From Reconnection to Relaxation: A Pedagogical Tale of Two Taylors

or: The Physics Assumptions Behind the Color VG

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This talk focuses on:

- what is the connection between local reconnection and global relaxation?

- how do highly localized reconnection processes, for large Rm, Re, produce global self-organization and structure formation?
We attempt to:

- describe both magnetic fields and flows with similar concepts
- connect and relate to talks by H. Ji, D. Hughes, H. Li, O.D. Gurcan...
- describe self-organization principles
Outline

i.) Preamble: From Reconnection to Relaxation and Self-Organization
   → What ‘Self-Organization’ means
   → Why Principles are important
   → Examples of turbulent self-organization
   → Preview

ii.) Focus I: Relaxation in R.F.P. (J.B. Taylor)
   → RFP relaxation, pre-Taylor
   → Taylor Theory  - Summary
      - Physics of helicity constraint + hypothesis
      - Outcome and Shortcomings
   → Dynamics  → Mean Field Theory  - Theoretical Perspective
      - Pinch’s Perspective
      - Some open issues
   → Lessons Learned and Unanswered Questions
iii.) Focus II: PV Transport and Homogenization (G.I. Taylor)

→ Shear Flow Formation by (Flux-Driven) Wave Turbulence

→ PV and its meaning; representative systems

→ Original Idea: G.I. Taylor, Phil. Trans, 1915, ‘Eddy Motion in the Atmosphere’
  - Eddy Viscosity, PV Transport and Flow Formation
  - Application: Rayleigh from PV perspective

→ Relaxation: PV Homogenization (Prandtl, Batchelor, Rhines, Young)
  - Basic Ideas
  - Proof of PV Homogenization
  - Time Scales
  - Relation to Flux Expulsion
  - Relation to Minimum Enstrophy states
Outline

→ Does PV Homogenize in Zonal Flows?
  - Physical model and Ideas
  - PV Transport and Potential Enstrophy Balance
  - Momentum Theorems (Charney-Drazin) and Incomplete Homogenization
  - RMP Effects
  - $B_0$ Effects
  - Lessons Learned and Unanswered Questions

→ Discussion and General Lessons Learned
I.) Preamble

→ From Reconnection to Relaxation

- Usually envision as localized event involving irreversibility, dissipation etc. at a singularity

\[
S.-P. \quad V = V_A / Rm^{1/2}
\]

- ??? - how describe global dynamics of relaxation and self-organization

- multiple, interacting/overlapping reconnection events

→ turbulence, stochastic lines, etc
I.) Preamble, cont’d

→ What does ‘Self-Organization’ mean?
  - context: driven, dissipative, open system
  - turbulence/stochasticity - multiple reconnection states
  - Profile state (resilient, stiff) attractors
  - usually, multiple energy channels possible
  - bifurcations between attractor states possible
  - attractor states macroscopically stable, though may support microturbulence

→ Elements of Theory

  - universality (or claims thereof)

  - coarse graining - i.e., diffusion

  - constraint release - i.e., relaxation of freezing-in law

  - selective decay hypothesis
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<td>$L \rightarrow H$</td>
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<td>nearly marginal $m = 1$ ’s + resistive interchange +...</td>
<td>ITG, CTEM, ... Issue: ELMs?! (domain limited)</td>
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- **Universality:**

Taylor State (Clear)

\[ H_M = \int d^3 x \mathbf{A} \cdot \mathbf{B} \]

only constraint

Magnetic energy dissipated as conserved

\[ H_M \]

Profile Consistency (especially pedestal) (soft)

PV mixed, subject dynamical constraints

Enstrophy (Turbulence) mixed, dissipated, as macroscopic flow emerges
Why Principles?

→ INSIGHT

→ Physical ideas necessary to guide both physical and digital experiments

→ Principles + Reduced Models required to extract and synthesize lessons from case-by-case analysis

→ Principles guide approach to problem reduction
Examples of Self-Organization Principles

→ Turbulent Pipe Flow: (Prandtl → She)

\[ \sigma = -\nu T \frac{\partial \langle v_y \rangle}{\partial x} \quad \nu_T \sim v_* x \quad \Rightarrow \langle v_y \rangle \sim v_* \ln x \]

Streamwise Momentum undergoes scale invariant mixing

→ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc) Minimize \( E_M \) at conserved global \( H_M \) ⇒ Force-Free RFP profiles

→ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

(Focus 2) → PV tends to mix and homogenize
→ Flow structures emergent from selective decay of potential enstrophy relative energy

→ Shakura-Sunyaev Accretion

→ disk accretion enabled by outward viscous angular momentum flux
Preview

- Will show many commonalities - though NOT isomorphism - of magnetic and flow self-organization

- Will attempt to expose numerous assumptions in theories thereof

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II.) Focus I - Magnetic Relaxation

→ Prototype of RFP’s: *Zeta* (UK: late 50’s - early 60’s)

- toroidal pinch = vessel + gas + transformer
- initial results → violent macro-instability, short life time
- weak $B_T$ → stabilized pinch $\leftrightarrow$ sausage instability eliminated
- $I_p > I_{p, crit}$ ($\theta > 1 +$) → access to “Quiescent Period”

→ Properties of Quiescent Period:

- macrostability - reduced fluctuations
- $\tau_E \sim 1 \text{ msec} \quad T_e \sim 150 \text{ eV}$
- $B_T(a) < 0$ → reversal

→ Quiescent Period is origin of RFP
Further Developments

- Fluctuation studies:
  
  \[ \text{turbulence} = \begin{cases} \text{m} = 1 & \text{kink-tearing} \rightarrow \text{tend toward force-free state} \\ \alpha \text{E} & \text{resistive interchange, ...} \end{cases} \]

- Force-Free Bessel Function Model
  
  \[ B_\theta = B_0 J_1(\mu r) \quad B_z = B_0 J_0(\mu r) \]
  \[ \textbf{J} = \alpha \textbf{B} \]

  observed to correlate well with observed B structure

- L. Woltjer (1958): Force-Free Fields at constant \( \alpha \)
  
  \[ \rightarrow \text{follows from minimized } E_M \text{ at conserved } \int d^3x \textbf{A} \cdot \textbf{B} \]

- steady, albeit modest, improvement in RFP performance, operational space

  \[ \rightarrow \text{Needed: Unifying Principle} \]
Theory of Turbulent Relaxation  
(J.B. Taylor, 1974)

→ hypothesize that relaxed state minimizes magnetic energy subject to constant global magnetic helicity

i.e. profiles follow from:

\[ \delta \left[ \int d^3 x \frac{B^2}{8\pi} + \lambda \int d^3 x A \cdot B \right] = 0 \]

\[ \nabla \times B = \mu B ; \quad \frac{J \cdot B}{B^2} = \text{const} \]

Taylor state is:

- force free
- flat/homogenized \( \frac{J ||}{B} \)
- recovers BFM, with reversal for \( \theta = \frac{2I_p}{aB_0} > 1.2 \)
- Works amazingly well
Result:

\[ \theta = \frac{\mu a}{2} = \frac{2I_p}{aB_0} \]

\[ F = \frac{B_{z,\text{wall}}}{\langle B \rangle} \]

and numerous other success stories

→ Questions:
- what is magnetic helicity and what does it mean?
- why only global magnetic helicity as constraint?
- Theory predicts end state → what can be said about dynamics?
- What does the pinch say about dynamics?

