Zonal Flows:
From Wave Momentum and Potential Vorticity Mixing to Shearing Feedback Loops and Enhanced Confinement

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Outline:

I. Some Preliminaries

II. Heuristics of Zonal Flows
   - Wave Transport and Flows

III. Momentum Theorems, Potential Enstrophy Balance, and the Role of Mixing
   - PV Dynamics and Charney-Drazin Theorems
   - Implications for evolution of flows

IV. Why should I care?: Momentum Theorem → Feedback Loops → Shearing and Energetics
   - From Momentum to Feedback Loops and Shearing
   - Predator-Prey: Theory and Reality
   - Multi-Predator and Prey -> towards the LH transition
Outline:

V. REAL MEN do gyrokinetics...
   - GK and PV
   - Momentum Theorems
   - Energetics
   - Granulations

VI. The Current Challenge: Avalanches, Spatial structure and the PV staircase

VII. Open Issues and Questions
Philosophy of the Talk

If one computes but does not think, one will be bewildered.
(GYRO, GTC 而不思則罔)  (after Confucius)
For extended background material
(reviews, notes, key articles, book chapters):

http://physics.kaist.ac.kr/xе/ph742_f2010
Zonal Flows

Tokamaks

planets
The Fundamentals

- **Kelvin’s Theorem** for rotating system

\[
\omega \rightarrow \omega + 2\Omega
\]

\[
\int \mathbf{v} \cdot d\mathbf{l} = \int d\mathbf{a} \cdot (\omega + 2\Omega) \equiv C
\]

\[
\dot{C} = 0
\]

- \( Ro = V/(2\Omega L) \ll 1 \)

\[
\mathbf{V} \approx -\nabla_\perp p \times \hat{z}/(2\Omega)
\]

geostrophic balance

\[\rightarrow 2D\text{ dynamics} \]

- Displacement on beta plane

\[
\dot{C} = 0 \quad \rightarrow \quad \frac{d}{dt} \omega \approx -\frac{2\Omega}{A} \sin \theta_0 \frac{dA}{dt}
\]

\[
= -2\Omega \frac{d\theta}{dt} = -\beta V_y
\]

\[
\omega = \nabla^2 \phi \quad \beta = 2\Omega \sin \theta_0 / R
\]
Fundamentals II

- Q.G. equation \( \frac{d}{dt}(\omega + \beta y) = 0 \)

- Locally Conserved PV \( q = \omega + \beta y \)

- Latitudinal displacement \( \rightarrow \) change in relative vorticity

- Linear consequence \( \rightarrow \text{Rossby Wave} \)

\[ \omega = -\beta k_x/k^2 \]

observe: \( v_{g,y} = 2\beta k_x k_y/(k^2)^2 \) \( \rightarrow \text{Rossby wave intimately connected to momentum transport} \)

- Latitudinal PV Flux \( \rightarrow \text{circulation} \)
- Obligatory re: 2D Fluid

- \( \omega \) Fundamental: \[ \partial_t \omega = \nabla \times (\mathbf{V} \times \omega) \]

\[ \frac{d \omega}{dt} = \frac{\omega}{\rho} \cdot \nabla \mathbf{V} \rightarrow \text{Stretching} \]

- 2D \( \frac{d\omega}{dt} = 0 \) \rightarrow \[ E = \langle v^2 \rangle \] conserved

\[ \Omega = \langle \omega^2 \rangle \]

![Graph of energy range](image)

Inverse energy range \( E(k) \sim k^{-5/3} \)

forward enstrophy range \( E(k) \sim k^{-3} \)

\( k_R \) \( k_f \)

How?

\[ \partial_t \langle \Delta k^2 \rangle_E > 0 \text{ with } \dot{E} = \dot{\Omega} = 0 \]

\[ \partial_t \langle \Delta k^2 \rangle_E = -\partial_t \bar{k}_E^2 \]

\[ \therefore \partial_t \bar{k}_E^2 < 0 \rightarrow \text{large scale accumulation} \]
→ **Caveat Emptor:**

- often said `Zonal Flow Formation \( \simeq \) Inverse Cascade’

but

- anisotropy crucial \( \rightarrow \) \( \langle \tilde{V}^2 \rangle, \, \beta, \, \text{forcing} \rightarrow \text{ZF scale} \)

- numerous instances with:
  \( \left\langle \begin{array}{c}
    \text{no inverse inertial range} \\
    \text{ZF formation} \leftrightarrow \text{quasi-coherent}
  \end{array} \right. \)

all really needed:

\[
\langle \tilde{V}_y \tilde{q} \rangle \rightarrow \text{PV Flux} \rightarrow \langle \tilde{V}_y \tilde{V}_x \rangle \rightarrow \text{Flow}
\]

→ transport of PV is fundamental element of dynamics
Isn’t this Talk re: Plasma?

2 Simple Models

a.) Hasegawa-Wakatani (collisional drift inst.)

b.) Hasegawa-Mima (DW)

\[ \mathbf{V} = \frac{c}{B} \mathbf{\hat{z}} \times \nabla \phi + \mathbf{V}_{pol} \]
\[ \rightarrow m_s \]

\[ L > \lambda_D \rightarrow \nabla \cdot \mathbf{J} = 0 \rightarrow \nabla_{\perp} \cdot \mathbf{J}_{\perp} = -\nabla_{\parallel} J_{\parallel} \]

\[ J_{\perp} = n_e |e|V_{pol}^{(i)} \]

\[ J_{\parallel} : \eta J_{\parallel} = -(1/c) \partial_t A_{\parallel} - \nabla_{\parallel} \phi + \nabla_{\parallel} p_e \]

n.b.
MHD: \( \partial_t A_{\parallel} \) v.s. \( \nabla_{\parallel} \phi \)

DW: \( \nabla_{\parallel} p_e \) v.s. \( \nabla_{\parallel} \phi \)

b.) \( \frac{dn_e}{dt} = 0 \)

\[ \rightarrow \frac{dn_e}{dt} + \frac{\nabla_{\parallel} J_{\parallel}}{-n_0 |e|} = 0 \]
\[ \rho_s^2 \frac{d}{dt} \nabla^2 \hat{\phi} = -D_{||} \nabla^2 (\hat{\phi} - \hat{n}/n_0) + \nu \nabla^2 \nabla^2 \hat{\phi} \]

\[ \frac{d}{dt} n - D_0 \nabla^2 \hat{n} = -D_{||} \nabla^2 (\hat{\phi} - \hat{n}/n_0) \]

\[ D_{||} k_||^2 / \omega \]

is key parameter

b.) \[ D_{||} k_||^2 / \omega \gg 1 \rightarrow \hat{n}/n_0 \sim e\hat{\phi}/T_e \quad (m, n \neq 0) \]

\[ \frac{d}{dt} (\phi - \rho_s^2 \nabla^2 \phi) + v_* \partial_y \phi = 0 \quad \rightarrow \text{H-M} \]

n.b. \[ PV = \phi - \rho_s^2 \nabla^2 \phi + \ln n_0(x) \quad \frac{d}{dt} (PV) = 0 \]
An infinity of models follow:

- MHD: ideal ballooning resistive $\rightarrow$ RBM
- HW + $A_\parallel$: drift - Alfven
- HW + curv.: drift - RBM
- HM + curv. + Ti: Fluid ITG
- gyro-fluids
- GK

N.B.: Most Key advances appeared in consideration of simplest possible models
Behind the Color VG: An Overview

- Paradigm of “self-regulating DWT + sheared/zonal flow”
  > 15 + years old success story in MFE theory, experiment

- generic structure: ‘generation’ + ‘feedback’ → ‘predator-prey’ system

  - generation → perturbation from presumed state + Reynolds stress modelling
  - Coherent:
    parametric - variations on Mathieu
    envelope - variations on NLS
    → assume few initial modes, narrow spectrum
  - stochastic:
    ~ linearized Boltzmann equation, $N(k, x, t)$ in wave kinetics
    → assume eikonal description, spectrum structure

- feedback → simple shearing rule, linear/diffusive?
- final states → dynamical system theory
Issues:

- theoretical approach for generation is effectively “Linear Theory”
  - given presumed, pre-existing state, do seed shears grow?
  - what of evolved state?
  - Is there a unified **general principle** and/or perspective?

- k-space vs real space?
  - little scale separation or true “inverse cascade” - PV mixing
    Fundamental!
  - *real space structure* of Z.F. is of practical interest
    for predictive modelling! \( \rightarrow \) SCALE

- relation to macroscopics?
  - fixed flux, instead of local growth, drives flow
  - relation to ‘non-locality phenomena’, i.e. turbulent entrainment and spreading \( \rightarrow \) PE budget

- Zonal flows and phase space structure dynamics?
  - role of Z.F. in phase space structure dynamics?
  - Z.F. impact on relaxation - beyond Q.L.T?
What We will Endeavor to Show

- Potential Vorticity conservation is a fundamental ‘freezing-in law’ constraint on zonal flow dynamics.
- PV conservation directly links transport (i.e. particles, heat) to flow and potential enstrophy (‘roton population’) evolution.
- Essential Elements in Z.F. Generation:
  - PV mixing in space (McIntyre and Wood, 2009)
  - translation symmetry in direction of flow

\[ \therefore \text{ “Inverse cascade,” “modulational instability” not central though modulational calculation is useful.} \]

- Charney-Drazin Momentum Theorem:
  - characterizes evolved flow → non-acceleration theorem
  - relates flow evolution directly to driving flux via potential enstrophy balance
Part II: Heuristics of Zonal Flows

→ Wave Transport and Flows

→ Critical Element: Potential Vorticity Flux
Heuristics of Zonal Flows a):
Simplest Possible Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow
  (c.f. Vallis '07, Held '01)

- Key Physics:

  - Rossby Wave:
    \[ \omega_k = -\frac{\beta k_x}{k_z^2} \]
    \[ v_{gy} = 2\beta \frac{k_x k_y}{k_z^2} \]
    \[ \therefore v_{gy} v_{phy} < 0 \]
    \[ \rightarrow \text{Backward wave!} \]
    \[ \Rightarrow \text{Momentum convergence at stirring location} \]

  - Energy radiation
  - Momentum convergence
“...the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region.” (I. Held, ’01)

Outgoing waves ⇒ incoming wave momentum flux

Local Flow Direction (northern hemisphere):

- eastward in source region
- westward in sink region
- set by $\beta > 0$

Some similarity to spinodal decomposition phenomena ⇒ both ‘negative diffusion’ phenomena
Key Point: Finite Flow Structure requires separation of excitation and dissipation regions.

=> Spatial structure and wave propagation within are central.

→ momentum transport by waves
Key Elements:

- Waves → propagation transports momentum ↔ stresses
  → modest-weak turbulence

- vorticity transport → momentum transport → Reynolds force
  → the Taylor Identity

- Irreversibility → outgoing wave boundary conditions

- symmetry breaking → direction, boundary condition
  → $\beta$

- Separation of forcing, damping regions
  → need damping region broads than source region
  → akin intensity profile...

All have obvious MFE counterparts...
2) MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure
  - couple to damping ↔ outgoing wave i.e. Pearlstein-Berk eigenfunction

\[ x > 0 \Rightarrow v_{gr} > 0 \]

\[ v_{gr} = -2\rho_s^2 \frac{k_0 k_r v_*}{(1 + k_\perp^2 \rho_s^2)^2} \quad v_* < 0 \Rightarrow k_r k_\theta > 0 \]

\[ \langle v_{rE} v_{\theta E} \rangle = -\frac{c^2}{B^2} |\phi_k|^2 k_r k_\theta < 0 \]

- outgoing wave energy flux → incoming wave momentum flux → counter flow spin-up!

- zonal flow layers form at excitation regions
So, if spectral intensity gradient $\rightarrow$ net shear flow $\rightarrow$ mean shear formation

\[ S_r = v_{gr} \varepsilon = -\frac{2k_r k_\theta V_t \rho_*^2}{(1 + k_\perp^2 \rho^2)} \varepsilon \]

\[ \langle \vec{v}_r \vec{v}_\theta \rangle \approx \sum_k -k_r k_\theta |\phi_k^-|^2 \]

Reynolds stress proportional radial wave energy flux $\vec{S}$, mode propagation physics (Diamond, Kim ‘91)

Equivalently:

\[ \partial_t E + \nabla \cdot \vec{S} + (\omega \text{Im} \omega) E = 0 \] (Wave Energy Theorem)

- $\therefore$ Wave dissipation coupling sets Reynolds force at stationarity

Interplay of drift wave and ZF drive originates in mode dielectric

Generic mechanism...
• One More Way:
• Consider:
  – Radially propagating wave packet
  – Adiabatic shearing field
• Wave action density $N_k = \frac{E(k)}{\omega_k}$ — adiabatic invariant
• $\therefore E(k) \downarrow \Rightarrow$ flow energy decreases, due Reynolds work $\Rightarrow$ flows amplified (cf. energy conservation)
• $\Rightarrow$ Further evidence for universality of zonal flow formation
Heuristics of Zonal Flows d.)

