\left(1 + e^{i\Delta\phi} + e^{i2\Delta\phi} + e^{i3\Delta\phi} + \dots + e^{i(N-1)\Delta\phi}\right)$$

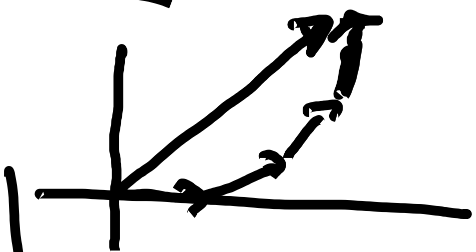
$$1 + \omega + \omega^2 + \omega^3 + \dots + \omega^{N-1} = \frac{1-\omega^N}{1-\omega}$$

$$(1+\omega+\omega^2+\omega^3+\dots\omega^{N-1})(1-\omega) =$$

$$= 1 - \cancel{\omega} + \cancel{\omega} - \omega^2 + \omega^2 \dots - \omega^N$$

$$\omega = e^{i\Delta\phi}$$

$$\left|\frac{1-w^N}{1-w}\right| = \left|\frac{1-e^{+iN\Delta\phi}}{1-e^{i\Delta\phi}}\right| =$$

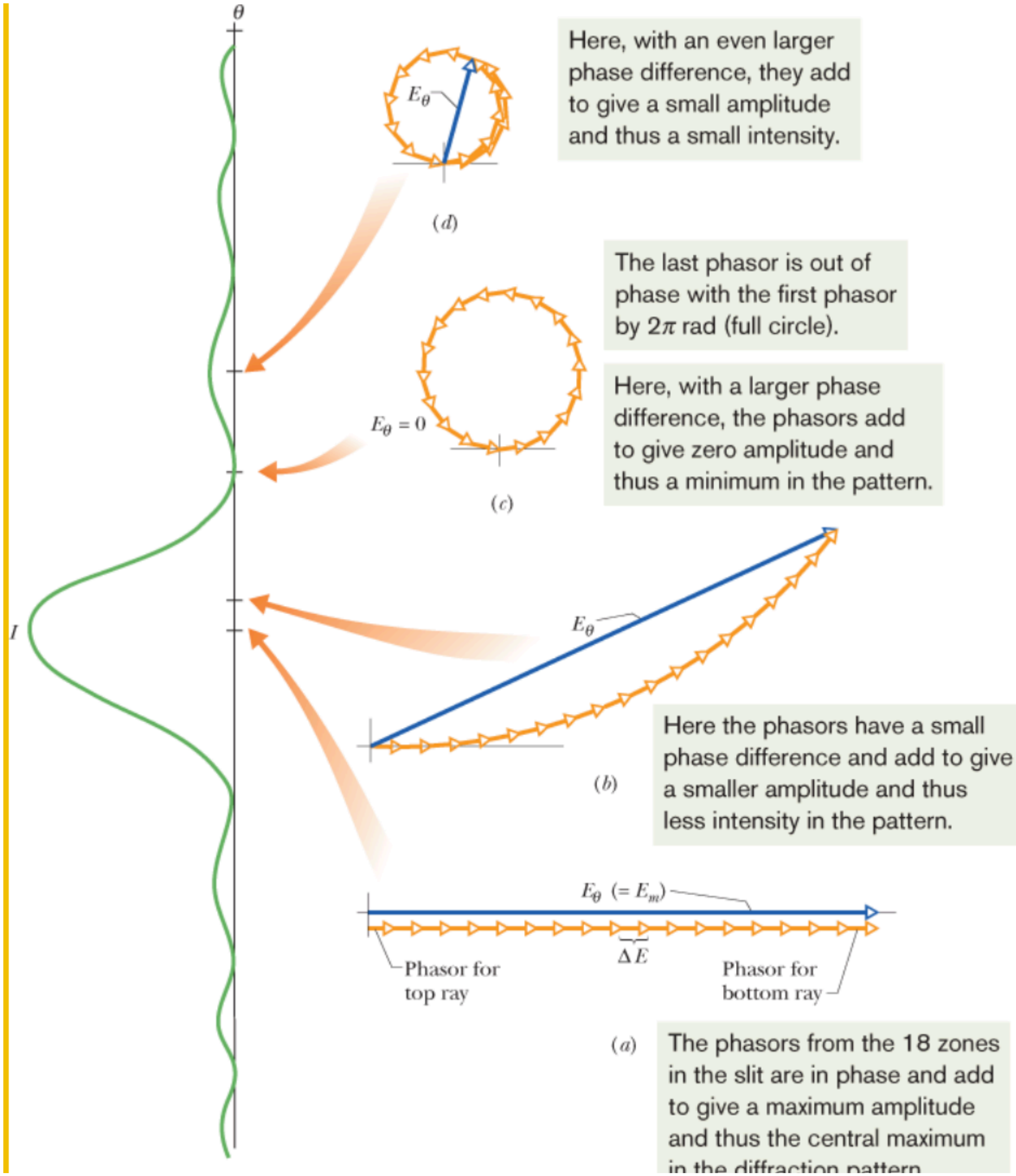


$$= \left|\frac{e^{i\frac{N}{2}\Delta\phi}}{e^{i\Delta\phi/2}}\right| \left|\frac{e^{-i\frac{N}{2}\Delta\phi} - e^{i\frac{N}{2}\Delta\phi}}{e^{-i\Delta\phi/2} - e^{i\Delta\phi/2}}\right| =$$

$$\frac{N\Delta\phi}{2} = \frac{\pi a}{\lambda}\sin\theta$$

$$= \frac{\sin\frac{N}{2}\Delta\phi}{\frac{\Delta\phi}{2}} = (\text{square})$$

$$= \left|2N\frac{\sin\left(\frac{\pi a}{\lambda}\sin\theta\right)}{\frac{\pi a}{\lambda}\sin\theta}\right|^2 = I_m\frac{\left(\sin\overbrace{\left(\frac{\pi a}{\lambda}\sin\theta\right)}^{=\alpha}\right)^2}{\left(\frac{\pi a}{\lambda}\sin\theta\right)^2}$$



