Quasilinear Theory

- how instabilities/turbulence modify the profiles which drive them

- mean field theory \( \rightarrow \) evolves \( \langle F \rangle \)

- useful as device for calculating turbulent transport coefficients, e.g., anomalous resistivity.

- special application: turbulent/anomalous resistivity

- see also: posted supplementary notes \( \rightarrow \) Chapter 3 of book manuscript.

Quasilinear Theory – Vlasov Plasma

ii) Motivation and Overview

Linear theory determines instantaneous stability of plasma
c.i.e. \[ \mathcal{E}(k, \omega) = 1 + \frac{\omega_p^2}{k^2} \int \frac{d^3 \mathbf{v}}{\omega - \omega_0} \]

\[ \Rightarrow \] growth/damping rate \[ \gamma_k = \gamma_0 \] \[ \langle f \rangle \]

but \[ \langle f \rangle \] evolves... If \[ \langle f \rangle \] evolves slowly:

\[ \langle f \rangle \text{ slowly } \Rightarrow \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t} < \gamma_0 \]

can consider: \[ \gamma_k = \gamma_0 \] \[ \langle f \rangle \] \[ \Rightarrow \text{ evolution driven by instabilities} \]

physics: mean distribution evolution... \[ \Rightarrow \text{ driven by relaxation} \]

\[ \Rightarrow \text{ quasilinear theory is concerned with describing and understanding the slow evolution of} \]

\[ \langle f \rangle \]...
3. quasilinear theory is mindless mean field theory \( \text{i.e.,} \)

\[
\langle \phi \rangle = \langle \phi (y, t) \rangle \quad \text{where} \quad \langle \rangle \quad \text{eliminates spatial dependence}
\]

\[
\Rightarrow \quad t \quad \text{understood} \quad \text{"slow"}
\]

\[
\frac{df}{dt} + \nabla \cdot \frac{df}{dx} + \frac{2}{m} E \frac{df}{dv} = 0
\]

then \( Q \), equation is simply \( \text{\textsuperscript{\(\text{above} \)}} \)

\[
\frac{d}{dt} \left( \frac{2}{m} \langle E \vec{f} \rangle \right) = 0
\]

\( \text{i.e.,} \) generic mean field equation \( \text{\textsuperscript{\(\text{for} \langle \phi \rangle \)}} \)

\( \) for mean of conserved order parameter \( \langle \phi \rangle \)

\[
\frac{d}{dt} \langle \phi \rangle + \nabla \cdot J_v = 0 \quad \to \quad \text{phase space continuity equation}
\]

\[
J_v = \nabla = \langle \frac{2}{m} E \phi \rangle
\]

elementary closure problem

\[
= \frac{d}{dt} \langle \tilde{E} \phi \rangle \quad \text{i.e., relate} \quad \langle \phi \rangle \quad \text{to} \quad \langle \tilde{E} \phi \rangle \quad \text{hierarchy?}
\]

\[
\text{for:} \quad E = \tilde{E}
\]

\[
\phi = \langle \phi \rangle + \tilde{\phi}
\]

\( \text{\textsuperscript{\(\text{simplest example of moment closure}\)}} \)

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then Q.L.T. simply takes form:

\[ f \rightarrow F_{\text{linear}} \quad (\text{i.e., linear response of } \quad \text{play on linear response of } f) \]

\[ \frac{df}{dt} + \frac{\eta}{m} \frac{df}{dx} + \frac{E}{m} \frac{df}{dv} = 0 \quad \text{v/avov} \]

\[ \Rightarrow -i(\omega - kv) \frac{F}{\omega} = -\frac{2}{m} \frac{E}{\omega} \frac{d\langle f \rangle}{dv} \]

\[ J = -\frac{2}{m^2} \sum_{\omega} \frac{1}{\omega} \frac{1}{\omega - kv} \frac{d\langle f \rangle}{dv} \]

and with \( \omega = \omega(k) \) only \( \Omega \) spectrum of\( \text{degemodes, only) } \quad (\text{i.e., condensation approach to criticality in usual phase transitions}) \)

Q.L.T. equation is:

\[ \frac{d\langle f \rangle}{dt} = \frac{2}{\nu} D \frac{d\langle f \rangle}{dv} \]

\[ D = \frac{2}{m^2} \sum_{\kappa} \left| \frac{\omega}{\omega - kv} \right|^2 \]

i.e. mindless mean field theory...

\[ \frac{\langle f \rangle}{E_n} = \lim_{n \to 0} \frac{\langle f \rangle}{E_n} \quad \text{adv. fields} \]
But, surprisingly: Q-L.T. works quite well.