Ambipolarity breaking \(\rightarrow\) polarization charge \(\rightarrow\) Reynolds stress : The critical connection

• Schematically:
  – Polarization charge

\[
\rho^2 \nabla^2 \phi = n_{i,GC}(\phi) - n_e(\phi)
\]

polarization length scale \(\Gamma_{i,GC} \neq \Gamma_e\)

SO \(\Gamma_{i,GC} \neq \Gamma_e\) \(\Rightarrow\) \(\rho^2 \langle \tilde{\nu}_E \nabla^2 \tilde{\phi} \rangle \neq 0\) \(\Leftarrow\) ‘PV mixing’

– If 1 direction of symmetry (or near symmetry):

\[
\langle \tilde{\nu}_E \nabla^2 \tilde{\phi} \rangle = -\partial_r \langle \tilde{\nu}_E \tilde{\nu}_L \rangle \quad \text{(Taylor, 1915)}
\]

– Vorticity Flux: \(\rho^2 \partial_r \langle \tilde{\nu}_E \tilde{\nu}_L \rangle\) \(\Rightarrow\) Reynolds force \(\Rightarrow\) Flow Drive
Heuristics of Zonal Flows d.) cont’d

• Implications:
  – ZF’s generic to drift wave turbulence in any configuration: electrons tied to flux surfaces, ions not
    • g.c. flux → polarization flux
    • zonal flow
  – Critical parameters
    • ZF screening (Rosenbluth, Hinton ‘98)
    • polarization length
    • cross phase → PV mixing

• Observe:
  – can enhance $e\varphi_{ZF}/T$ at fixed Reynolds drive by reducing shielding, $\rho^2$
  – typically: $\varepsilon / \varepsilon_0 \sim 1 + \rho_i^2 / \lambda_D^2 + f_i \rho_b^2 / \lambda_D^2 + f_d \delta_d^2 / \lambda_D^2$

  – Leverage (Watanabe, Sugama) → flexibility of stellerator configuration
    • Multiple populations of trapped particles
    • $\langle E_r \rangle$ dependence (FEC 2010)
Heuristics of Zonal Flows d.) cont’d

- Yet more: \( \frac{\partial}{\partial t} \langle v_\perp \rangle = -\partial_r \langle \tilde{v}_r E \tilde{v}_E \rangle - \gamma_d \langle v_\perp \rangle + \mu v_r^2 \langle v_\perp \rangle \)

- Reynolds force opposed by flow damping

- Damping:
  - Tokamak \( \gamma_d \sim \gamma_{ii} \)
    - trapped, untrapped friction
    - no Landau damping of (0, 0)
  - Stellerator/3D \( \gamma_d \leftrightarrow NTV \)
    - damping tied to non-ambipolarity, also
    - largely unexplored

- Weak collisionality \( \rightarrow \) nonlinear damping \( \rightarrow \) problematic
  \( \rightarrow \) tertiary \( \rightarrow \) ‘KH’ of zonal flow
  \( \rightarrow \) magnetic shear!?  
  \( \rightarrow \) other mechanisms?
4) GAMs Happen

- Zonal flows come in 2 flavors/frequencies:
  - $\omega = 0 \Rightarrow$ flow shear layer
  - GAM $\omega^2 \equiv 2c_s^2 / R^2 (1 + k_r^2 \rho_0^2) \Rightarrow$ frequency drops toward edge $\Rightarrow$ stronger shear
    - radial acoustic oscillation
    - couples flow shear layer (0,0) to (1,0) pressure perturbation
    - $R \equiv$ geodesic curvature (configuration)
    - Propagates radially

- GAMs damped by Landau resonance and collisions

$$\gamma_d \sim \exp[-\omega_{GAM}^2 / (v_{thi} / Rq)^2]$$
- q dependence!
- edge

- Caveat Emptor: GAMs easier to detect $\Rightarrow$ looking under lamp post ?!
Notable by Absence: Three “Usual Suspects”

- “Inverse Cascade”
  - Wave mechanism is essentially \textit{linear}
  - \(\Rightarrow\) scale separation often dubious
- “Rhines Mechanism”
  - requires very broad dynamic range
  - Waves \(\Leftrightarrow k_R \Leftrightarrow\) forced strong turbulence
  - \textit{strong turbulence model}
- “Modulational Instability”
  - coherent, quasi-coherent wave process
  - useful concept, but not \textit{fundamental}

\(\Rightarrow\) see P.D. et al. PPCF’05, CUP’10 for \textit{detailed} discussion

Lesson: Formation of zonal bands is \textit{generic} to the response of a rapidly rotating fluid to any localized perturbation
Inverse Cascade/Rhines Mechanism

\[ k \sim -\beta k_x / k^2 \]

1/\tau_k

transfer \iff triad couplings

eddying transfer: \( \omega_{MM} < 1/\tau_c \)

wave transfer: \( \omega_{MM} > 1/\tau_c \)

cross over: \( \omega_{MM} \sim 1/\tau_c \)

\[ \Rightarrow \text{Rhines Scale} - \text{emergent characteristic scale for ZF} \]

\[ l_R \sim (\tilde{u}/\beta)^{1/2} \sim \epsilon^{1/5} / \beta^{3/5} \]

Contrast: Rhines mechanism vs critical balance
Part III: Momentum Theorems for Zonal Flows:

⇒ How Do We Understand and Exploit PV Mixing?