θ
I
Here, with an even larger phase difference, they add to give a small amplitude and thus a small intensity.
E_θ
(d)
The last phasor is out of phase with the first phasor by 2π rad (full circle).
Here, with a larger phase difference, the phasors add to give zero amplitude and thus a minimum in the pattern.
$E_\theta = 0$
(c)
E_θ
(b)
Here the phasors have a small phase difference and add to give a smaller amplitude and thus less intensity in the pattern.
E_θ $(= E_m)$
ΔE
Phasor for top ray
Phasor for bottom ray
(a)
The phasors from the 18 zones in the slit are in phase and add to give a maximum amplitude and thus the central maximum in the diffraction pattern

$$\sin \phi/2 = \frac{E_\theta}{2R}$$

$$\phi = \frac{E_m}{R}$$

$$E_\theta = E_m \frac{\sin \phi/2}{\phi/2}$$

$$I \propto E_\theta^2 = \propto \left(\frac{\sin \phi/2}{\phi/2}\right)^2$$

$$\boxed{\phi/2 = \frac{\pi a \sin\theta}{\lambda} = \alpha}$$

$$\Delta L = a \sin\theta$$

$$\phi = 2\pi \frac{\Delta L}{\lambda} = \frac{2\pi a \sin\theta}{\lambda}$$

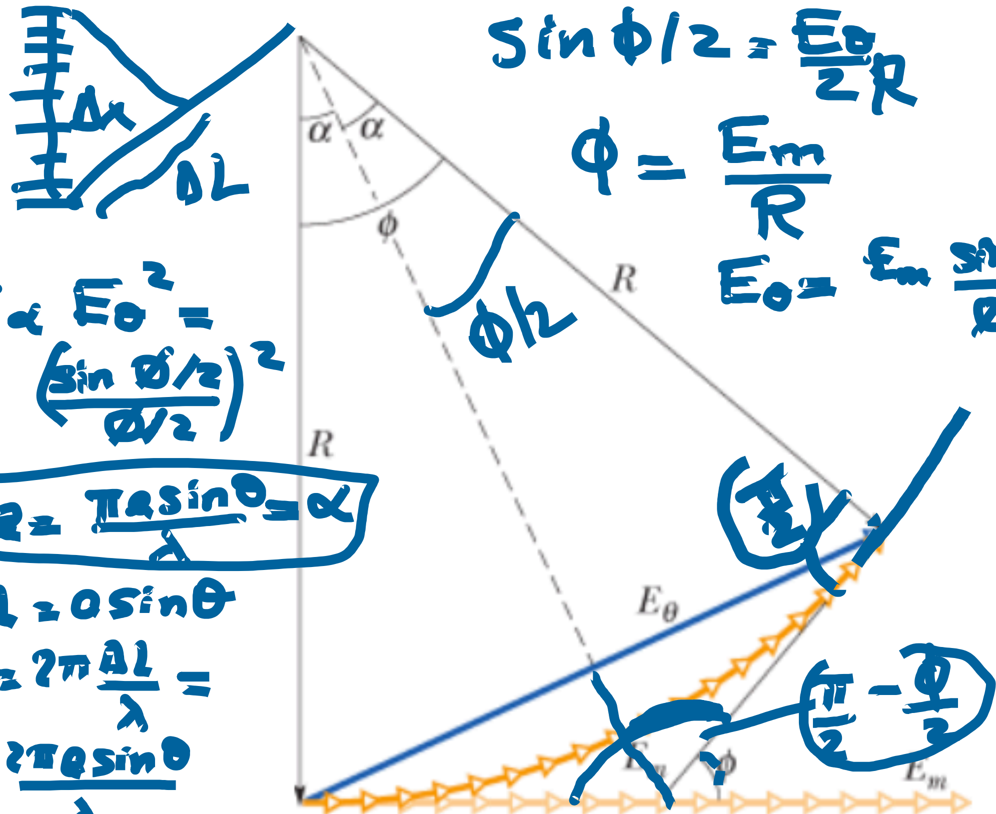


Figure 36-9
A construction used to calculate the intensity in single-slit diffraction. The situation shown corresponds to that of Fig. 36-7*b*.

Diffraction by a Circular Aperture

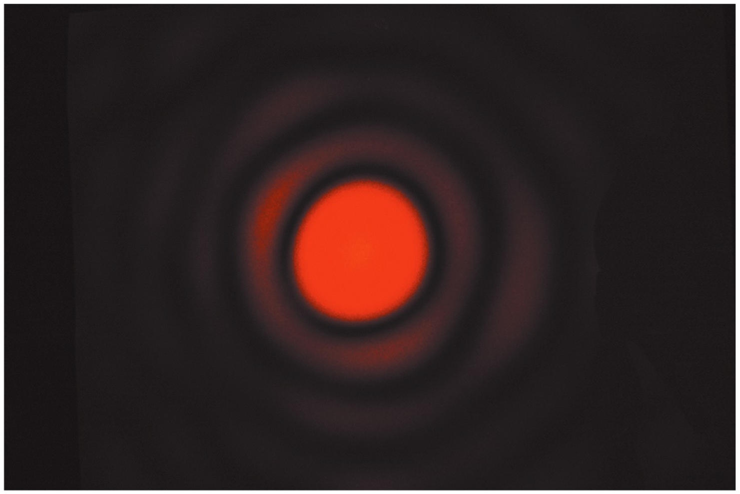
Diffraction by a circular aperture or a lens with diameter *d* produces a central maximum and concentric maxima and minima, given by

$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(first minimum—circular aperture).}$$

The angle ϑ here is the angle from the central axis to any point on that (circular) minimum.

$$\sin\theta = \frac{\lambda}{a} \quad \text{(first minimum—single slit),}$$

which locates the first minimum for a long narrow slit of width a. The main difference is the factor 1.22, which enters because of the circular shape of the aperture.