→ Central Issue: Origin of Irreversibility
Magnetic helicity - what is it?

- consider two linked, closed flux tubes

Tube 1: Flux $\phi_1$, contour $C_1$

Tube 2: Flux $\phi_2$, contour $C_2$

if consider tube 1:  

$$H_M^1 = \int_{V_1} d^3 x \mathbf{A} \cdot \mathbf{B} = \int_{C_1} dl \int_{A_1} dS \mathbf{A} \cdot \mathbf{B}$$

$$= \int_{C_1} dl_1 \cdot \mathbf{A} \int_{A_1} d\mathbf{a} \cdot \mathbf{B}$$

$$= \phi_1 \int_{C_1} dl_1 \cdot \mathbf{A} = \phi_1 \phi_2$$

similarly for tube 2:  

$$H_M^2 = \phi_1 \phi_2$$

so  

$$H_M = 2\phi_1 \phi_2$$

generally:  

$$H_M = \pm 2n\phi_1 \phi_2$$
- Magnetic helicity measures self-linkage of magnetic configuration

- conserved in ideal MHD - topological invariant

\[ \frac{d}{dt} H_M = -2\eta c \int d^3x \mathbf{J} \cdot \mathbf{B} \]

- consequence of Ohm’s Law structure, only

N.B.

- can attribute a finite helicity to each closed flux tube with non-constant \( q(r) \)

- in ideal MHD \( \rightarrow \infty \) number of tubes in pinch. Can assign infinitesimal tube to each field line

- \( \infty \) number of conserved helicity invariants

\[ \rightarrow \text{Follows from freezing in} \]
Question:

How many magnetic field lines in the universe?

(E. Fermi to M.N. Rosenbluth, oral exam at U. Chicago, late 1940’s...)
Why Global helicity, Only?

- in ideal plasma, helicity conserved for each line, tube

  i.e. \[ \mathbf{J} = \mu(\alpha, \beta) \mathbf{B} \quad \mu(\alpha', \beta') \neq \mu(\alpha, \beta) \]

- Turbulent mixing eradicates identity of individual flux tubes, lines!

  i.e.

- if turbulence s/t field lines stochastic, then ‘1 field line’ fills pinch.

  1 line $\leftrightarrow$ 1 tube $\rightarrow$ only global helicity meaningful.

- in turbulent resistive plasma, reconnection occurs on all scales, but: \[ \tau_R \sim l^\alpha \quad \alpha > 0 \]

  ( $\alpha = 3/2$ for S-P reconnection)

  Thus larger tubes persist longer. Global flux tube most robust

- selective decay: absolute equilibrium stat. mech. suggests possibility of inverse cascade of magnetic helicity (Frisch ’75) $\rightarrow$ large scale helicity most rugged.
Comments and Caveats

→ Taylor’s conjecture that global helicity is most rugged invariant remains a conjecture

→ unproven in any rigorous sense

→ many attempts to expand/supplement the Taylor conjecture have had little lasting impact (apologies to some present....)

→ Most plausible argument for global $H_M$ is stochastization of field lines → forces confinement penalty. No free lunch!

→ Bottom Line:

- Taylor theory, simple and successful

- but, no dynamical insight!
Dynamics I:

- The question of Dynamics brings us to mean field theory (c.f. Moffat ’78 and an infinity of others - see D. Hughes, Thursday Lecture)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$?
  → how relate to relaxation?

- Caveat: - MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

- MFT is often very useful, but often fails miserably

- Structural Approach (Boozer): (plasma frame)

  $\langle E \rangle = \eta \langle J \rangle + \langle S \rangle$

  → something → related to $\langle \tilde{v} \times \tilde{B} \rangle$

$\langle S \rangle$ conserves $H_M$  Note this is ad-hoc, forcing $\langle S \rangle$ to fit the conjecture. Not systematic, in sense of perturbation theory

$\langle S \rangle$ dissipates $E_M$
Now

\[ \partial_t H_M = -2c \eta \int d^3 x \langle J \cdot B \rangle - 2c \int d^3 x \langle S \cdot B \rangle \]

\[ \therefore \langle S \rangle = \frac{B}{B^2} \nabla \cdot \Gamma_H \]

Conservation \( H_M \) \[\rightarrow\] \( \langle S \rangle \sim \nabla \cdot \) (Helicity flux)

\[ \partial_t \int d^3 x \frac{B^2}{8\pi} = - \int d^3 x \left[ \eta J^2 - \Gamma_H \cdot \nabla \frac{\langle J \rangle \cdot B}{B^2} \right] \]

so

\[ \Gamma_H = -\lambda \nabla (J\parallel/B) \quad \text{, to dissipate} \quad E_M \]

\[ \rightarrow \text{simplest form consistent with Taylor hypothesis} \]

\[ \rightarrow \text{turbulent hyper-resistivity} \quad \lambda = \lambda [\langle \tilde{B}^2 \rangle] \quad \text{- can derive from QLT} \]

\[ \rightarrow \text{Relaxed state:} \quad \nabla (J\parallel/B) \rightarrow 0 \quad \text{homogenized current} \quad \rightarrow \text{flux vanishes} \]
Dynamics II: The Pinch’s Perspective

- Boozer model not based on fluctuation structure, dynamics

- Aspects of hyper-resistivity do enter, but so do other effects
  
  → Point: Dominant fluctuations controlling relaxation are m=1 tearing modes resonant in core → global structure

  → Issue: What drives reversal \( B_z \) near boundary?