⇒ Toward a Unifying Principle in the Zonal Flow Story via Potential Enstrophy Balance
Potential Vorticity Dynamics and Charney-Drazin Theorems

- Example: Simplest interesting system → Hasegawa-Wakatani
  
  Vorticity: \( \frac{d\nabla^2 \phi}{dt} = -D_\parallel \nabla^2 (\phi - n) + D_0 \nabla^2 \nabla^2 \phi \)
  
  Density: \( \frac{dn}{dt} = -D_\parallel \nabla^2 (\phi - n) + D_0 \nabla^2 n \)

- Locally advected PV: \( q = n - \nabla^2 \phi \)
  
  - Content of PV → charge density
    
    \( n \) → guiding centers → electrons
    
    \( -\nabla^2 \phi \) → polarization → ions

- Conserved on trajectories in inviscid theory \( dq/dt = 0 \)

- PV conservation → freezing-in law
  
  Kelvin’s theorem → dynamical constraint
Thm’s, cont’d

- Potential Enstrophy (P.E.) Balance

\[ \frac{d\langle q^2 \rangle}{dt} = 0 \]
\[ \rightarrow \delta_t \langle q^2 \rangle \equiv \partial_t \langle q^2 \rangle + \partial_r \langle \tilde{V}_r q^2 \rangle + D_0 \langle (\nabla q)^2 \rangle \rightarrow \text{P.E. evolution} \]
\[ = -\langle \tilde{V}_r q \rangle \langle q \rangle' \rightarrow \text{P.E. Production by PV mixing} / \text{flux} \]

- PV flux:

\[ \langle \tilde{V}_r \tilde{q} \rangle = \langle \tilde{V}_r \tilde{n} \rangle - \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle \]

but:

\[ \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{V}_r \tilde{V}_\theta \rangle \quad \text{(Taylor, 1915)} \]

(n.b : symmetry in \theta direction)

- P.E. production directly couples driving transport and flow drive

- Fundamental Relation for Vorticity flux

\[ \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = \langle \tilde{V}_r \tilde{n} \rangle + (\delta_t \langle q^2 \rangle)/\langle q \rangle' \]

Reynolds force relaxation Local PE decrement

\[ \therefore \text{Reynolds force locked to particle flux + P.E. decrement by PV conservation; transcends quasilinear theory} \]
Contrast: Implications of PV Freezing-in Law

\[
dn/dt = 0 \quad (?)
\]
\[
d\langle n \rangle / dr \neq 0
\]
\[
\tilde{n} \text{ grows} \rightarrow \langle \tilde{V}_r \tilde{n} \rangle \rightarrow :-(
\]

\[
dq/dt = 0 \quad (!)
\]
\[
d\langle q \rangle / dr \neq 0
\]
\[
\tilde{q} \text{ grows}
\]
\[
\rightarrow \left\{ \langle \tilde{V}_r \tilde{n} \rangle \rightarrow \text{transport} \rightarrow :-(
\right.
\]
\[
\left. \langle \tilde{V}_r \nabla^2 \phi \rangle \rightarrow \text{flow} \rightarrow :-ight)
\]

Lesson: Even if \( \langle q \rangle \cong \langle n \rangle \), PV conservation must channel free energy into zonal flows!

Key Question: Branching ratio of energy coupled to flow, vs transport-inducing fluctuations?
Combine:

\[
\partial_t \langle V_\theta \rangle = -\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle - \nu \langle V_\theta \rangle
\]

Charney-Drazin Momentum Theorem

(1960, et.seq., P.D., et.al. ’08, for HW)

\[
\partial_t \{(\text{WAD}) + \langle V_\theta \rangle\} = -\langle \tilde{V}_r \tilde{n} \rangle - \frac{\delta_t \langle \tilde{q}^2 \rangle}{\langle q \rangle'} - \nu \langle V_\theta \rangle
\]

\begin{array}{ll}
\text{Pseudomomentum} & \text{local P.E. decrement} \\
\Rightarrow & \\
\end{array}

\text{WAD} = \text{Wave Activity Density}, \frac{\langle \tilde{q}^2 \rangle}{\langle q \rangle'}

\begin{itemize}
    \item pseudomomentum in \( \theta \)-direction (Andrews, McIntyre ’78)
    \item Generalized Wave Momentum Density
        \begin{itemize}
            \item momentum of quasi-particle gas of waves, turbulence
            \item consequence of azimuthal/poloidal symmetry
            \item not restricted to linear response, but reduces correctly
        \end{itemize}
\end{itemize}
What Does it Mean? → “Non-Acceleration Theorem”:

\[ \partial_t \{(WAD) + \langle V_\theta \rangle \} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle/\langle q \rangle' - \nu \langle V_\theta \rangle \]

- absent \[ \langle \tilde{V}_r \tilde{n} \rangle \), driving flux
- \[ \delta_t \langle \tilde{q}^2 \rangle \), local potential enstrophy decrement

→ cannot accelerate Z. F. with stationary fluctuations!

- Essential physics is PV conservation and translational invariance in \( \theta \) → freezing quasi-particle gas momentum into flow → relative “slippage” required for zonal flow growth

- obvious constraint on models of stationary zonal flows!

↔ need explicit connection to relaxation, dissipation
Aside: H-M

C-D Theorem for HM

\[ \frac{\partial_t}{\partial_t} \{ \text{WAD} + \langle V_\theta \rangle \} = \frac{\langle \tilde{\tau}^2 \rangle \tau_c}{\langle q \rangle'} - \frac{1}{\langle q \rangle'} \left\{ \partial_r \langle \tilde{V_r} \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} - \nu \langle V_\theta \rangle \]

C-D prediction for \( \langle V_\theta \rangle \) at stationary state, HM model

\[ \langle V_\theta \rangle = \frac{1}{\nu \langle q \rangle'} \left\{ \langle \tilde{\tau}^2 \rangle \tau_c - \partial_r \langle \tilde{V_r} \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} \]

- Note: Flow direction set by: \( \langle q \rangle' \), source, sink distribution
- Forcing, damping profiles determine shear
- Potential Enstrophy Transport impact flow structure
In More Depth: What Really Determines Zonal Flow?

- driving flux: $\langle \tilde{V}_r \tilde{n} \rangle = \Gamma_0 - \Gamma_{col} = \int dr' S_n(r') - \Gamma_{col}$
  - Total flux $\Gamma_0$ fixed by sources, $S_n \rightarrow$ flux driven system
  - Collisional flux in turbulent system, $\Gamma_{col}$ (computed with actual profiles)

- $\Gamma_o - \Gamma_{col} \rightarrow$ available flux

- P.E. decrement: $\delta_t \langle \tilde{q}^2 \rangle = \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle$
  - change in roton intensity (PE) changes flow profile
    - roton dissipation
    - P.E. flux, direction increment, according to convergence (> 0) or divergence (< 0) of pseudomomentum, locally

So: P.E. transport and “spreading” intrinsically linked to flow structure, dynamics

Net $\delta$(P.E.) can generate net spin-up

$\therefore$ Zonal flow dynamics intrinsically “non-local” $\leftrightarrow$ couple to turbulence spreading (fast, meso-scale process)
Clarifying the Enigma of Collisionless Zonal Flow Saturation