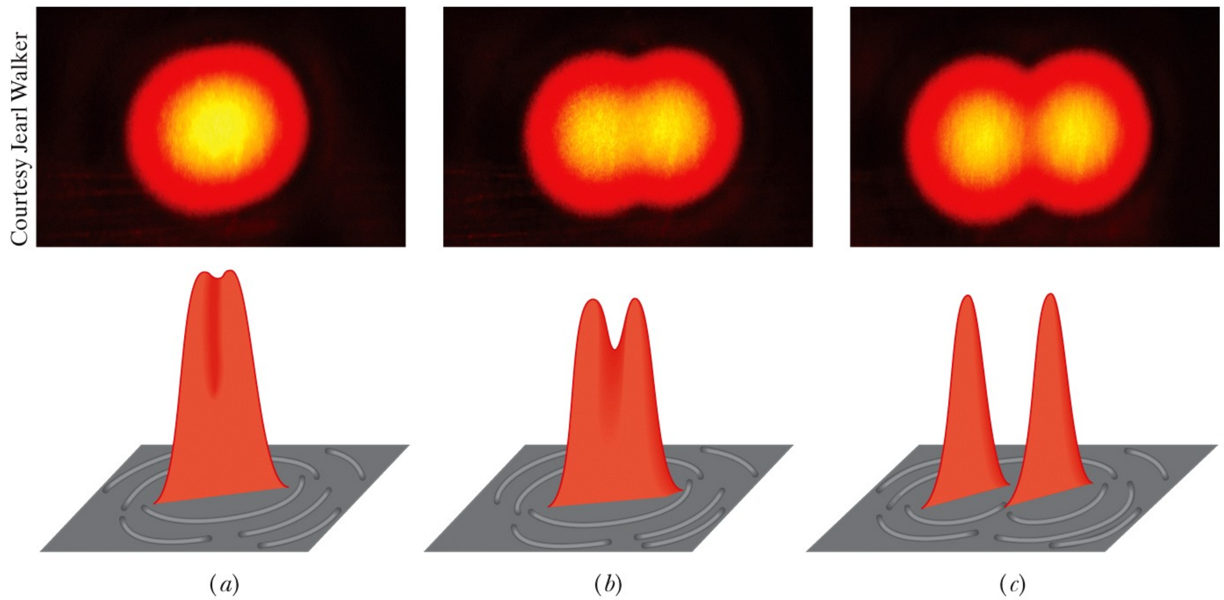


Courtesy Jearl Walker

The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum.

Diffraction by a Circular Aperture

Resolvability

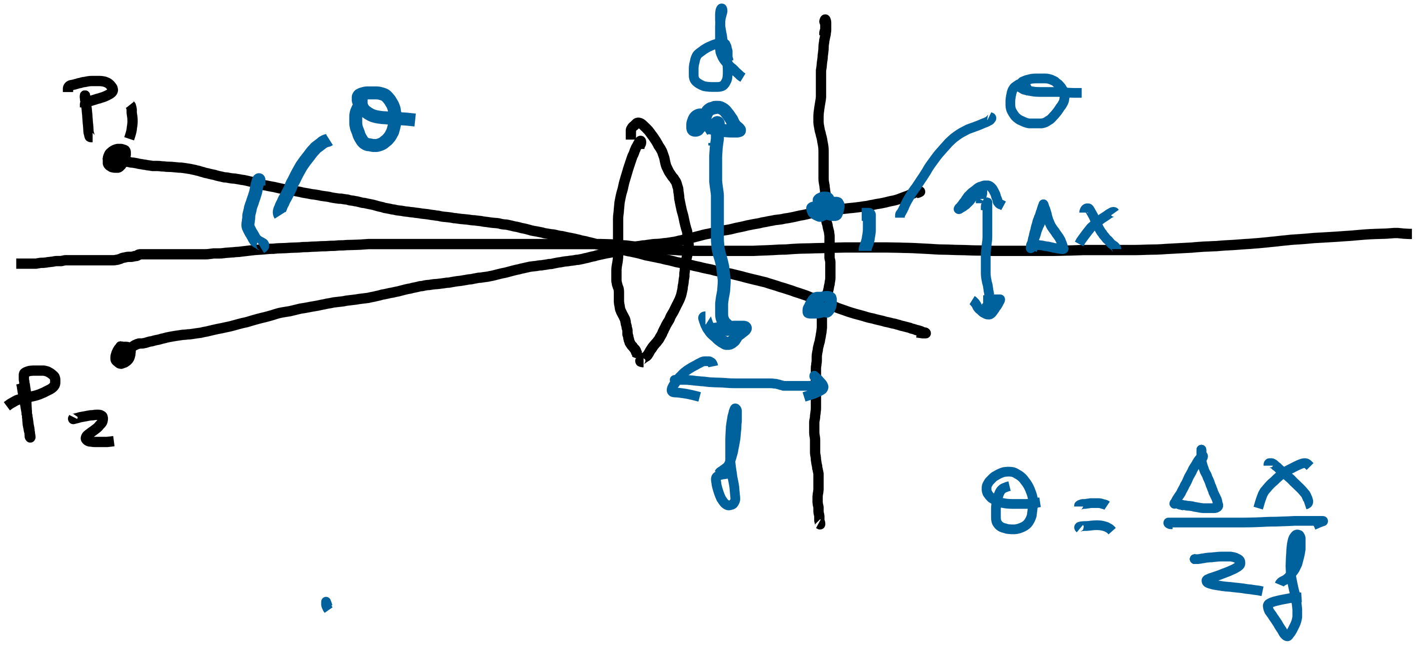


The images of two point sources (stars) formed by a converging lens. At the bottom, representations of the image intensities. In (a) the angular separation of the sources is too small for them to be distinguished, in (b) they can be marginally distinguished, and in (c) they are clearly distinguished. Rayleigh's criterion is satisfied in (b), with the central maximum of one diffraction pattern coinciding with the first minimum of the other.