Key issue: why? 

N.B.: In contrast to critical phenomena, external noise ignored \[ \rightarrow \text{instability driven} \]

@ Some questions to keep in mind: deterministic

1. Why is Q.L. equation a diffusion equation? When is this valid?

2. Nature of irreversibility...

3. Can Q.L. equation be derived from Fokker-Planck theory?

4. Also "irreversibility" related...

5. How does Q.L. equation balance the energy-momentum budgets?

6. When does Q.L. theory fail?

7. Related. 1. What is "Sinfarg criterion" for Q-L-Ti. Can such a criterion be formulated?

8. What is dynamics of quasilinear relaxation?

c.i.e. physics?
Basic Scales / Regime Definition

1. Generally, T, L, T, concerned with
   a) broad spectrum of:
   b) unstable wave

2. For current-driven non-acoustic (QIA)
turbulence:

   ![Diagram of spectrum]

   unstable spectrum → why?

3. Infinite system, k quantized, i.e.
   \[ k_m = \frac{m\pi}{L}, \text{ etc.} \]

   - sec have spectrum of phase velocities
     \[ \omega_m/k_m = \omega(k_m) / k_m = \omega_k \]

   - wave-particle resonance occurs when
     \[ v = \omega_k \text{ m} \]
Then Sin Isaac ⇒

\[ m^2 = \sum_{m} q E_m \cos (k_m x - \omega_m t) \]

\[ \text{n.b.} \]

\[ \text{deterministic} \]

\[ \text{no RPA} \]

and 1 resonance dominant ⇒

\[ m^2 = q E_0 \cos (k_0 x + (v_m x - \omega_0) t) \]

Each resonant velocity defines an

phase space island

\[ V = v_{ph m} \]

\[ AV \sim (2 \phi_0)^{1/2} \]

separatrix

diagram

circulating

\[ \text{QLT is concerned with the case of:} \]

Multiple overlapping resonances ⇒

\[ \text{separatrix's proximity} \]

\[ \text{destruction} \]

⇒

circulating trapping wandering

\[ V = v_{m_0} \]

\[ AV \sim (2 \phi_0)^{1/2} \]

separatrix

overlapping resonances

of overlap

hop

separated resonances

particle can wander stochastically

from resonance to resonance, i.e., hopping

⇒ diffusion in \[ V \]

\[ \text{diffusion with } \text{time} \]

⇒ what is it?
Overlap condition (B.V. Chirikov):
\[
\frac{1}{2} (\Delta V_m + \Delta V_{m+1}) \geq V_{m+1} - V_m
\]

\[\rightarrow\] particle motion stochastic

\[\rightarrow\] fundamental irreversibility \[\rightarrow\] orbit stochasticity (not dissipation, Landau damping \[\rightarrow\] Landau critical phenomenon)

\[\rightarrow\] underpinning of diffusion equation

But, a swindle \[\int_0^\infty \rightarrow \] use of unperturbed orbit in estimate!

\[\text{i.e. } \rightarrow x \rightarrow x + v \cdot t \text{ valid?}\]
Consider: linear unperturbed orbit \[\rightarrow\]

have: \[E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]\]

\[\rightarrow\] particle "sees" instantaneous pattern of electric field from modal superposition

\[\text{i.e. } \sum_k E_k(x,t)\]}
relevant comparison is:

\[ T_L \rightarrow \text{life time of instantaneous pattern} \]
\[ T_b \rightarrow \text{bounce time of particle in pattern} \]

Obviously, 1. \( T_L \ll T_b \rightarrow \text{unperturbed orbit} \)
(pattern changes prior to bouncing) is satisfactory approximation.

2. \( T_L \gg T_b \rightarrow \text{particle bounces prior to pattern changes} \)
so must consider orbit perturbation...