Approach: QL \( \langle \tilde{v} \times \tilde{B} \rangle \) in MHD exterior - exercise: derive!

\[
\langle \tilde{v} \times \tilde{B} \rangle \approx \sum_k |\gamma_k| \frac{R}{r} (q_{res} - q(r)) \langle B_\theta \rangle \partial_r (|\tilde{\xi}_r|^2)_k
\]

i.e. \( \langle J_\theta \rangle \) driven opposite \( \langle B_\theta \rangle \) → drives/sustains reversal
What of irreversibility - i.e. how is kink-driven reversal ‘locked-in’?

\[ \text{drive } J_{\parallel}/B \text{ flattening, so higher } n\text{'s destabilized by relaxation front} \]

\[ \rightarrow \text{global scattering } \rightarrow \text{propagating reconnection front} \]

\[ m=1, \quad \begin{align*}
  n &\rightarrow m=0, \\
  n+1 &\rightarrow \text{driven current sheet, at } r_{rev}
\end{align*} \]

\[ \sum \text{beat } \begin{align*}
  m=2, \\
  2n+1
\end{align*} \]

(difference beat)

\[ \text{but then } m=1, \quad \begin{align*}
  n+2 &\rightarrow \text{tearing activity, and relaxation} \\
  &\text{region, broadens}
\end{align*} \]

\[ \rightarrow \text{Bottom Line: How Pinch ‘Taylors itself’ remains unclear, in detail} \]
Summary of Magnetic Relaxation

concept: topology

process: stochastization of fields, turbulent reconnection

constraint released: local helicity

players: tearing modes

Mean Field: \(^{\text{EMF}} = \langle \tilde{\vec{v}} \times \tilde{\vec{B}} \rangle\)

Global Constraint: \(\int d^3x \vec{A} \cdot \vec{B}\)

NL: Helicity Density Flux

Outcome: B-Profile

Shortcoming: Rates, confinement \(\rightarrow\) turbulent transport
Focus II: Potential Vorticity Mixing ↔ Iso-vorticity Contour Reconnection

→ Prandtl-Batchelor Theorem and PV Homogenization

→ Self-Organization of Zonal Flows
PV and Its Meaning: Representative Systems

The Fundamentals

- **Kelvin’s Theorem** for rotating system

\[ \omega \rightarrow \omega + 2\Omega \]

relative \hspace{1cm} planetary

\[ \oint \mathbf{v} \cdot d\mathbf{l} = \int d\mathbf{a} \cdot (\omega + 2\Omega) \equiv C \]

\[ \dot{C} = 0 \], to viscosity (vortex reconnection)

- \( Ro = V/(2\Omega L) \ll 1 \) \quad \rightarrow \quad \mathbf{V} \cong -\nabla_\perp p \times \hat{z}/(2\Omega) \]

\rightarrow \text{2D dynamics}

- Displacement on beta plane

\[ \dot{C} = 0 \quad \rightarrow \quad \frac{d}{dt} \omega \cong -\frac{2\Omega}{A} \sin \theta_0 \frac{dA}{dt} \]

\[ = -2\Omega \frac{d\theta}{dt} = -\beta V_y \]

\[ \omega = \nabla^2 \phi, \quad \beta = 2\Omega \sin \theta_0 / R \]
Fundamentals II

- Q.G. equation
  \[ \frac{d}{dt} (\omega + \beta y) = 0 \]
  n.b. topography

- Locally Conserved PV
  \[ q = \omega + \beta y \]
  \[ q = \omega / H + \beta y \]

- Latitudinal displacement \( \rightarrow \) change in relative vorticity

- Linear consequence \( \rightarrow \) **Rossby Wave**

  \[ \omega = -\beta k_x / k^2 \]

  observe:
  \[ v_{g,y} = 2\beta k_x k_y / (k^2)^2 \]
  Rossby wave intimately connected to momentum transport

- Latitudinal PV Flux \( \rightarrow \) circulation
- **Obligatory re: 2D Fluid**

- $\omega$ Fundamental: 
  $$ \partial_t \omega = \nabla \times (V \times \omega) $$

  $$ \frac{d}{dt} \frac{\omega}{\rho} = \frac{\omega}{\rho} \cdot \nabla V \rightarrow \text{Stretching} $$

- **2D** $d\omega/dt = 0$  
  $$ E = \langle v^2 \rangle \quad \text{conserved} $$
  $$ \Omega = \langle \omega^2 \rangle $$

**Inverse energy range** $E(k) \sim k^{-5/3}$

**Forward enstrophy range** $E(k) \sim k^{-3}$

**How?**

$$ \partial_t \langle \Delta k^2 \rangle_E > 0 \quad \text{with} \quad \dot{E} = \dot{\Omega} = 0 $$

$$ \partial_t \langle \Delta k^2 \rangle_E = -\partial_t k_E^2 $$

**Dual cascade**

$$ \left\{ \begin{array}{l}
\partial_t k_E^2 < 0 \\
\partial_t k_\Omega^2 > 0
\end{array} \right. \rightarrow \text{large scale accumulation} $$

$$ \rightarrow \text{flow to small scale dissipation} $$
→ Isn’t this Meeting about Plasma?

→ 2 Simple Models

- a.) Hasegawa-Wakatani (collisional drift inst.)
- b.) Hasegawa-Mima (DW)

\[ \mathbf{V} = \frac{c}{B} \hat{z} \times \nabla \phi + \mathbf{V}_{pol} \]
\[ \sim (\omega/\Omega) \]

\[ L > \lambda_D \rightarrow \nabla \cdot \mathbf{J} = 0 \rightarrow \nabla_\perp \cdot \mathbf{J}_\perp = -\nabla_\parallel \mathbf{J}_\parallel \]

\[ J_\perp = n|e|V^{(i)}_{pol} \]

\[ J_\parallel : \eta J_\parallel = -(1/c) \partial_t A_\parallel - \nabla_\parallel \phi + \nabla_\parallel p_e \]

\[ \frac{dn_e}{dt} = 0 \]

\[ \rightarrow \frac{dn_e}{dt} + \frac{\nabla_\parallel \mathbf{J}_\parallel}{-n_0|e|} = 0 \]

n.b.