Rayleigh's criterion suggests that two objects are on the verge of resolvability if the central diffraction maximum of one is at the first minimum of the other. Their angular separation can then be no less than

$$\theta_R = 1.22\,\frac{\lambda}{d} \quad \text{(Rayleigh's criterion).}$$

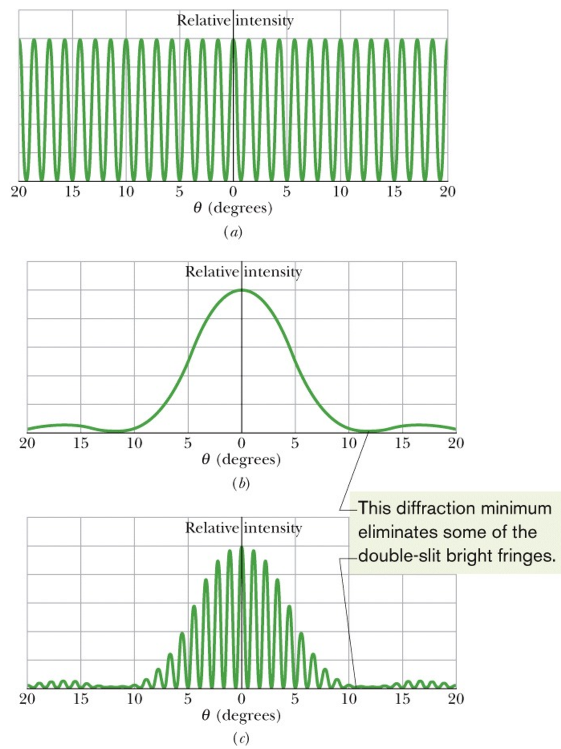
in which *d* is the diameter of the aperture through which the light passes.

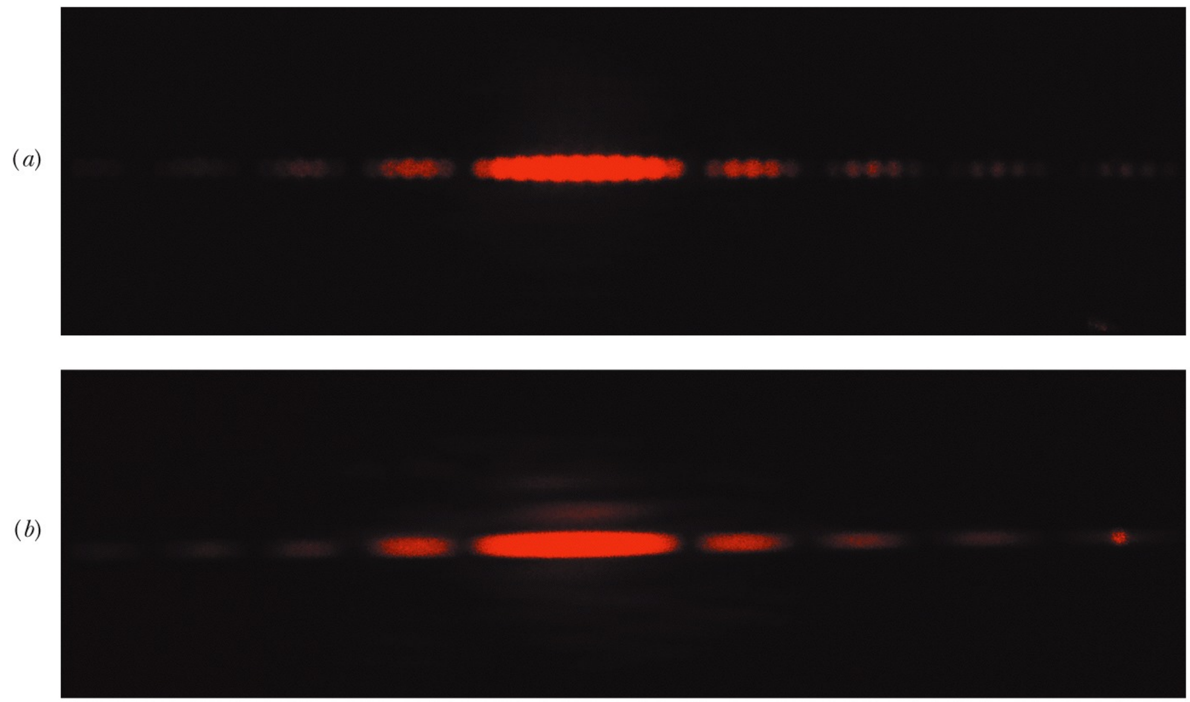


$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(first minimum—circular aperture).}$$

Diffraction by a Double Slit

Waves passing through two slits produce a combination of double-slit interference and diffraction by each slit.



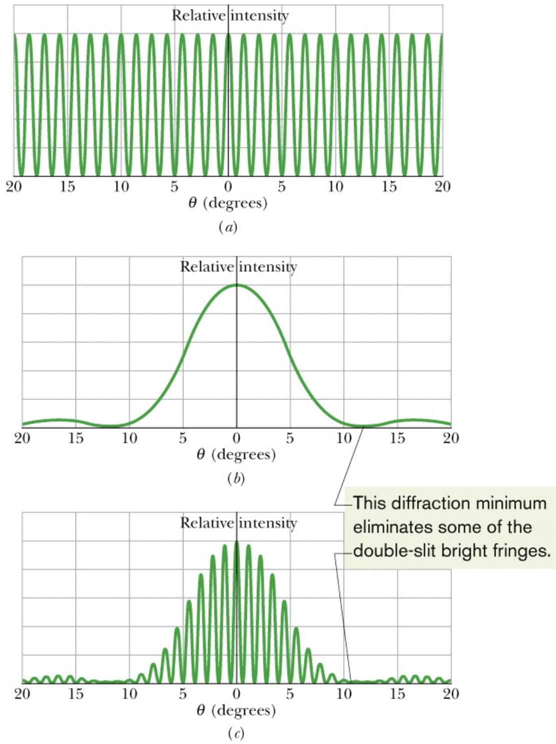


Courtesy Jearl Walker

(a) The intensity plot to be expected in a double-slit interference experiment with vanishingly narrow slits. (b) The intensity plot for diffraction by a typical slit of width a (not vanishingly narrow). (c) The intensity plot to be expected for two slits of width a. The curve of (b) acts as an envelope, limiting the intensity of the double-slit fringes in (a). Note that the first minima of the diffraction pattern of (b) eliminate the double-slit fringes that would occur near 12° in (c).

Diffraction by a Double Slit

Waves passing through two slits produce a combination of double-slit interference and diffraction by each slit.