(i.e.,
\[ T = T_L \rightarrow T_b \]

vs.
\[ \text{quasilinear theory relevant to evolution when:} \]

1. \( \rightarrow \text{orbits stochastic (Chirikov condition satisfied)} \)
2. \( T_{\text{Life}} < T_{\text{bounce}} \rightarrow \text{unperturbed orbits valid.} \)
But, how relate \( \tau \) to physical quantities?

**Key point:** Superposition patterns disperse!

\[
E(x,t) = \sum_k E_k e^{i(kx - \omega_k t)}
\]

\[
= \sum_k E_k \exp \left[ i \left( k \frac{x - (v_k/k) t}{k} \right) \right]
\]

\[
\Lambda(\nu/k) = \text{spread in phase velocity} \quad v_p(k)
\]

so dispersion rate is (time)\(^{-1}\) to disperse by one wavelength

\[
\sqrt{\tau} = k \Lambda(\nu/k)
\]

\[
= \left( \frac{d\nu_k}{dk} \frac{\Delta k}{k} - \frac{\nu_k \Delta k}{k^2} \right)
\]

\[
= \left( \frac{d\nu_k}{dk} - \frac{\nu_k}{k} \right) \Delta k = \left( \nu_p(k) - v_p(k) \right) \Delta k
\]

\[
\text{n.b.} \quad \tau \rightarrow \infty \text{ for non-dispersive waves!}
\]

Generally, QLT/weak turbulence encounters trouble for non-dispersive, weakly dispersive waves.
Consider: \[ \langle E(x_1, t_1) E(x_2, t_2) \rangle_{x_1t_1} = C \]

electric field correlation function

\[ C = C(x_2, T) \], for \[ \text{homogeneous stationary fluctuations} \]

\[ x_1 = x_4 + x_2 \quad t_1 = t_4 + t_2 \]
\[ x_2 = x_4 - x_2 \quad t_2 = t_4 - t_2 \]

\[ \langle x_4 \rangle = \langle x_4 \rangle t_4 \]

\[ C(x_2, T) = \langle \sum_{k, k'} F_k F_{k'} e^{-i(k+k')x_2 - \left(k \cdot \omega_k + k' \cdot \omega_{k'}\right) t_4} e^{i(k-k')x_2 - \left(k \cdot \omega_k - k' \cdot \omega_{k'}\right) t_4} \rangle_{x_4 t_4} \]

\[ x_4, t_4 \text{ average} \Rightarrow k = -k', \quad \omega_k = -\omega_{k'} \]

\[ C(x_2, T) = \sum_k |F_k|^2 e^{i\omega_k x_2 - i\omega_k t_4} \]
Now:

- Assume continuous spectrum

- For simplicity, take model

\[ |E_k|^2 = \frac{E_0^2}{[(k - k_0)^2 + \Delta k^2]^{1/2}} \]

- Evaluate on U.P.O.

\[ x = x_0 + \nu \sqrt{T} \]

\[ \langle E^2 \rangle = \int \frac{dk}{[\Delta k^2]^{1/2}} \frac{E_0^2 e^{ikx_0} e^{-ik(k - k_0)T}}{[(k - k_0)^2 + \Delta k^2]^{1/2}} \]

integrating:

phase factor - irrelevant

\[ \sim E_0^2 e^{ikx_0} e^{-\Delta k x_0} \]

\[ e^{(\Delta k \nu - \omega k_0)T} - e^{-\Delta (\Delta k \nu - \omega k_0)T} \]

correlation decay & interplay with resonance

n.b.: note that spread is doppler-shifted if critical
\[A(\kappa V - u_k) = V \Delta \kappa - V_{gn} \Delta \kappa\]

\[V_{gn} = \frac{dw}{d\kappa}\]

\[\langle E^2 \rangle = C(x, y)\]

\[= E_0^2 e^{-k_0 x} e^{i(k_0 V - u_k)T} \frac{1}{|\Delta \kappa x_0|} \times \exp \left\{ i (V - V_{gn}) \Delta \kappa \right\}^2\]

\[\text{set lifetime}\]