**MHD:** \[ \partial_t A_\parallel \text{ v.s. } \nabla_\parallel \phi \]

**DW:** \[ \nabla_\parallel p_e \text{ v.s. } \nabla_\parallel \phi \]
So H-W

\[ \rho_s^2 \frac{d}{dt} \nabla^2 \hat{\phi} = -D_\parallel \nabla_\parallel^2 (\hat{\phi} - \hat{n}/n_0) + \nu \nabla^2 \nabla^2 \hat{\phi} \]

\[ \frac{d}{dt} n - D_0 \nabla^2 \hat{n} = -D_\parallel \nabla_\parallel^2 (\hat{\phi} - \hat{n}/n_0) \]

\[ D_\parallel k_\parallel^2 / \omega \]

is key parameter

n.b. \quad PV = n - \rho_s^2 \nabla^2 \phi \quad \frac{d}{dt} (PV) = 0

\rightarrow \text{total density}

b.) \quad D_\parallel k_\parallel^2 / \omega \gg 1 \rightarrow \hat{n}/n_0 \sim e^\hat{\phi}/T_e \quad (m, n \neq 0)

\[ \frac{d}{dt} (\phi - \rho_s^2 \nabla^2 \phi) + v_* \partial_y \phi = 0 \quad \rightarrow \text{H-M} \]

n.b. \quad PV = \phi - \rho_s^2 \nabla^2 \phi + \ln n_0(x)
An infinity of models follow:

- MHD: ideal ballooning resistive → RBM

- HW + $A_{||}$: drift - Alfven

- HW + curv.: drift - RBM

- HM + curv. + Ti: Fluid ITG

- gyro-fluids

- GK

N.B.: Most Key advances appeared in consideration of simplest possible models
Homogenization Theory (Prandtl, Batchelor, Rhines, Young)

\[ \partial_t q + \nabla \phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q \]

Now: \( t \rightarrow \infty \) \quad \partial_t q \rightarrow 0

For \( \nu = 0 \) \quad q = q(\phi)

\[ \rightarrow q = q(\phi) \quad \text{is arbitrary solution} \]

\[ \rightarrow \text{can develop arbitrary fine scale} \quad q = q(\phi) \]

\[ \rightarrow \text{closed stream lines,} \quad \nu = 0 \]

\[ \rightarrow \text{no irreversibility} \]
Now \( \nu \neq 0 \)

\[ q(\phi) \to \text{const} \quad t \to \infty \quad \text{small} \ \nu \to \text{global behavior} \]

- non-diffusive stretching produces arbitrary fine scale structure
- for small, but finite \( \nu \), instead of fine scale structure, must have:

\[ \nu \rightarrow \text{finite} \quad \text{at large} \quad \text{PV homogenization} \]

i.e. finite \( \nu \) at large \( Re \to \text{PV homogenization} \)

analogy in MHD? \( \rightarrow \) Flux Expulsion
Prandtl - Batchelor Theorem:

Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline $C_0$. Then, if diffusive dissipation, i.e. 
$$\partial_t q + \nabla \phi \times \hat{z} \cdot \nabla q = \nu \nabla^2 q$$
then vorticity $\rightarrow$ uniform (homogenization), as $t \rightarrow \infty$ within $C_0$

→ underpins notion of PV mixing → basic trend

→ fundamental to selective decay to minimum enstrophy state in 2D fluids (analogue of Taylor hypothesis)
Proof:

\[ \int_{A_n} \nabla \cdot (v q) = 0 \quad \text{(closed streamlines)} \]

\[ 0 = \int_{A_n} \nabla \cdot (\nu \nabla q) \]

\[ = \nu \int_{C_n} dl \hat{n} \cdot \nabla q \quad \text{(form of dissipation relevant!)} \]

For \( q = q(\phi) \)

\[ 0 = \nu \int_{C_n} dl \hat{n} \cdot \nabla \phi_n \frac{\delta q}{\delta \phi_n} \]

\[ = \nu \frac{\delta q}{\delta \phi_n} \int_{C_n} dl \hat{n} \cdot \nabla \phi_n \]

\[ \therefore 0 = \nu \frac{\delta q}{\delta \phi_n} \Gamma_n \]

\[ \therefore \frac{\delta q}{\delta \phi_n} = 0 \quad \rightarrow \text{q homogenized, within } C_0 \]

\[ \quad \rightarrow \text{q' tends to flatten!} \]
How long to homogenize? ↔ What are the time scales?

Key: synergism between shear and diffusion

\[
\frac{1}{\tau_{mix}} \sim \frac{1}{\tau_c (Re)^{-1/3}}
\]

\[\tau_c \equiv \text{circulation time}\]

PV homogenization occurs on hybrid decorrelation rate

but \[\tau_{mix} \ll \tau_D \quad \text{for} \quad Re \gg 1\] \[\rightarrow\] time to homogenize is finite

Point of the theorem is global impact of small dissipation - akin Taylor
PV Transport and Potential Enstrophy Balance → Zonal Flow

Preamble

• Zonal Flows Ubiquitous for:
  ~ 2D fluids / plasmas \( R_0 < 1 \)
  Rotation \( \tilde{\Omega} \), Magnetization \( \tilde{B}_0 \), Stratification
  Ex: MFE devices, giant planets, stars...
Preamble II

• What is a Zonal Flow?
  – $n = 0$ potential mode; $m = 0$ (ZFZF), with possible sideband (GAM)
  – toroidally, poloidally symmetric $ExB$ shear flow

• Why are Z.F.’s important?
  – Zonal flows are secondary (nonlinearly driven):
    • modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. ‘78)
    • modes of minimal damping (Rosenbluth, Hinton ‘98)
    • drive zero transport ($n = 0$)
  – natural predators to feed off and retain energy released by gradient-driven microturbulence
Heuristics of Zonal Flows a):
Simplest Possible Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow (c.f. Vallis '07, Held '01)
- Key Physics:

\[
\begin{align*}
\omega_k &= -\frac{\beta k_x}{k^2} \\
\nu_{gy} &= 2\beta \frac{k_x k_y}{k^2} \\
\therefore \nu_{gy} \nu_{phy} &< 0
\end{align*}
\]

\[
\langle \tilde{v}_y \tilde{v}_x \rangle = \sum_k -k_x k_y |\hat{\phi}_k|^2
\]

→ Backward wave!

⇒ Momentum convergence at stirring location
... “the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region.” (I. Held, ’01)

- Outgoing waves ⇒ incoming wave momentum flux

- Local Flow Direction (northern hemisphere):
  - eastward in source region
  - westward in sink region
  - set by $\beta > 0$

- Some similarity to spinodal decomposition phenomena
  → both ‘negative diffusion’ phenomena
Key Point: Finite Flow Structure requires *separation* of excitation and dissipation regions.

\[ \text{equator} \quad \text{mid- latitude} \quad \text{pole} \]

\[ \text{dissipation} \quad \text{stirring} \quad \text{dissipation} \]

\[ \rightarrow \text{momentum transport by waves} \]
Key Elements:

- Waves $\rightarrow$ propagation transports momentum $\leftrightarrow$ stresses
  $\rightarrow$ modest-weak turbulence

- vorticity transport $\rightarrow$ momentum transport $\rightarrow$ Reynolds force
  $\rightarrow$ the Taylor Identity

- Irreversibility $\rightarrow$ outgoing wave boundary conditions

- symmetry breaking $\rightarrow$ direction, boundary condition
  $\rightarrow\beta$

- Separation of forcing, damping regions
  $\rightarrow$ need damping region broads than source region
  $\rightarrow$ akin intensity profile...