- Flow evolution with: $\nu \to 0$, $S_n \neq 0$ and nearly stationary turbulence

$$\partial_t \langle V_\theta \rangle = - \left( \int dr' S_n(r') - \Gamma_{\text{col}} \right) - \left( \partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle + D_0 \langle (\nabla \tilde{q})^2 \rangle \right) / \langle q \rangle$$

Possible Outcomes:

- $\langle q \rangle' \to 0$, locally $\rightarrow$ shear flow instability (the usual)
  $\leftrightarrow$ limit cycle of burst and recovery, effective viscosity?
  $\rightarrow$ problematic with magnetic shear

- $\langle \tilde{V}_r \tilde{n} \rangle$ v.s. $\partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle$ $\rightarrow$ potential enstrophy transport and inhomogeneous turbulence, with $\tilde{n}/n \sim \text{M.L.T}$
  $\rightarrow$ flux drive vs. roton population flux
  $\rightarrow$ novel saturation mechanism

- $\langle q \rangle' \to 0$, globally $\rightarrow$ homogenized PV state (Rhines, Young, Prandtl, Batchelor)
  $\rightarrow$ decouples mean PV, PE evolution

- homogeneous marginality, i.e. $\int dr' S_n(r') = \Gamma_{\text{col}} \leftrightarrow \text{ala'}$ stiff core

N.B.: $\langle q \rangle' = 0 \Rightarrow \partial_r \langle n \rangle = \partial_r^2 \langle V_E \rangle = \partial_r \langle \omega_E \rangle \rightarrow$ particular profile relation!
Partial Summary

- A Unifying Perspective: C-D theorem for zonal flow momentum derived based on
  - PV conservation on trajectories
  - PV mixing $\rightarrow$ (i.e. forward, enstrophy cascade!) $\rightarrow$ mean relaxation
  - symmetry in flow direction
  - C-D theorem $\iff$ freezing-in law for flow + Q.P./wave gas
    - rigorous non-acceleration theorem constraint on theory
    - identifies $\langle \tilde{V}_r \tilde{n} \rangle$ and $\delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle$ as key elements determining flow evolution $\rightarrow$ links ZF for flux drive
    - allows useful calculations of flow shear $\langle V_\theta \rangle'$ and profile structure
  - PE transport identified as novel collisionless flow regulation mechanism
  - C-D theorems proved for HW, resistive interchange, GF ITG ...
An Application: Self-Acceleration and Intrinsic Rotation in Basic Experiment

- Intrinsic rotation (Ida, Rice) now central focus of MFE research on turbulence, transport self-organization i.e. Rice scaling, $\Delta \bar{V}_\phi \sim \Delta W / I_p$
- for intrinsic rotation, Reynolds stress $\langle \bar{V}_r \bar{V}_\theta \rangle$ is key, i.e.
  $$\langle \bar{V}_r \bar{V}_\theta \rangle = -\chi_\phi \frac{\partial \langle V_\phi \rangle}{\partial r} + \Pi_{r,\phi}^{\text{res}}, \quad \Pi_{r,\phi}^{\text{res}} = \begin{cases} \text{residual stress} \\ \text{wave driven, non-diffusive} \end{cases}$$
  (Gurcan, P.D., McDevitt, et.al. ’07, ’08, ’09)
- $\Pi_{r,\phi}^{\text{res}}$
  - physics: wave momentum transport, symmetry breaking
  - critical to intrinsic rotation, spin-up, i.e.
  $$\partial_t \int_0^a \langle p_\phi \rangle = -\Pi_{r,\phi}^{\text{res}}|_a, \quad \partial_r \langle V_\phi \rangle|_a = (\Pi_{r,\phi}^{\text{res}}/\chi_\phi)|_a$$

residual stress, $\Pi_{r,\phi}^{\text{res}}|_a$, on boundary is essential
- akin engine: converts $\nabla p$, $\nabla T$ to $\nabla V_\phi$ via turbulence
- boundary condition on flow critically important
Intrinsic rotation observed in CSDX (Z. Yan, et.al., '09)

- CSDX
  - linear device $\rightarrow$ symmetry is azimuthal
  - $T_i < T_e$, low temperature $\rightarrow$ well described by collisional DWT and H-W system
  - edge neutrals $\rightarrow$ strong drag $\sim$ no slip B.C.
- Intrinsic azimuthal rotation $\rightarrow$ surely linked to PV dynamics
  - electron direction
  - exceeds $v_{de}$
  - exhibits prominent edge shear layer
- $\Pi_{r,\phi}^{res}$ (Residual Stress) directly measured
  - $\langle \tilde{V}_r \tilde{V}_\theta \rangle$ measured
  - $-\chi_\phi \partial \langle V_\phi \rangle / \partial r$ synthesized $\rightarrow$ significant residual found
  - $\Pi_{r,\phi}^{res} / \chi_\phi \neq 0$, especially significant in edge shear layer
Residual & Diffusive Stress Decomposition Consistent with Averaged Flow Profiles in Basic Experiment

**Total Stress**

\[
\langle \delta V / \delta r \rangle \left( 10^6 \text{ cm}^2 / \text{s}^2 \right)
\]

- Graph showing total stress distribution.

**Sheared Flow**

\[
\langle V_s \rangle \left( 10^4 \text{ cm/s} \right)
\]

- Graph showing sheared flow distribution.

**Residual Stress Amplifies Flow**

- Graph showing residual stress amplification.

**Diffusive Stress => Inward Flux**

- Graph showing diffusive stress and inward flux.

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*Z. Yan et al, Submitted*
What does PV conservation tell us about Residual Stress and Self-Acceleration?