\[\frac{1}{\tau_L} = \frac{1}{|V - V_{gn}(k)| \Delta \kappa} \equiv \frac{\text{Autocorrelation Time}}{\text{Time}}\]

\[\text{Note:} \quad \tau_L = \frac{1}{\tau_{ac}}\]

\[\text{for resonant particles,} \quad V = \frac{u_k}{k}\]

\[\frac{1}{\tau_L} = \frac{1}{|V_{ph} - V_{gn}| \Delta \kappa} \rightarrow \text{recoherence earlier!}\]

\[\text{Can think} \quad |V_{ph} \Delta \kappa| \rightarrow \frac{1}{\tau_{ac}} \text{wave-particle}\]

\[|V_{gn} \Delta \kappa| \rightarrow \frac{1}{\tau_{ac}} \text{wave}.\]
generally, shorter time dominated except for non-dispersive waves.

So, can enumerate key time scales:

\[ \tau_a = \frac{1}{\Delta k (v_{ph} - v_{gr})} \]

\( \tau_a \) = persistence of E pattern (\( R^2 \) auto-correlation) for resonant particles.

\( \tau_g = \frac{1}{\Delta k} \) = growth/damping time.

\[ \tau_{Tr} = \frac{1}{k \sqrt{g/m}} \]

\( \tau_{Tr} \) = trapping time.

\[ \tau_{relax} = \frac{1}{\langle \langle \Delta E \rangle \rangle} \]

\( \tau_{relax} \) = avg. distribution relaxation time.

\( \tau_a < \tau_{Tr} \) \( \implies \) u.p.o. valid

\( \tau_a < \tau_{relax} \) \( \implies \) \( \langle E \rangle \) closure meaningful

\( \tau_a < \tau_g < \tau_{relax} \) \( \implies \) QL.T. valid.

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(iii) Energy - Momentum Budgets

A key point: There are two ways of implementing the book-keeping and accounting.

\[\begin{align*}
\text{resonant particles} & \quad \text{vs. 'waves'} \\
\text{particles} & \quad \text{vs. fields}
\end{align*}\]

Keep in mind: Wave = Field + Non-resonant Particle.
(i.e. for plasma oscillation) \( E(\omega) = 1 - \frac{q^2}{\omega^2} \)

Wave Energy = \[W = \frac{2}{\omega} \left(\omega \delta\right) \left| \frac{E}{\delta t} \right|^2 \]

\[= \frac{\omega \delta}{\omega} \left| \frac{E}{\delta t} \right|^2 \]

\[= 2 \left( \frac{\left| E \right|^2}{\delta t} \right) \]

Field - non-resonant particle

(Show!)

...
\[ \frac{\partial \langle f \rangle}{\partial t} = - \frac{3}{\hbar} \frac{\partial}{\partial \nu} \frac{\langle f \rangle}{m} \]

\[ \frac{\partial}{\partial t} \int d\nu \frac{m \nu^2}{2} \langle f \rangle = - \int d\nu \frac{m \nu^3}{2} \frac{\partial}{\partial \nu} \frac{\langle f \rangle}{m} \]

\[ = \int d\nu \frac{m \nu^2}{2} \langle f \rangle \]

Using \( f_k \) for \( f \):

\[ \Sigma_{k, \nu} = - \int d\nu \frac{\nu^2}{m} \sum_L \frac{1}{E_L} \frac{1}{2} \left( \frac{\hbar k}{m} \right) \frac{\partial}{\partial \nu} \frac{\langle f \rangle}{m} \]

\[ = - \int d\nu \frac{\pi^2}{m} \sum_{k, \nu} \frac{1}{E_k} \frac{\partial}{\partial \nu} \frac{\langle f \rangle}{m} \]

\[ = - \frac{\pi^2}{m} \sum_{k, \nu} \frac{\partial}{\partial \nu} \frac{\langle f \rangle}{m} \frac{1}{E_k} \]

As resonant particles stabilize/deshibolize waves, expect resonant particles conserve energy against waves.
wave energy evolution:

Recall: \( E = \frac{1}{4} \mathcal{C} \int d\omega \frac{dE}{d\omega} \)