All have obvious MFE counterparts...
2) MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure

- outgoing wave energy flux → incoming wave momentum flux → counter flow spin-up!

- zonal flow layers form at excitation regions

\[
\begin{align*}
\text{Heuristics of Zonal Flows b.)} & \\
\text{\textbullet} \quad \text{couple to damping } \leftrightarrow \text{ outgoing wave} & \\
& \text{i.e. Pearlstein-Berk eigenfunction} \\
& x > 0 \quad \Rightarrow \quad v_{gr} > 0 \\
& v_{*} < 0 \quad \Rightarrow \quad k_{r}k_{\theta} > 0 \\
& \langle v_{rE}v_{0E} \rangle = -\frac{c^{2}}{B^{2}} |\Phi_{k}|^{2} k_{r}k_{\theta} < 0 \\
& x=0 \quad \text{radial structure}
\end{align*}
\]
So, if spectral intensity gradient $\rightarrow$ net shear flow $\rightarrow$ mean shear formation

$$S_r = v_{gr} \varepsilon \equiv -\frac{2 k_r k_\theta V_t \rho_*^2}{(1 + k_\perp^2 \rho^2)} \varepsilon$$

$$\langle \tilde{v}_r \tilde{v}_\theta \rangle \approx \sum_k -k_r k_\theta \left| \phi_k \right|^2$$

Reynolds stress proportional radial wave energy flux $\vec{S}$, mode propagation physics (Diamond, Kim ‘91)

Equivalently: $\partial_t E + \nabla \cdot S + (\omega \text{Im} \omega) E = 0$ (Wave Energy Theorem)

$\therefore$ Wave dissipation coupling sets Reynolds force at stationarity

Interplay of drift wave and ZF drive originates in mode dielectric

Generic mechanism…
Towards Calculating Something: Revisiting Rayleigh from PV Perspective

- G.I. Taylor’s take on Rayleigh criterion

  - consider effect on (zonal) flow by displacement of PV: $\delta y$

    $$\frac{\partial}{\partial t} \langle v_x \rangle = \langle \tilde{v}_y \tilde{q} \rangle$$

    $$\tilde{q} = (\text{PV of vorticity blob at } y) - (\text{mean PV at } y)$$

    $$\langle q(y) \rangle = \langle q(y_0) \rangle + (y - y_0) \left. \frac{d \langle q \rangle}{dy} \right|_{y_0}$$

    Small displacement

    $$\therefore \frac{\partial}{\partial t} \langle v_x \rangle = -\langle \tilde{v}_y \delta y \rangle \frac{d \langle q \rangle}{dy} = -\left( \partial_t \frac{\langle \tilde{\varepsilon}^2 \rangle}{2} \frac{d \langle q \rangle}{dy} \right)$$

    Flow driven by PV Flux
So, for instability

\[ \partial_t \langle \tilde{E}^2 \rangle > 0 \quad ; \text{growing displacement} \]
\[ \frac{\partial}{\partial t} \int_{-a}^{a} dy \langle v_x \rangle = 0 \quad ; \text{momentum conservation} \]

\[-\int_{-a}^{a} dy \left( \partial_t \frac{\langle \tilde{E}^2 \rangle}{2} \right) \frac{d\langle q \rangle}{dy} = 0 \]

\[
\frac{d\langle q \rangle}{dy} \quad \text{must change sign within flow interval}
\Rightarrow \text{inflection point}
\]

also,

\[
\frac{\partial}{\partial t} \left\{ \langle v_x \rangle + \frac{\langle \tilde{E}^2 \rangle}{2} \frac{d\langle q \rangle}{dy} \right\} = 0
\]

\[
\tilde{q} = -\tilde{\varepsilon} \frac{d\langle q \rangle}{dy}
\]

\[
\frac{\partial}{\partial t} \left\{ \langle v_x \rangle - \left( -\frac{\langle \tilde{q}^2 \rangle}{2\partial\langle q \rangle / \partial y} \right) \right\} = 0
\]

\[-\langle \tilde{q}^2 \rangle / 2\partial\langle q \rangle / \partial y \equiv \text{Pseudomomentum for QG system} \]

\[ \rightarrow \text{no slip condition of flow + quasi-particle gas} \]

\[ \rightarrow (\text{significant}) \text{ step toward momentum theorem} \]

i.e. ties flow to wave momentum density
Zonal Flows I

- Fundamental Idea:
  - Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  - Kelvin’s theorem is ultimate foundation

- G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  - Polarization charge
    \[ \rho^2 \nabla^2 \phi = n_{i,GC}(\phi) - n_e(\phi) \]
    polarization length scale
  - so \( \Gamma_{i,GC} \neq \Gamma_e \)
    \[ \rho^2 \langle \dot{v}_{rE} \nabla^2 \tilde{\phi} \rangle \neq 0 \]
    ‘PV transport’
  - polarization flux    → What sets cross-phase?
  - If 1 direction of symmetry (or near symmetry):
    \[ -\rho^2 \langle \dot{v}_{rE} \nabla^2 \tilde{\phi} \rangle = -\partial_r \langle \dot{v}_{rE} \tilde{v}_{\perp E} \rangle \]  (Taylor, 1915)
    \[ -\partial_r \langle \dot{v}_{rE} \tilde{v}_{\perp E} \rangle \]
    Reynolds force    → Flow
Notable by Absence: Three “Usual Suspects”

- “Inverse Cascade”
  - Wave mechanism is essentially linear
    → scale separation often dubious
  - PV transport is sufficient / fundamental

- “Rhines Mechanism”
  - requires very broad dynamic range
  - Waves ↔ $k_R$ ↔ forced strong turbulence
  - strong turbulence model

- “Modulational Instability”
  - coherent, quasi-coherent wave process
  - useful concept, but not fundamental

→ see P.D. et al. PPCF’05, CUP’10 for detailed discussion

Lesson: Formation of zonal bands is generic to the response of a rapidly rotating fluid to any localized perturbation
Inverse Cascade/Rhines Mechanism

\[ k \sim -\beta k_x/k^2 \]

\[ 1/\tau_k \]

transfer \(\leftrightarrow\) triad couplings

\[
\begin{align*}
 k & \quad k' \\
 k'' &
\end{align*}
\]

eddy transfer: \( \omega_{MM} < 1/\tau_c \)

wave transfer: \( \omega_{MM} > 1/\tau_c \)

cross over: \( \omega_{MM} \sim 1/\tau_c \)

\[ \Rightarrow \quad \text{Rhines Scale} - \text{emergent characteristic scale for ZF} \]

\[ l_R \sim (\bar{v}/\beta)^{1/2} \sim \epsilon^{1/5}/\beta^{3/5} \]