For identical slits with width a and center-to-center separation d, the intensity in the pattern varies with the angle ϑ from the central axis as

$$I(\theta) = I_m(\cos^2\beta)\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \text{(double slit)},$$

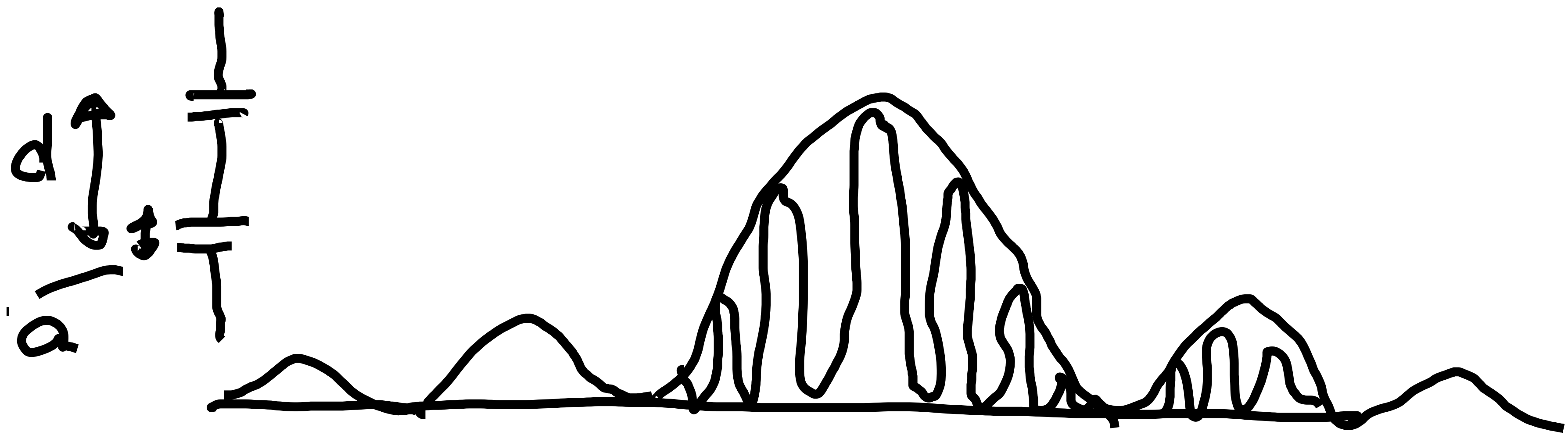
in which

$$\beta = \frac{\pi d}{\lambda}\sin\theta$$

and

$$\alpha = \frac{\pi a}{\lambda}\sin\theta.$$

Note carefully that the right side of double slit equation is the product of I_m and two factors. (1) The interference factor $\cos^2\beta$ is due to the interference between two slits with slit separation d. (2) The diffraction factor $[(\sin a)/a]^2$ is due to diffraction by a single slit of width a.



$$I(\theta) = I_m \cos^2\beta \left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$\boxed{\begin{gathered}\beta = \frac{\pi d}{\lambda}\sin\theta \\ \alpha = \frac{\pi a}{\lambda}\sin\theta\end{gathered}}$$

diff mins: $a\sin\theta = m\lambda$

interference max: $d\sin\theta = m\lambda$

diff max: $a\sin\theta \approx (m+\frac{1}{2})\lambda$

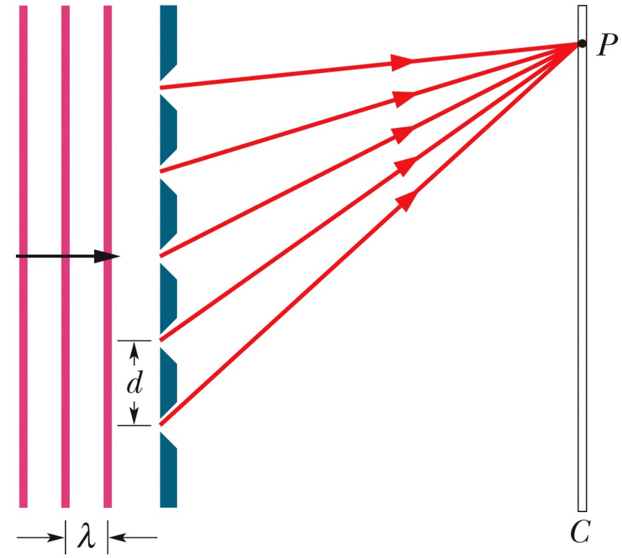
Diffraction Gratings

A diffraction grating is a series of "slits" used to separate an incident wave into its component wavelengths by separating and displaying their diffraction maxima. Diffraction by *N* (multiple) slits results in maxima (lines) at angles ϑ such that

$$d \sin \theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—lines)},$$

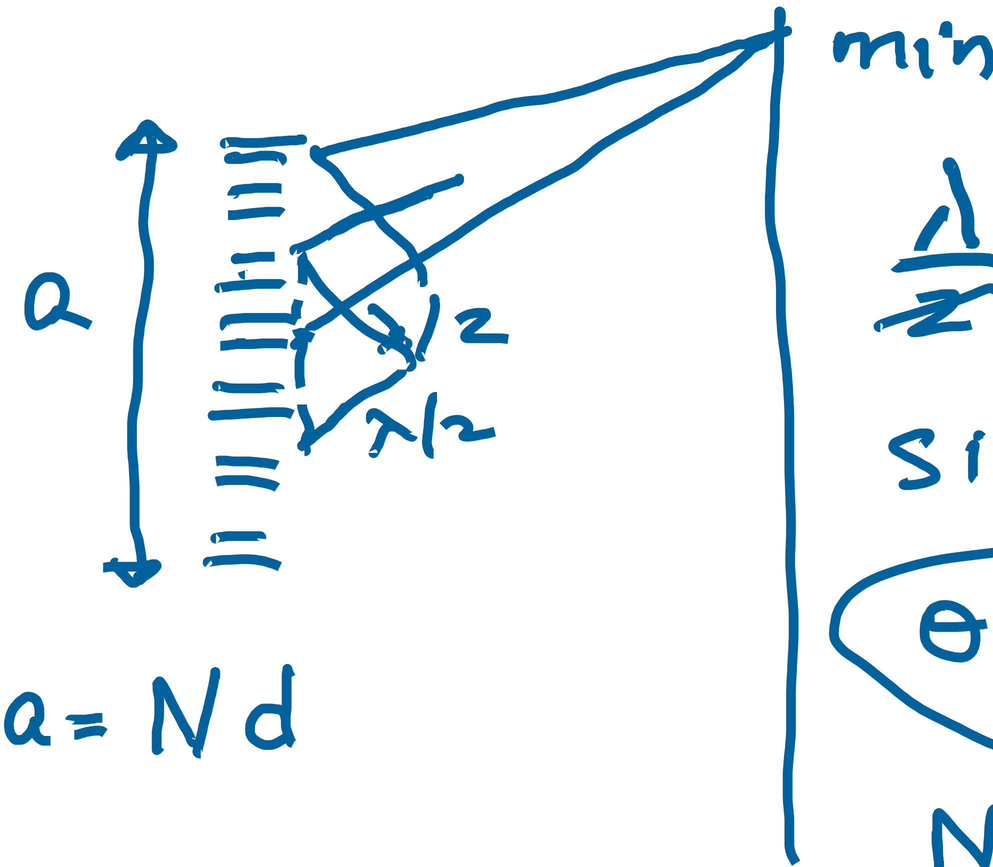
A line's **half-width** is the angle from its center to the point where it disappears into the darkness and is given by