\( E(\omega_{\text{y}} + i\delta_{\text{y}}) + iE_{\text{IM}} = 0 \)

\( i\delta_{\text{y}} = -\frac{\partial E_{\text{IM}}}{\partial E(\omega_{\text{y}})} \quad \text{and} \quad \frac{\partial E_{\text{IM}}}{\partial E(\omega_{\text{y}})} = -\frac{\partial E}{\partial \omega_{\text{y}}} \)

Now, \( W = \text{Wave Energy Density} \)

\( W = \sum \sum \omega (\omega \mathcal{C}) \frac{|E|}{8\pi} \)

\( = \sum \omega \frac{\partial}{\partial \omega} \frac{|E|}{8\pi} \)

\( \frac{\partial W}{\partial t} = \sum \sum 2 \delta_{\text{y}} \omega_{\text{y}} \frac{\partial E}{\partial \omega} \frac{|E|}{8\pi} \)

\( = \sum 2 (\frac{\partial E_{\text{IM}}}{\partial E(\omega_{\text{y}})}) \omega_{\text{y}} \frac{\partial E_{\text{IM}}}{\partial E(\omega_{\text{y}})} \frac{|E|}{8\pi} \)

\( = \sum -E_{\text{IM}} (k\delta_{\text{y}}) \omega_{\text{y}} \frac{|E|}{8\pi} \)

\( \frac{|E|}{8\pi} \)
\[ \mathcal{E}_{IM} = \frac{\omega^3}{k^2} \frac{\partial^2 \phi}{\partial \nu^2} \text{ (-ii)} \]

\[ (\lambda_0 = 1) \]

\[ \frac{\partial W}{\partial t} = \sum_{m} \sum_{k_{||} I M} \frac{\omega^2}{m^2 \omega_{k_{||} I M}} \frac{\partial^2 \phi}{\partial \nu^2} \frac{\left| E_{k I M} \right|^2}{\omega_{k_{||} I M}} \]

\[ = \pi \rho^2 \sum_{m} \frac{\omega^2}{m^2} \frac{\partial^2 \phi}{\partial \nu^2} \frac{\left| E_{k I M} \right|^2}{\omega_{k_{||} I M}} \]

\[ = \sum_{\text{resonant}} \frac{\partial E_{k I M}}{\partial t} + \frac{\partial W}{\partial t} = 0 \]

**Note:**

- This is essentially a re-write of the Pointing theorem for plasma waves, i.e.

\[ \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{Q} = 0 \]

\[ \nabla \cdot \mathbf{E} \cdot \mathbf{j} \text{ coupling} \]

\[ \nabla \cdot \text{ divergence of wave energy} \]

\[ \text{energy density flux} \]
For homogeneous system: \[ \nabla \cdot \mathbf{S} = 0 \]

\[ \frac{\partial \mathbf{W}}{\partial t} + \int \mathbf{q} \, d\omega = 0 \]

\[ \langle E \cdot J \rangle \text{ mediated by resonant particles (DC field)} \]

\[ \frac{\partial \mathbf{W}}{\partial t} + \int \mathbf{q} \, d\omega (\text{RPKEO}) = 0 \]

Energy Thm I

\[ \text{Waves and resonant particles conserve energy!} \]

? What is the fate of RPKEO for saturate waves. What must happen?

Now can observe:

\[ \mathbf{W} = \mathbf{NRPKEO} + \mathbf{FEO} \]

\[ \int \mathbf{q} \, d\omega (\text{RPKEO}) + \int \mathbf{q} \, d\omega (\text{RPKEO}) = 0 \]

So, simply re-grouping terms:

\[ \frac{\partial \mathbf{W}}{\partial t} + \int \mathbf{q} \, d\omega (\text{RPKEO} + \text{NRPKEO}) = 0 \]

\[ \mathbf{FEO} \]
\[
\frac{\partial F_{\text{ED}}}{} + \frac{\partial}{\partial t} \left( \rho K_{\text{ED}} \right) = 0
\]

\text{Energy Thm.}

Fields and particles conserve energy.