Contrast: Rhines mechanism vs critical balance

Rhines Scale

Inverse energy range

forward enstrophy range

“Waves + ZF”

triads: 2 waves + ZF

The crux:

- 3 wave resonance requires 1 wave with \( k_x = 0 \)

- ZF's appear at \( k_R \)

- coupling maximal at \( k_R \)

\[ \Rightarrow \quad k_R \quad \text{Z.F. dominates} \]
→ **Caveat Emptor:**

- often said ‘Zonal Flow Formation ⪅ Inverse Cascade’

**but**

- anisotropy crucial → $\langle \tilde{V}^2 \rangle$, $\beta$, forcing → ZF scale

- numerous instances with: 
  
  no inverse inertial range
  
  ZF formation ↔ quasi-coherent

all really needed:

$$\langle \tilde{V}_y \tilde{q} \rangle \rightarrow \text{PV Flux} \rightarrow \langle \tilde{V}_y \tilde{V}_x \rangle \rightarrow \text{Flow}$$

→ transport and mixing of PV are fundamental elements of dynamics
Zonal Flows II

- Potential vorticity transport and momentum balance
  - Example: Simplest interesting system → Hasegawa-Wakatani
    - Vorticity: \( \frac{d}{dt} \nabla^2 \phi = -D_\parallel \nabla_\parallel^2 (\phi - n) + D_0 \nabla^2 \nabla^2 \phi \)
    - Density: \( \frac{dn}{dt} = -D_\parallel \nabla_\parallel^2 (\phi - n) + D_0 \nabla^2 n \)
  - Locally advected PV: \( q = n - \nabla \phi^2 \)
    - PV: charge density \( n \rightarrow \) guiding centers
    - \(-\nabla \phi^2 \rightarrow\) polarization
    - conserved on trajectories in inviscid theory \( dq/dt = 0 \)
    - PV conservation → Freezing-in law
      Kelvin’s theorem \( \rightarrow\) Dynamical constraint

\[ D_0 \quad \text{classical, feeble} \]
\[ \text{Pr} = 1 \quad \text{for simplicity} \]
Zonal Flow II, cont’d

- Potential Enstrophy (P.E.) balance
  \[ \frac{d}{dt} \langle \tilde{q}^2 \rangle = \text{flux} - \text{dissipation} \]
  \[ \langle \tilde{q}^2 \rangle \rightarrow \text{coarse graining} \]
  \[ \text{LHS} \Rightarrow \frac{d}{dt} \langle \tilde{q}^2 \rangle = \partial_t \langle \tilde{q}^2 \rangle + \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle \]
  \[ \text{RHS} \Rightarrow \text{P.E. evolution} - \langle \tilde{V}_r \tilde{q} \rangle = \text{P.E. Production by PV mixing / flux} \]

- PV flux:
  \[ \langle \tilde{V}_r \tilde{q} \rangle = \langle \tilde{V}_r \tilde{n} \rangle - \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle ; \text{ but } \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{V}_r \tilde{\phi} \rangle \]
  \[ \therefore \text{P.E. production directly couples driving transport and flow drive} \]

- Fundamental Stationarity Relation for Vorticity flux
  \[ \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = \langle \tilde{V}_r \tilde{n} \rangle + (\partial_t \langle \tilde{q}^2 \rangle) / \langle q \rangle' \]
  \[ \text{Reynolds force} \quad \text{Relaxation} \quad \text{Local PE decrement} \]
  \[ \therefore \text{Reynolds force locked to driving flux and P.E. decrement; transcends quasilinear theory} \]
Contrast: Implications of PV Freezing-in Law

\[
\begin{align*}
\frac{dn}{dt} &= 0 \quad (?) \\
\frac{d\langle n \rangle}{dr} &= 0 \\
\tilde{n} \text{ grows} &\rightarrow \langle \tilde{V}_r \tilde{n} \rangle \rightarrow :-( \\
\end{align*}
\]

\[
\begin{align*}
\frac{dq}{dt} &= 0 \quad (!) \\
\frac{d\langle q \rangle}{dr} &= 0 \\
\tilde{q} \text{ grows} &\rightarrow \left\{ \begin{array}{l}
\langle \tilde{V}_r \tilde{n} \rangle \rightarrow \text{transport} \rightarrow :-( \\
\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle \rightarrow \text{flow} \rightarrow :-) 
\end{array} \right. \\
\end{align*}
\]

Lesson: Even if \( \langle q \rangle \cong \langle n \rangle \), PV conservation must channel free energy into zonal flows!

Key Question: Branching ratio of energy coupled to flow vs transport-inducing fluctuations?
Combine: \[
\partial_t \langle V_\theta \rangle = -\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle - \nu \langle V_\theta \rangle
\]
yields...

Charney-Drazin Momentum Theorem
(1960, et.seq., P.D., et.al. ’08, for HW)

Pseudomomentum  local P.E. decrement
\[
\Rightarrow \quad \partial_t \{ \langle \text{WAD} \rangle + \langle V_\theta \rangle \} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle
\]
driving flux  drag

WAD = Wave Activity Density, \( \langle \tilde{q}^2 \rangle / \langle q \rangle' \)

- pseudomomentum in \( \theta \)-direction (Andrews, McIntyre ’78)
- Generalized Wave Momentum Density

i) momentum of quasi-particle gas of waves, turbulence
ii) consequence of azimuthal/poloidal symmetry
iii) not restricted to linear response, but reduces correctly
What Does it Mean? → “Non-Acceleration Theorem”:

\[ \partial_t \{ (WAD) + \langle V_\theta \rangle \} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle \]

absent

\[ \begin{align*}
\langle \tilde{V}_r \tilde{n} \rangle, & \text{ driving flux} \\
\delta_t \langle \tilde{q}^2 \rangle, & \text{ local potential enstrophy decrement}
\end{align*} \]

cannot

\[ \begin{align*}
\text{accelerate} \\
\text{maintain}
\end{align*} \]

Z.F. with stationary fluctuations!