$$\Delta\theta_{hw} = \frac{\lambda}{Nd \cos \theta} \quad \text{(half-width of line at } \theta\text{)}.$$



An idealized diffraction grating, consisting of only five rulings, that produces an interference pattern on a distant viewing screen *C*.

Note that for light of a given wavelength λ and a given ruling separation *d*, the widths of the lines decrease with an increase in the number *N* of rulings. Thus, of two diffraction gratings, the grating with the larger value of *N* is better able to distinguish between wavelengths because its diffraction lines are narrower and so produce less overlap.



(first minimum)

$$\frac{\lambda}{\cancel{2}} = \frac{Nd}{\cancel{2}} \sin\theta$$

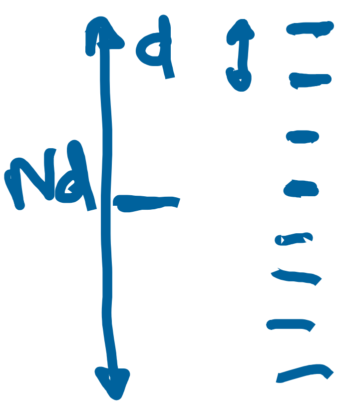
$$\sin\theta = \frac{\lambda}{Nd}$$

$$\theta = \frac{\lambda}{Nd}$$

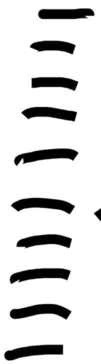
N large

$$a = Nd$$

first m



same algebra as for single slit diffraction (see earlier slides) $Nd = a$



$d \updownarrow = \Delta x$

$\frac{d}{\lambda}(\sin\theta_2 - \sin\theta_1) = \frac{1}{N}$

$\sin\theta_2 - \sin\theta_1 = \frac{\lambda}{dN}$

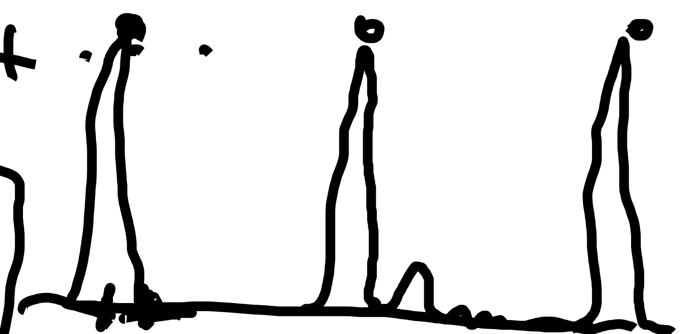
$d\sin\theta = m\lambda$

$\frac{d}{\lambda}\sin\theta_1 = m$ max

$\frac{d}{\lambda}\sin\theta_2 = m + \frac{1}{N}$

$\Delta\phi = 2\pi\frac{\Delta L}{\lambda} = \frac{2\pi}{\lambda} d\sin\theta$

$\Sigma = 1 + e^{i\Delta\phi} + e^{i2\Delta\phi} + \cdots$



$I = I_0 \frac{\sin^2\left(\frac{N\pi d}{\lambda}\sin\theta\right)}{\sin^2\left(\frac{\pi d}{\lambda}\sin\theta\right)}$

$\frac{\pi}{\lambda} d\sin\theta_1 = m\pi$

$\theta = 0 : I = N^2 I_0$

$\frac{N\pi d}{\lambda}\sin\theta_2 = \pi(mN+1)$

$\frac{N\pi d}{\lambda}\sin\theta_1 = m\pi N = \pi(mN)$

$$\boxed{\sin\theta_2 - \sin\theta_1 = \frac{\lambda}{Nd}}$$

$$\theta_2 = \theta_1 + \Delta\theta$$

$$\sin\theta_2 = \sin(\theta_1 + \Delta\theta) = \sin(\theta_1) + \cos(\theta_1)\Delta\theta$$

$$\sin\theta_2 - \sin\theta_1 = \cos\theta_1 \Delta\theta$$

maxima:

$$d\sin\theta = m\lambda$$

$$\boxed{d\cos\theta \cdot \Delta\theta = m\,\Delta\lambda}$$

$$\boxed{\Delta\theta = \frac{\lambda}{Nd\cos\theta}}$$

$$\cancel{d}\cancel{\cos\theta}\frac{\lambda}{\cancel{Nd}\cancel{\cos\theta}} = m\,\Delta\lambda$$