What is the physics of all this?

\[ D = \rho \sum \frac{\varepsilon_0^2}{m^2} |E_k|^2 \left( c/\omega - \text{hv} \right) \]

QL diffusion

For general, weakly non-stationary state...

\[
= \sum \frac{\varepsilon_0^2}{m^2} |E_k|^2 \left( \frac{\text{Im} \omega}{(\omega - \text{hv})^2 + \delta_k^2} \right)
\]

\[
= \sum \frac{\varepsilon_0^2}{m^2} |E_k|^2 \left\{ \frac{\text{Re} \omega}{(\omega - \text{hv})^2 + \delta_k^2} \right\}
\]

\text{Resonant diffusion non-resonant diffusion}

Resonant diffusion \to \text{reversible} - \text{resonance overlap is underpinning}

\[ \to \text{rooted in particle stochasticity} \]
Resonant diffusion can be obtained from Fokker-Planck calculation (show this!)

In principle, can persist in steady state (but how balance energy...?)

Non-Resonant Diffusion:

\[ D^{NR} = \sum_{k} \frac{g^2}{m^2} \frac{1}{u_k^2} \frac{\sigma_k}{u_k} \]

\[ = \frac{4}{\Delta t} \sum_{k} |v_k| \]

where \( |v_k|^2 = \frac{g^2}{m^2} \frac{1}{u_k^2} \)

Corresponds to 'sloshing' motion energy of particles in wave

\[ \text{i.e. } D^{NR} \sim \Delta t \mathcal{E}_{\text{quiver}} \]

Thus reversible can't be obtained from Fokker-Planck theory \( \Rightarrow \text{'fake diffusion'} \)

Vanishes in stationary state
Point is that can counterrepresent diffusion as:

\[ D_{\text{non-resonant}} = \frac{\text{part of wave energy density}}{\text{part of total particle kinetic energy density}} \]

so two forms of energy conservation.

Note: Physically, the picture of plasma as
\[ \text{gas} \quad \sum \text{resonant particles} = \quad \text{waves} \]
\[ \text{resonant particles} + \text{quasi-particles} \]
\[ \text{waves} < \text{WHE, etc.} \]
is appealing and will pervade this course.

N.B.: Direct proof of \( \Delta (\text{PKE} + \text{FEO}) = 0 \)
From QL equation:

\[
\frac{\partial \omega}{\partial t} (\text{PKED}) = -\frac{\sum k}{\hbar} \int dv \frac{\omega^2}{k} \frac{E_n^2}{\hbar^2} \frac{1}{4\pi} \frac{\omega^2}{\omega - \hbar v} \frac{d\xi d\delta}{\omega - \hbar v} dv
\]

\[
\xi (k, \omega) = 1 + \frac{\omega^2}{\hbar} \int dv \frac{d\xi d\delta}{\omega - \hbar v} dv
\]

\[
\frac{\partial \omega}{\partial t} (\text{PKED}) = -c \sum \frac{|E_n|^2}{\hbar} \int dv \frac{\omega^2}{k} \frac{\hbar v}{\omega - \hbar v} \frac{d\xi d\delta}{\omega - \hbar v} dv
\]

Using \( \xi (k, \omega) = 0 \):

\[
\omega_n = \omega_n^0 + \gamma\nu
\]

\[
\frac{\partial}{\partial t} (\text{EED})
\]
(v.) Applications of Quasilinear Theory

$\omega_n = \omega_0 \left(1 + \frac{3}{2} k^2 N_0^2\right)^{1/2}$

Quasilinear Equations:

$\varepsilon(k, \omega_n) = 0 \Rightarrow \psi(k), \phi(k)$ from $\langle \phi \rangle$

$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial v} \left( D \frac{\partial \phi}{\partial v} \right)$

$0 = D^R + D^NR$

$= \sum_n \frac{q^2}{m^2} |E_n|^2 \left\{ \pi \delta(w - \omega_n) + \frac{\delta \omega}{\delta N_n} \right\}$

$\frac{\partial}{\partial t} \left( \frac{|E_n|^2}{8\pi} \right) = 2 \delta N_n \frac{|E_n|^2}{8\pi}$
Observe: resonant diffusion describes dynamics of tail particles.

Non-resonant diffusion describes dynamics of bulk Maxwellian.