Essential physics is PV conservation and translational invariance in \( \theta \) → freezing quasi-particle gas momentum into flow → relative “slippage” required for zonal flow growth

obvious constraint on models of stationary zonal flows!

\[ \leftrightarrow \text{ need explicit connection to relaxation, dissipation} \]

N.B. Inhomogeneous dissipation → incomplete homogenization!!?
Aside: H-M

C-D Theorem for HM

\[ \partial_t \{ \text{WAD} + \langle V_\theta \rangle \} = \frac{\langle \tilde{f}^2 \rangle^c}{\langle q \rangle'} \frac{1}{\langle q \rangle'} \left\{ \partial_r \langle \tilde{V}_r \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} - \nu \langle V_\theta \rangle \]

C-D prediction for \( \langle V_\theta \rangle \) at stationary state, HM model

\[ \langle V_\theta \rangle = \frac{1}{\nu \langle q \rangle'} \left\{ \langle \tilde{f}^2 \rangle^c \partial_r \langle \tilde{V}_r \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} \]

→ Note: Flow direction set by: \( \langle q \rangle' \), source, sink distribution

→ Forcing, damping profiles determine shear

→ Potential Enstrophy Transport impact flow structure
In More Depth: What Really Determines Zonal Flow?

- driving flux: $\langle \tilde{V}_r \tilde{n} \rangle = \Gamma_0 - \Gamma_{\text{col}} = \int dr' S_n(r') - \Gamma_{\text{col}}$
  - Total flux $\Gamma_0$ fixed by sources, $S_n \rightarrow$ flux driven system
  - Collisional flux in turbulent system, $\Gamma_{\text{col}}$ (computed with actual profiles)

$\Gamma_0 - \Gamma_{\text{col}} \rightarrow$ available flux

- P.E. decrement: $\delta_t \langle \tilde{q}^2 \rangle = \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle$
  $\rightarrow$ change in roton intensity (P.E) changes flow profile
  - roton dissipation
  - P.E. flux, direction increment, according to convergence ($> 0$) or divergence ($< 0$) of pseudomomentum, locally

So: P.E. transport and “spreading” intrinsically linked to flow structure, dynamics

Net $\delta(\text{P.E.})$ can generate net spin-up

$\therefore$ Zonal flow dynamics intrinsically “non-local” $\leftrightarrow$ couple to turbulence spreading (fast, meso-scale process)
Clarifying the Enigma of Collisionless Zonal Flow Saturation

- Flow evolution with: $\nu \to 0$, $S_n \neq 0$ and nearly stationary turbulence

$$\partial_t \langle V_\theta \rangle = - \left( \int dr' S_n(r') - \Gamma_{\text{col}} \right) - \left( \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle \right) / \langle q \rangle'$$

Possible Outcomes:

- $\langle q \rangle' \to 0$, locally $\to$ shear flow instability (the usual)
  $\leftrightarrow$ limit cycle of burst and recovery, effective viscosity?
  $\to$ problematic with magnetic shear

- $\langle \tilde{V}_r \tilde{n} \rangle$ v.s. $\partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle$ $\to$ potential enstrophy transport and inhomogeneous turbulence, with $\tilde{n}/n \sim M.L.T$
  $\to$ flux drive vs. roton population flux
  $\to$ novel saturation mechanism

- $\langle q \rangle' \to 0$, globally $\to$ homogenized PV state (Rhines, Young, Prandtl, Batchelor)
  $\to$ decouples mean PV, PE evolution

- homogeneous marginality, i.e. $\int dr' S_n(r') = \Gamma_{\text{col}}$ $\leftrightarrow$ ala’ stiff core

N.B.: $\langle q \rangle' = 0 \Rightarrow \partial_r \langle n \rangle = \partial_r^2 \langle V_E \rangle = \partial_r \langle \omega_E \rangle$ $\to$ particular profile relation!
Summary of Flow Organization

concept: symmetry

process: PV mixing, transport

constraint released: Enstrophy conservation

players: drift waves

Mean Field: $\Gamma_{PV} = \langle \tilde{v}_r \tilde{q} \rangle$

Global Constraint: Bounding circulation

NL: Pseudomomentum Flux

Outcome: Zonal Flow Formation

Shortcoming: ZF pattern structure and collisionless saturation
Summary of comparison

- Many commonalities between magnetic and flow relaxation apparent.

- Common weak point is limitation of mean field theory
  \(\rightarrow\) difficult to grapple with strong NL, non-Gaussian fluctuations.

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Heuristics of Zonal Flows c.)

• One More Way:
  • Consider:
    – Radially propagating wave packet
    – Adiabatic shearing field
  • Wave action density $N_k = \frac{E(k)}{\omega_k}$ adiabatic invariant
    $\therefore E(k) \downarrow \Rightarrow$ flow energy decreases, due Reynolds work $\Rightarrow$
    flows amplified (cf. energy conservation)
  • $\Rightarrow$ Further evidence for universality of zonal flow formation
Heuristics of Zonal Flows d.) cont’d

• Implications:
  – ZF’s generic to drift wave turbulence in any configuration: electrons tied to flux surfaces, ions not
    • g.c. flux \(\rightarrow\) polarization flux
    • zonal flow
  – Critical parameters
    • ZF screening (Rosenbluth, Hinton ‘98)
    • polarization length
    • cross phase \(\rightarrow\) PV mixing

• Observe:
  – can enhance \(e\varphi_{ZF}/T\) at fixed Reynolds drive by reducing shielding, \(\rho^2\)
  – typically:
    \[
    \frac{\varepsilon}{\varepsilon_0} \approx 1 + \rho_i^2 / \lambda_D^2 + f_b \rho_b^2 / \lambda_D^2 + f_d \delta_d^2 / \lambda_D^2
    \]
    \(\varepsilon/\varepsilon_0\) total screening response
    \(\rho_b^2/\lambda_D^2\) banana width
    \(\delta_d^2/\lambda_D^2\) banana tip excursion
  – Leverage (Watanabe, Sugama) \(\rightarrow\) flexibility of stellerator configuration
    • Multiple populations of trapped particles
    • \(\langle E_r \rangle\) dependence (FEC 2010)
Heuristics of Zonal Flows d.) cont’d

• Yet more: \( \frac{\partial}{\partial t} \langle v_\perp \rangle = -\partial_r \langle \tilde{v}_r \tilde{v}_\perp \rangle - \gamma_d \langle v_\perp \rangle + \mu \nabla_r^2 \langle v_\perp \rangle \)

• Reynolds force opposed by flow damping

• Damping:
  – Tokamak \( \gamma_d \sim \gamma_{ii} \)
    • trapped, untrapped friction
    • no Landau damping of \((0, 0)\)
  – Stellerator/3D \( \gamma_d \leftrightarrow NTV \)
    • damping tied to non-ambipolarity, also
    • largely unexplored
  – RMP
    • zonal density, potential coupled by RMP field
    • novel damping and structure of feedback loop