- momentum balance: \( \partial_t \langle P_\theta \rangle = - \int_0^a \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle - \int_0^a \nu_n \langle V_\theta \rangle \)

C-D theorem: \( \int_0^a \nu_n \langle V_\theta \rangle = - \int_0^a \int d r' S_n (r') + \int_0^a \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' \)

\( \Rightarrow \) then for total Reynolds Stress on boundary:
\( \langle \tilde{V}_r \tilde{V}_\theta \rangle |_a = \int_0^a \left( - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' + \int d r' S_n (r') \right) \)

P.E. decrement \hspace{1cm} \text{Particle source drive}

\( \rightarrow \) exact expression via C-D theorem

\( \rightarrow \) interesting to compare to QL result \( (c > 1 \text{ HW}) \)

\[ \langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle = - \sum_k \frac{|\gamma_k| \langle \tilde{V}_r^2 \rangle_k}{(\omega - k_\theta V_\theta)^2} \left[ \frac{\partial}{\partial r} \langle \nabla^2 \phi \rangle - \frac{\partial}{\partial r} \langle n \rangle \right] \]

\( \text{turbulent viscosity} \hspace{1cm} \text{off-diagonal residual} \)

\( \rightarrow \) vorticity diffusion \hspace{1cm} \nabla n \text{ driven} \)

\( \therefore \nabla n \text{ drives mean flow vs turbulent viscosity} \)
Vortex Generation, Propagation & Broadening in DWT/ZF System

See M. Xu, et. al.
Lessons

- Reynolds force, intrinsic rotation set by:
  - particle fueling profile $\leftrightarrow \nabla n$ residual in QLT
  - PE increment (i.e. roton intensity out flow)
    $\leftrightarrow$ turbulent viscosity in QLT

- Fueling:
  - controls $\nabla n \rightarrow$ drives $\Pi_{r,\phi}^{res}/\chi_{\phi}$
  - not simply change in moment of inertia
  - consistent with rotating plasma as turbulence-mediated engine

- PE increment (with $\langle q \rangle'$):
  - $\partial_r \langle \tilde{V}_r \tilde{q}^2 \rangle \rightarrow$ boundary flux can produce net spin, either sign
  - only means for flow reversals to occur

- net $\langle V_\theta \rangle \leftrightarrow$ Fueling vs. PE increment competition i.e. equivalent to
  branching ration of \[
  \begin{cases}
  \text{particle flux!} \\
  \text{vorticity}
  \end{cases}
  \]
Part IV: Why Care? Practical Implication!

Momentum Theorems ↔ Feedback Loops
↔ Shearing and Energetics
Why care?: Shearing and Energetics

- ZF ‘shear suppression’ is really mode coupling from DW’s ⇒ ZF’s
  - Coupling conserves energy, momentum
  - Energy deposited in weakly damped mode with n=0 (i.e. no transport)
  - $\gamma_L \sim \gamma_{ExB}$ ‘rule’ inapplicable to ZF dynamics ⇔ rather, accessibility of state with increased energy partition $E_{ZF}/E_{DW} ⇔ \text{LRC} \sim E_{ZF}/E_{ZF} + E_{DW}$

N.B. Momentum Thm. is underpinning of `feedback loop’ structure
→ “Suppression” and “stress” locked together

⇒ need address all aspects of the problem
N.B. FEC2010:

- Mounting discussion that \( \left\langle V_E \right\rangle \) ’ changes not well correlated with L → H and other transition

But also:

- More observations of predator-prey interaction (also Zweben, APS) as harbingers of transition
Overview of TJ-II experiments

The L-H transition appears more correlated with the development of fluctuating $E_r$ than steady-state $E_r$ effects

(T. Estrada et al., PPCF-2009).

Doppler Reflectometer

$\rho=0.8$
Self-Regulation and Predator-Prey Models

- DW-ZF turbulence ‘nominally’ described by predator-prey
  \[
  \frac{\partial}{\partial t} N = \gamma N - \alpha V^2 N - \Delta \omega N^2, \\
  \frac{\partial}{\partial t} V^2 = \alpha N V^2 - \gamma_d V^2 - \gamma_{NL}(V^2) V^2. 
  \]
  - growth
  - suppression
  - self-NL

  Prey \equiv DW’s ( N ) \leftrightarrow forward enstrophy scattering
  Predator \equiv ZF’s ( V^2 ) \leftrightarrow inverse energy scattering

  stress drive  ZF damping  NL ZF damping

  Configuration \Rightarrow coupling coeffs.

- Can have:
  \[
  (\gamma / \Delta \omega); \quad (\gamma_d / \alpha, (\gamma - \Delta \omega \gamma_d / \alpha)^2) 
  \]
  - Fixed point
  - Limit cycle states,
  - depends on ratios of V dampings \Rightarrow phase lag

  N.B. Suppression + Reynolds terms $\alpha V^2 N$ cancel for TOTAL momentum, energy

- Major concerns/omissions
  - Mean ExB coupling?
  - Turbulence drive $\gamma \Rightarrow$ flux drive $\Leftrightarrow$ avalanching? $\Rightarrow$ not a local process
  - 1D \Rightarrow spatio-temporal problem (fronts, NL waves)? \Rightarrow barrier width
  - NL flow damping?
Self-Regulation and Predator-Prey Models

- **∇P coupling**
  \[ \gamma_L \text{ drive } \langle \nabla V_E \rangle, \]

\[ \partial_t \mathcal{E} = \mathcal{E} \mathcal{N} - a_1 \mathcal{E}^2 - a_2 V^2 \mathcal{E} - a_3 V_{ZF}^2 \mathcal{E}, \]

\[ \partial_t V_{ZF} = b_1 \frac{\mathcal{E} V_{ZF}}{1 + b_2 V^2} - b_3 V_{ZF}, \]

\[ \partial_t \mathcal{N} = -c_1 \mathcal{E} \mathcal{N} - c_2 \mathcal{N} + Q. \]

- **Simplest example of 2 predator + 1 prey problem**
  i.e. prey sustains predators \{ useful feedback \}
  predators limit prey
  But: - 2 predators (ZF, ∇ ⟨P⟩ ) compete
  - ∇ P enters drive → trigger

- **Relevance**: LH transition, ITB
  - ZF ⇒ triggers ⇒ rapid growth

\[ \mathcal{E} \equiv \text{DW energy} \]

\[ V_{ZF} \equiv \partial_r N_{ZF} \equiv \text{ZF shear} \]

\[ \mathcal{N} \equiv \nabla \langle P \rangle \equiv \text{Pressure gradient} \]

E. Kim, P.D., 2003
Self-Regulation and Predator-Prey Models

- **Observations:**
  - ZF’s trigger transition, $\nabla \langle P \rangle$ locks it in
  - Period of dithering, pulsations .... during ZF, $\nabla \langle P \rangle$ coexistence as $Q$ increases
  - Phase between $\xi$, $V_{ZF}$, $\nabla \langle P \rangle$ varies as $Q$ increases
  - $\nabla \langle P \rangle \Leftrightarrow$ ZF interaction $\Rightarrow$ effect on wave form

\[\text{Solid - } \xi\]
\[\text{Dotted - } V_{ZF}\]
\[\text{Dashed } \nabla \langle P \rangle\]
Self-Regulation and Predator-Prey Models