Expect: tail flattening with adjustment of core/bulk profile (i.e. effective temperature).

Now first consider resonant particles (i.e. on a bump):

\[
\frac{\partial <f>}{\partial t} = \frac{\partial}{\partial x} \left[ D \frac{\partial <f>}{\partial x} \right]
\]

\[
\text{generalization: } \text{Zeldovich Thm}
\]

Stationarity \( \Rightarrow \)

\[
D \frac{\partial^2 <f>}{\partial x^2} = 0
\]

Now "res" \( \Rightarrow \) some finite interval of phase velocities.
Stationarity $\implies D^R = 0$, i.e. fluctuations decay and damp

$$D^0 \implies \Delta <\phi^2>/\Delta \nu = a_j$$ plateau terms removing growth

N.B.: - In 1D $\rightarrow$ plateau
- can generalize

To resolve:

$$D^R = 8 \pi \frac{\sigma^2}{m^2} \sum_i |E_i|^2 \delta(\omega - k \nu)$$

$$= 16 \pi \frac{\sigma^2}{m^2} \int \frac{dk}{k} \Sigma (k) \delta(\omega - k \nu)$$

$$D^R = \frac{16 \pi \sigma^2}{m^2 \nu} \Sigma \left( \frac{\omega \nu}{\nu} \right)$$

$$\Delta \cdot D^R = \frac{16 \pi \sigma^2}{m^2 \nu} \left( \Delta - \gamma \nu / \nu \right) \Sigma \left( \frac{\omega \nu}{\nu} \right)$$
Now \[ \gamma_{ll} = \frac{-\epsilon_{TM}}{\omega} \frac{\partial \phi}{\partial \theta} \]

\[ \gamma_{ll} = \gamma_{ll}^p = \frac{\pi V^2 \omega_p}{\omega} \frac{\partial \phi}{\partial \theta} \]

\[ \frac{\partial \phi}{\partial \theta} = \frac{16\pi^2 \sigma^2}{m^2 V} \left( 2\pi V^2 \omega_p \frac{\partial \phi}{\partial \theta} \right) \hat{z} \left( \frac{4t}{V} \right) \]

\[ = \left( \frac{\pi V^2 \omega_p}{\omega} \frac{\partial \phi}{\partial \theta} \right) D^R \quad \text{using } D^R \text{ definitions.} \]

\[ D^R (v,t) = D^R (v_0) \exp \left[ \frac{\pi \omega_p V^2}{\omega} \int_{v_0}^{v} \frac{dt}{\omega} \frac{\partial \phi}{\partial \theta} \right] \]

\[ \frac{\partial \phi}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{\partial \phi}{\partial \theta} \]

\[ = \frac{\partial}{\partial t} \frac{\partial \phi}{\partial \theta} \quad \text{using } \gamma_{ll}, D \]

\[ \text{definitions} \]
\[ \langle f(y_t) \rangle - \langle f(y_0) \rangle = \frac{\partial}{\partial y} \left[ \frac{D^R(y_t) - D^R(y_0)}{\pi \nu^2} \right] \]

We have:

\[ D^R = D^R(y_0) \exp \left[ \int_{y_0}^{y_t} \frac{dH}{\nu} \right] \]

\[ \langle f(y_t) \rangle = \langle f(y_0) \rangle + \frac{\partial}{\partial y} \left[ \frac{D^R(y_t) - D^R(y_0)}{\pi \nu^2} \right] \]

Now recall we wish to know if:

i. \( D^R \to 0 \Rightarrow \Delta f(y) \to 0 \) (Fluctuations decay)

ii. \( \Delta f(y) \to 0 \Rightarrow \) finite \( D^R \) distribution plateaus.