Dispersion: $\boxed{\frac{\Delta\theta}{\Delta\lambda} = D}$

$$\boxed{\frac{\lambda}{\Delta\lambda} = N \cdot m = R}$$

Resolving power: $\boxed{R = \frac{\lambda_{av}}{\Delta\lambda}}$

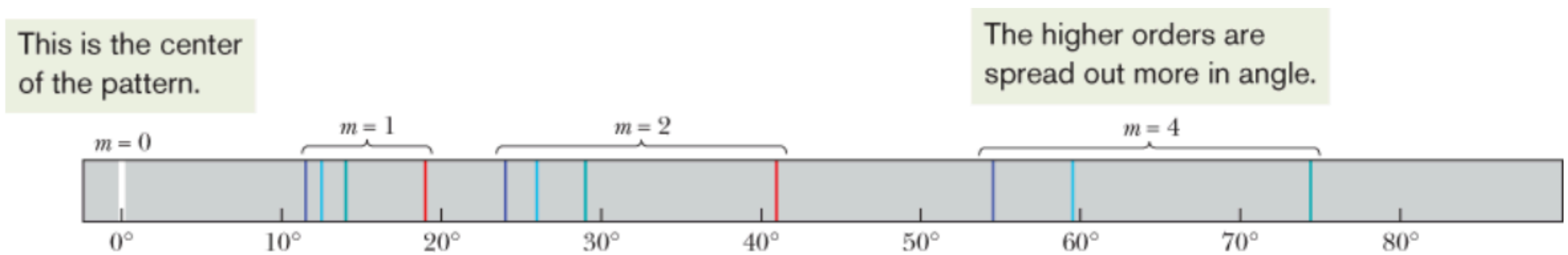


Figure 36-24
The zeroth, first, second, and fourth orders of the visible emission lines from hydrogen. Note that the lines are farther apart at greater angles. (They are also dimmer and wider, although that is not shown here.)

Gratings: Dispersion and Resolving Power

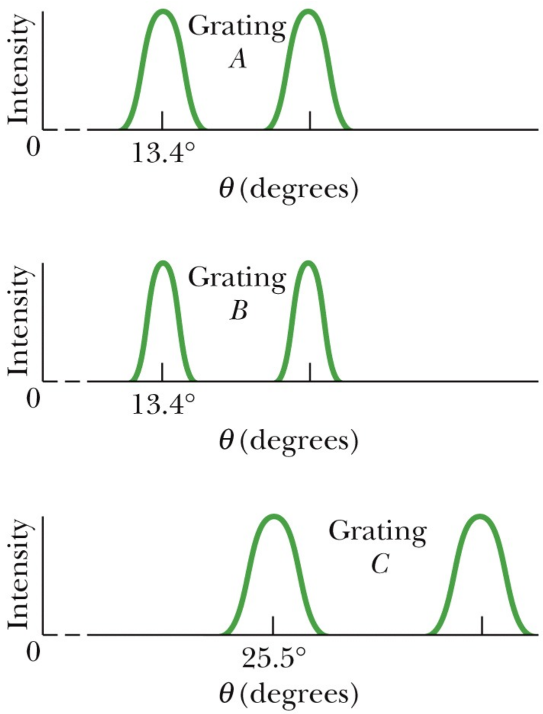
The resolving power R of a diffraction grating is a measure of its ability to make the emission lines of two close wavelengths distinguishable. For two wavelengths differing by $\Delta\lambda$ and with an average value of λ_{avg}, the resolving power is given by

$$R = \frac{\lambda_{avg}}{\Delta\lambda} = Nm$$

Table 36-1 Three Gratings[a]

Grating	N	d (nm)	θ	D (°/μm)	R
A	10 000	2540	13.4°	23.2	10 000
B	20 000	2540	13.4°	23.2	20 000
C	10 000	1360	25.5°	46.3	10 000

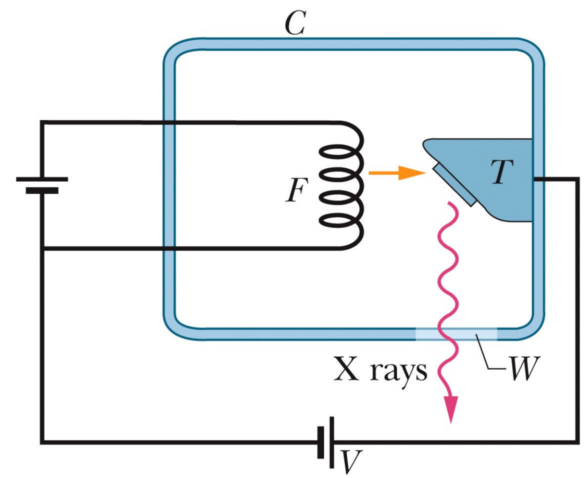
[a]Data are for $\lambda = 589$ nm and $m = 1$.

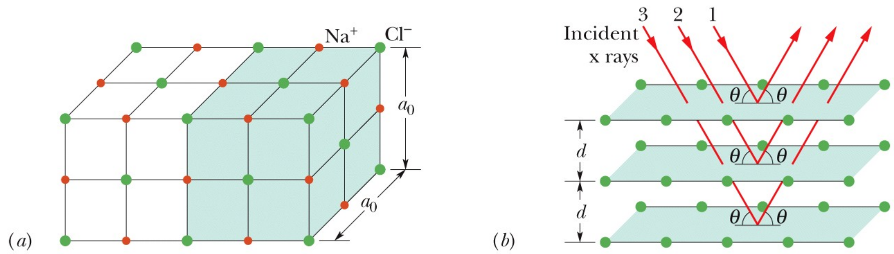


The intensity patterns for light of two wavelengths sent through the gratings of Table 36-1. Grating B has the highest resolving power, and grating C the highest dispersion.

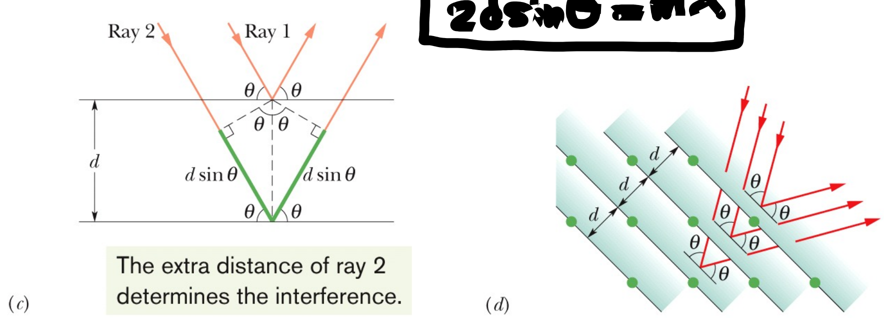
X-Ray Diffraction

X rays are electromagnetic radiation whose wavelengths are of the order of 1 Å (=10^{-10} m). Figure (right) shows that x rays are produced when electrons escaping from a heated filament F are accelerated by a potential difference V and strike a metal target T.