• Weak collisionality → nonlinear damping – problematic
  → tertiary → ‘KH’ of zonal flow →
  magnetic shear!?
  → other mechanisms?
4) GAMs Happen

- Zonal flows come in 2 flavors/frequencies:
  - $\omega = 0 \Rightarrow$ flow shear layer
  - GAM $\omega^2 \equiv 2c_s^2 / R^2 (1 + k_r^2 \rho_0^2) \Rightarrow$ frequency drops toward edge $\Rightarrow$ stronger shear
    - radial acoustic oscillation
    - couples flow shear layer (0,0) to (1,0) pressure perturbation
    - $R \equiv$ geodesic curvature (configuration)
    - Propagates radially

- GAMs damped by Landau resonance and collisions
  $$\gamma_d \sim \exp[-\omega_{GAM}^2 / (\nu_{thi} / Rq)^2]$$
  - $q$ dependence!
  - edge

- Caveat Emptor: GAMs easier to detect $\Rightarrow$ looking under lamp post ?!
Progress I: ZF’s with RMP (with M. Leconte)

- ITER ‘crisis du jour’: ELM Mitigation and Control
- Popular approach: RMP
- ? Impact on Confinement?

Y. Xu ‘11

$\Rightarrow$ RMP causes drop in fluctuation LRC, suggesting reduced Z.F. shearing
$\Rightarrow$ What is “cost-benefit ratio” of RMP?

- Physics:
  - in simple H-W model, polarization charge in zonal annulus evolves according:
    $$\frac{dQ}{dt} = -\int dA \left[ \bar{\nu}_x \bar{\rho}_{pol} \right] + \left( \frac{\delta B_r}{B_0} \right)^2 D_\parallel \frac{\partial}{\partial x} \left( \langle \phi \rangle - \langle n \rangle \right) \right]$$
  - Key point: $\delta B_r$ of RMP induces radial electron current $\rightarrow$ enters charge balance
Progress I, cont’d

- Implications
  - $\delta B_r$ linearly couples zonal $\phi$ and zonal $\hat{n}$
  - Weak RMP → correction, strong RMP $\to \langle E_r \rangle_{ZF} \approx -T_e \partial_r \langle n \rangle / |e|$

- Equations:
  \[
  \frac{d}{dt} \delta n_q + D_T q^2 \delta n_q + ib_q (\delta \phi_q - (1-c)\delta n_q) - D_{RMP} q^2 (\delta \phi_q - \delta n_q) = 0
  \]
  \[
  \frac{d}{dt} \delta \phi_q + \mu \delta \phi_q - a_q (\delta \phi_q - (1-c)\delta n_q) + \frac{D_{RMP}}{\rho_s^2} (\delta \phi_q - \delta n_q) = 0
  \]

- Results:
  \[
  \gamma > \gamma_c(\mu_{\delta B}) \quad \mu_{\delta B} > 0
  \]

Transitions in presence of RMP

$E_{ZF}/\mathcal{E}_L$ vs $\mathcal{E}/\mathcal{E}_L$ for various RMP coupling strengths
Progress II: $\beta$-plane MHD (with S.M. Tobias, D.W. Hughes)

Model

- Thin layer of shallow magneto fluid, i.e. solar tachocline
- $\beta$-plane MHD $\sim$ 2D MHD + $\beta$-offset  i.e. solar tachocline

\[
\partial_t \nabla^2 \phi + \nabla \times \hat{z} \cdot \nabla \nabla^2 \phi - \nu \nabla^2 \nabla^2 \phi = \beta \partial_x \phi + B_0 \partial_x \nabla^2 A + \nabla A \times \hat{z} \cdot \nabla \nabla^2 A + \tilde{f}
\]

\[
\partial_t A + \nabla \phi \times \hat{z} \cdot \nabla A = B_0 \partial_x \phi + \eta \nabla^2 A \quad \tilde{B}_0 = B_0 \hat{x}
\]

- Linear waves: Rossby – Alfven  $\omega^2 + \omega \beta \frac{k_x}{k^2} - k_x^2 V_A^2 = 0$  (R. Hide)
Progress II, cont’d

Observation re: What happens?

- Turbulence → stretch field → $\langle \vec{B}^2 \rangle \gg B_0^2$ i.e. $\langle \vec{B}^2 \rangle / B_0^2 \sim R_m$
  (ala Zeldovich)

- Cascades: - forward or inverse?
  - MHD or Rossby dynamics dominant !?

- PV transport: $\frac{dQ}{dt} = - \int dA \langle \vec{v} \vec{q} \rangle \rightarrow \text{net change in charge content}$
  due PV/polarization charge flux

Now $\frac{dQ}{dt} = - \int dA \left[ \langle \vec{v} \vec{q} \rangle - \langle \vec{B}_r \vec{J} \parallel \rangle \right] = - \int dA \partial_x \left\{ \langle \vec{v}_x \vec{v}_y \rangle - \langle \vec{B}_x \vec{B}_y \rangle \right\} \rightarrow \text{Reynolds mis-match}$

$\uparrow$ PV flux $\uparrow$ current along tilted lines

Taylor: $\langle \vec{B}_x \vec{J} \parallel \rangle = - \partial_x \langle \vec{B}_x \vec{B}_y \rangle \rightarrow \text{vanishes for Alfvenized state}$
Progress II, cont’d

- With Field

\[ B_0 = 10^{-1} \]

\[ B_0 = 10^{-2} \]

\[ B_0 = 0 \]

\[ B_0 = 10^{-3} \]
Progress II, cont’d

• Control Parameters for $\widetilde{B}$ enter Z.F. dynamics
  Like RMP, Ohm’s law regulates Z.F.

• Recall
  $- \langle \tilde{v}^2 \rangle \text{ vs } \langle \widetilde{B}^2 \rangle$
  $- \langle \widetilde{B}^2 \rangle \sim B_0^2 R_m \rightarrow \text{origin of } B_0^2 / \eta \text{ scaling !?}$

• Further study $\rightarrow$ differentiate between :
  $- \text{cross phase in } \langle \tilde{v}, \tilde{q} \rangle \text{ and O.R. vs J.C.M}$
  $- \text{orientation : } \tilde{B} \parallel \vec{V} \text{ vs } \tilde{B} \perp \vec{V}$
  $- \text{spectral evolution}$

• $+$ = zonal flow state
  $\Diamond$ = no zonal flow state

ZF observed

No ZF observed