Now, if \( D^R \to 0 \),

\[ \langle f(y_t) \rangle = \langle f(y_0) \rangle - \frac{\partial}{\partial y} \left[ \frac{D^R(y_0)}{\pi \nu^2} \right] \]

\[ D^R(0) = \frac{16 \pi^2 \varepsilon^2 \Sigma(U^0, \nu, 0)}{m^2 \nu} \]
Fluctuation energy

\[ \frac{16\pi^2}{m^2} \frac{S(0)}{H^4 U^2} = \frac{2E_F(0)}{(\hbar m)^2/2} \ll 1, \quad \text{as} \quad n \gg n_0 \]

\[ \langle f(y, t) \rangle = \langle f(y, 0) \rangle, \quad \text{to good approx.} \]

but, for resonant velocities,

\[ \rightarrow \text{linear instability} \Rightarrow \frac{df}{dV} > 0 \]

\[ R \rightarrow 0 \Rightarrow \frac{df}{dV} < 0 \]

but have (for \( R \rightarrow 0 \)) \[ \langle f(0, t) \rangle = \langle f(0) \rangle \]

\[ \text{Contradiction follows from assumption of} \quad R \rightarrow 0 \]

\[ \text{have established that} \]

\[ \frac{df}{dV} \rightarrow 0 \Rightarrow \text{plateau forms} \]
For plateau formation, one immediately determines saturation levels from:

\[
\frac{\partial (R\phi^2 E_0)}{\partial t} + \frac{\partial (wE_0)}{\partial t} = 0
\]

\[\text{i.e.} \quad f = 0\]

\[\text{as} \quad t = \infty\]

\[v_1 \quad \quad \quad v_2\]

\[\therefore \quad \Delta \left( \frac{v_2}{2} \mu v^2 \langle f \rangle \right) = - \Delta \int_{k_1}^{k_2} w_k \, dk\]

but \( w_k = 2 \Sigma(k) \)

\[\therefore \quad \Delta \left( \frac{v_2}{2} \mu v^2 \langle f \rangle \right) = - 2 \Delta \int_{k_1}^{k_2} \Sigma(k) \, dk\]
can estimate A (R.P.K.E.D.) analytically via construction

\[ \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \nu} \left( \frac{\partial \rho}{\partial \nu} \right) \]

\[ = \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \left( \frac{1}{m^2} \sum \frac{E_i P_i}{(\omega - \nu P_i)} \frac{\partial \rho}{\partial \nu} \right) \right) \]

\[ = \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \left( \frac{1}{m^2} \int \delta(k) \frac{\partial \rho}{\partial \nu} \right) \right) \]
\[ \frac{\partial \langle E \rangle}{\partial t} = \left( \frac{1}{2m} \int \frac{\partial^2 \langle E \rangle}{\partial \mathbf{k}^2} \right) \frac{\partial^2 \langle E \rangle}{\partial V^2} \]

Now define \( f(t) = \frac{3}{N} \int d\mathbf{k} \langle E \rangle(t) \)

Thus

\[ \frac{\partial \langle E \rangle}{\partial t} = \frac{1}{2m} \frac{\partial^2 \langle E \rangle}{\partial V^2} \]

Thus, for initial Maxwellian:

\[ \langle E \rangle = \left[ \frac{m}{2\pi} \right]^{\frac{1}{2}} \exp \left[ -\frac{mv^2}{T_0 + T_m} \right] \]

Thus, for non-resonant particles

at saturation

\[ T/2 \rightarrow \frac{T}{2} + \frac{4}{n} \int d\mathbf{k} \left[ E(k, \infty) - E(k, 0) \right] \]

i.e., electrons heated by net increase in field energy.
can also note:

\[ \frac{\partial}{\partial t} (R_{PKED}) + \frac{\partial}{\partial t} (WEO) = 0 \]

For plasma waves,

\[ \frac{\partial}{\partial t} (R_{PKED}) = -2 \frac{\partial}{\partial t} (FEO) \]

so

\[ \Delta (R_{PKED}) = -2 \Delta (FEO) \]

but

\[ \Delta (PKED) = -\Delta (FEO) \]

so

\[ \Delta (R_{PKED}) = +2 (\Delta (PKED)) \]

\[ \Rightarrow 0 = \Delta (R_{PKED}) + 2\Delta (NR_{PKED}) \]

and

\[ \Delta (PKED) - \Delta (RP_{PKED}) = -\Delta (FEO) - 2 \Delta (FEO) \]

\[ \Delta (NR_{PKED}) = \Delta (FEO) \]

as shown above.

\[ \frac{\partial}{\partial t} (FEO) \]
heating is one-sided to conserve momentum.