- Comparison with and without $\langle V_E \rangle' \iff ZF- \langle V_E \rangle'$ mode competition $\Rightarrow$ evolution as probe of theory ?!

\begin{itemize}
  \item \textbf{with}
  \begin{itemize}
  \item \includegraphics[width=0.45\textwidth]{with.png}
  \end{itemize}
  \item \textbf{without}
  \begin{itemize}
  \item \includegraphics[width=0.45\textwidth]{without.png}
  \end{itemize}
\end{itemize}
Self-Regulation and Predator-Prey Models

P.D., et al., FEC 1994

Stage, $V_E'$ is primarily due to $V_{\theta}'$, and the ambient transport is reduced, but not quenched. Hence, there is some constraint upon $\nabla P$-steepening, so that an ELM-free H-mode is possible at modest power. In the second stage, for which $P > P_{\text{thresh}}$, the fluctuations are quenched. As a consequence, the poloidal flow decays, and the pressure gradient is the dominant contributor to $E_r'$. In this stage, the ambient transport is reduced to feeble levels, so that the pressure gradient will surely steepen to the ballooning limit, resulting in the onset of ELMs, which are discussed in Section (IV) of this paper. A second aspect of the evolution is that the ratio of poloidal flow shear to diamagnetic velocity shear is given by

$$\frac{V_d'}{V_{\theta}'} = \frac{b/a - \bar{E}}{\bar{E}}$$

which further illustrates the dominance of $V_{\theta}'$ near threshold $b/a = \bar{E}$, and the dominance of $V_d'$ at high power ($\bar{E} \to 0$). A third notable aspect of the evolution is that the temporal duration of the “flow dynamo” phase is sensitive to the rate at which the external power input is “ramped.” Specifically, a rapid power ramp will compress the time duration of the flow-dynamo phase, and thus may render it unobservable to diagnostics without sufficient temporal evolution[17]. Also, as with any bifurcation, the transition time diverges at the power threshold. Thus, the detailed transition dynamics are best studied at modest power levels. A fourth interesting aspect of the model is the fact that the ambient L-mode pressure gradient serves as the “seed” for the transition, by driving a diamagnetic velocity which is amplified by the flow dynamo, once the power threshold is exceeded. The sign of the seed $V_E'$ is determined by the relative magnitudes of $L_n$ and $L_{Ti}$. For $L_n < L_{Ti}$, the sign is
Recent Events

• TJII (FEC 2010)
  – Gradual Transitions ($P \sim P_{\text{Thresh}}$)
  – Appearance of Limit Cycle $E_r, n$

• Conway (FEC 2010)
  – Cycles / Pulsations in I-phase
  – 3 players: $GAM, ZF, \langle U_{\perp} \rangle$
  – GAM as LH trigger

• Miki, Diamond (FEC 2010)
  – ZF, GAM multi-predator problem
  – Pulsation as co-existance
Flows and turbulence dynamics, Gradual L-H transitions

Gradual transitions happen for
\( P \sim P_{\text{threshold}} \)
And/or
Non optimal \( \tau \) range

Overview of TJ-II experiments
Flows and turbulence dynamics

Doppler reflectometry

The time evolution shows a predator-prey behaviour:

Periodic evolution of $E_r$ and $\tilde{n}$ with the $E_r$ following $\tilde{n}$ with a phase delay of $90^\circ$.

T. Estrada et al., 2010
FIG. 5: (a) $f_D$ plus (b) $u_\perp$ & $S_D$ time traces over several I-phase pulses showing strong GAM oscillation, plus synchronized Doppler spectra from (c) low and (d) high I-phases, (e) L-mode earlier in same #24906 and (f) H-mode from similar discharge #24570.

FIG. 6: Evolution of GAM amplitude and mean $E \times B$ velocity across L to I-phase, plus long range (toroidal) coherence $\gamma^2$ of GAM $f_D$ and $S_D$ peaks.
FIG. 7: Radial profiles of GAM p.t.p amplitude (b) and long range correlation $\gamma^2(f_D)$ (c) during the L-I-H transition.
Multi-predator-prey model for ZF/GAM system

Nonlinear coupling treated by wavekinetic theory.

Geodesic Curvature: Leakage by ExB flow conservation in toroidal plasmas

Sound wave Propagation: Leakage by toroidal flow

Drift wave turbulence $N$

Zonal Flow $U=\langle v_E \rangle$

Anisotropic pressure (GAM) $G=\langle p \sin \theta \rangle$

Anisotropic parallel flow $V=\langle v_{||} \cos \theta \rangle$

Prey Depends on mode frequencies

Predators ⇒ multiple competition for ‘ecological niche’ to feed on prey…
GAM shearing [Miki ‘10 PoP] shows different population and dynamics for different frequency shear flows must be considered for turbulence suppressions.

\[
\frac{\partial \langle \varepsilon \rangle}{\partial t} \sim -2 \sigma \langle k^2 \rho_i^2 / (1 + k_\perp^2 \rho_i^2) \rangle \langle \varepsilon \rangle |\vec{V}'|^2 \tau_{ac} \sim -|\vec{V}'|^2 \tau_{ac} \langle \varepsilon \rangle.
\]

where

\[
\tau_{ac,\omega,k} \sim \left| \left( \frac{\partial \Omega_q}{\partial q_r} \right)_{\text{GAM}} - v_{gr}(k) \right| \Delta q_r \quad \text{Auto–coherence time of GAM wave packet propagating shear!}
\]

(cf. effective reduction of time varying ExB shearing rate [Hahm ‘99 PoP] )

GAM shearing can be estimated by the autocorrelation times representing resonances between drift wave and GAM group velocity – “GAM shearing”

Shorter GAM autocorrelation reduces the efficiency of turbulence suppression

Therefore, in discussion of turbulence suppression by the GAM, comparison of shearing partition is necessary

Ratio of SHEARING

Shearing partition of GAM to total ZFs

\[
\eta(r) \equiv \frac{\tau_{ac,GAM} E_\omega}{\tau_{ac,ZF} E_0 + \tau_{ac,GAM} E_\omega}
\]
Predator–prey model with nonlinear multi-shearing comprehends two new roles and reveals:

\[
\frac{\partial N}{\partial t} = N(\gamma_L - \Delta \omega N - \alpha_0 E_0 - \alpha_\omega E_\omega) + \text{n.o.t.}
\]