$\boxed{2d\sin\theta = m\lambda}$



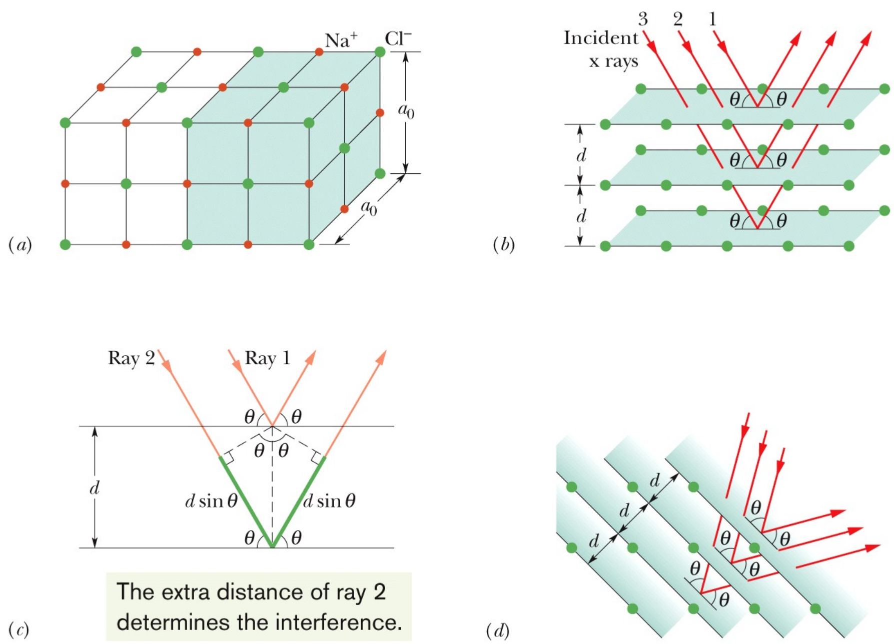
(a) The cubic structure of NaCl, showing the sodium and chlorine ions and a unit cell (shaded). (b) Incident x rays undergo diffraction by the structure of (a). The x rays are diffracted as if they were reflected by a family of parallel planes, with angles measured relative to the planes (not relative to a normal as in optics). (c) The path length difference between waves effectively reflected by two adjacent planes is $2d\sin\vartheta$. (d) A different orientation of the incident x rays relative to the structure. A different family of parallel planes now effectively reflects the x rays.

X-Ray Diffraction

As shown in figure below if x rays are directed toward a crystal structure, they undergo Bragg scattering, which is easiest to visualize if the crystal atoms are considered to be in parallel planes.

For x rays of wavelength λ scattering from crystal planes with separation d, the angles u at which the scattered intensity is maximum are given by

$$2d \sin \theta = m\lambda, \qquad \text{for } m = 1, 2, 3, \ldots$$



(a) The cubic structure of NaCl, showing the sodium and chlorine ions and a unit cell (shaded). (b) Incident x rays undergo diffraction by the structure of (a). The x rays are diffracted as if they were reflected by a family of parallel planes, with angles measured relative to the planes (not relative to a normal as in optics). (c) The path length difference between waves effectively reflected by two adjacent planes is *2d*sin *ϑ*. (d) A different orientation of the incident x rays relative to the structure. A different family of parallel planes now effectively reflects the x rays.

35-5 Michelson's Interferometer

An interferometer is a device that can be used to measure lengths or changes in length with great accuracy by means of interference fringes.

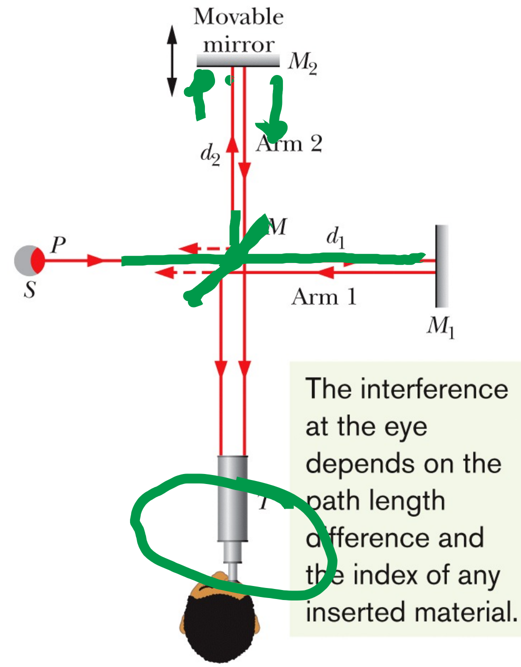
In Michelson's interferometer, a light wave is split into two beams that then recombine after traveling along different paths.

The interference pattern they produce depends on the difference in the lengths of those paths and the indexes of refraction along the paths.

If a transparent material of index n and thickness L is in one path, the phase difference (in terms of wavelength) in the recombining beams is equal to

$$\text{phase difference} = \frac{2L}{\lambda}(n-1),$$

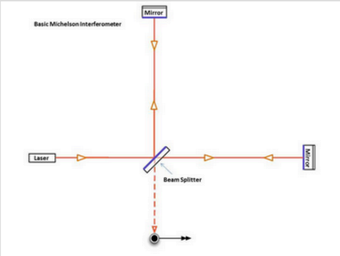
where λ is the wavelength of the light.



Michelson's interferometer, showing the path of light originating at point P of an extended source S. Mirror M splits the light into two beams, which reflect from mirrors M_1 and M_2 back to M and then to telescope T. In the telescope an observer sees a pattern of interference fringes.

LIGO's Interferometer

Although much more sophisticated, at their cores, LIGO's **interferometers** are fundamentally Michelson Interferometers, a device invented in the 1880's. We can say this because both Michelson and LIGO interferometers share these traits:



A basic Michelson interferometer

- They are both are L-shaped (not all interferometers are this shape)
- They both have mirrors at the ends of the arms to reflect light in order to combine light beams and create an interference pattern
- They both measure patterns and intensity of a resulting light beam after two beams have been superimposed or forced to 'interfere'

But this is where the similarities end. The size and added complexity of LIGO's interferometers are far beyond anything Michelson could have envisioned or that his original interferometer could have achieved.

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