Introduce competition between ZF and GAM:

\[
\begin{align*}
\frac{\partial E_0}{\partial t} &= A_0 E_0 (\alpha_0 N (1 - \gamma_0 E_0 - \gamma_\omega E_\omega) - \gamma_0) \\
\frac{\partial E_\omega}{\partial t} &= A_\omega E_\omega (\alpha_\omega N (1 - \gamma_\omega E_0 - \gamma_\omega E_\omega) - \gamma_\omega)
\end{align*}
\]
Possible Fixed points in the multiple shearing predator–prey

1. L-mode state

\[(N, E_0, E_\omega) = (N_L, 0, 0)\]

2. ZF only state

\[(N, E_0, E_\omega) = (N_{*0NL}, E_{*0NL}, 0),\]

3. GAM only state

\[(N, E_0, E_\omega) = (N_{*\omega NL}, 0, E_{*\omega NL}),\]

4. Coexisting state (ZF+GAM)

\[(N, E_0, E_\omega) = (N_{*0\omega NL}, E_{0*0\omega NL}, E_{\omega*0\omega NL}),\]

Which states are stable is determined by system parameters – γ_L (gradient), q(r), ν, etc.
We observe hysteretic behaviors in the $E_0/E_\omega$ ratio with respect to $L_T^{-1}$, related to bistability.

Bistability in shear field of low frequency and high frequency ZF due to different shearing effects.

For some parameters

Criterion:

$$\frac{\alpha_0}{\alpha} < \frac{\gamma_{0\omega}}{\gamma_{00}}$$

• Application of noise can affect transition path? (cf. [Itoh ’03 PPCF])
• Possibly mean flow can change states.

NOTE: This is NOT the hysteresis seen in L–H transition!
Lessons

1. Broadband shearing has coherence time, as well as strength
   \[ \tau_c \langle V_E'^2 \rangle \rightarrow \eta \rightarrow \text{shearing partition} \]

2. ZF/GAM interaction → multi-shearing competition
   → Minimal: 1 prey + 2 predators \((\omega \sim 0, \omega_{GAM})\)

   ➢ Mode competition required

4. Considered one mechanism for mode competition via coupling higher order wavekinetics.
   ➢ Turbulence mediation is central

5. States: L, ZF/GAM only, coexistence

6. States and sequence of progress selected by \((R/L_{T crit} - R/L_T)\) evolution and parameters.
   ZF → coexistence → GAM, transition

7. Bistability in shearing field (envelope) possible → jumps/transitions between GAM/ZF state possible

8. To characterize competition, compare → \( \gamma, \alpha, \text{damping}, \tau_c \).
V.) But REAL Men Do Gyrokinetics...?!
Comparison of QG, GK dynamics

QG, GK systems structurally similar, i.e.

<table>
<thead>
<tr>
<th>Dynamical variable</th>
<th>QG system</th>
<th>GK system</th>
</tr>
</thead>
<tbody>
<tr>
<td>PV, ( q(x, t) )</td>
<td>( \partial_t q + { q, \phi } = 0 )</td>
<td>distribution function, ( f(x, v, t) )</td>
</tr>
<tr>
<td>Time evolution</td>
<td>( \frac{dq}{dt} = \partial_t q + { q, \phi } = 0 )</td>
<td>( \frac{df}{dt} = \partial_t f + { f, H } = 0 )</td>
</tr>
<tr>
<td>Circulation</td>
<td>( \Gamma = \oint (V + 2\Omega a \sin \theta) dl )</td>
<td>( \Gamma = \int \mathbf{v} \cdot d\mathbf{x} )</td>
</tr>
<tr>
<td>Kelvin’s Thm.</td>
<td>Yes</td>
<td>Yes (Lynden-Bell, ’67)</td>
</tr>
<tr>
<td>Vorticity</td>
<td>PV, ( q = \nabla^2 \phi + F(\phi, n) )</td>
<td>GK Poisson, Pol Charge</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \int d^3v f + \rho_s^2 \nabla^2 \phi = g(\phi, n_e, ...) )</td>
</tr>
<tr>
<td>ZF Generation</td>
<td>Vorticity Flux</td>
<td>Pol. Charge Flux</td>
</tr>
<tr>
<td>C-D Theorem</td>
<td>Yes</td>
<td>???</td>
</tr>
</tbody>
</table>

Some general observations:

- GK Poisson equation links fluid vorticity to kinetic dynamics
- Spatial flux of polarization charge is underpinning of Z.F. generation mechanism in GK systems
- C-D Theorem for GK systems!? Yes, as has Kelvin’s Theorem!
Example: Darmet Model, A Simplified Interesting Prototype

- Darmet '06: Trapped Ion Induced ITG

- Bounce Averaged DKE for Trapped Ions + GK Poisson Equations
  \[ \partial_t f + v_d \partial_y f + \{ \phi, f \} = C(f) \]
  \[ \alpha_e (\phi - \langle \phi \rangle_\theta) - \rho^2 \nabla^2 \phi = \frac{2}{n_{eq} \sqrt{\pi}} \int_0^\infty dE \sqrt{E} f - 1 \]

- Drive: \( Q = -\chi_{col} \langle T \rangle' + \int dE \sqrt{E} E \langle v_r \delta f \rangle \)
  to match applied heat flux

- Irreversibility
  - trapped ion drift resonance
  - \( \sim 1D \) resonance dynamics \( (v_{ph\phi} \leftrightarrow v_d) \)

\( \rightarrow \) possibility of long wave-ion coherence time, \( K(Kubo \#) \gg 1 \)

\( \therefore \) phase space structure formation, failure of QLT are both likely
Charney-Drazin Thm. for GK Turbulence

- Simple Test Case: Trapped Ion Induced ITG, Darmet '06
  DKE for trapped ions + GK Poisson Equations

\[ \partial_t f + v_d \partial_y f + \{\phi, f\} = C(f) \]

\[ \alpha_e (\phi - \langle \phi \rangle_0) - \rho^2 \nabla^2 \phi = \frac{2}{n_{eq} \sqrt{\pi}} \int_0^\infty dE \sqrt{E} f - 1 \]

→ Polarization Charge as Fluid Vorticity!

- $\delta f^2$ Balance (Recall: $\langle \delta q^2 \rangle$ for fluid model)

\[ \partial_t \langle \delta f^2 \rangle + \partial_r \langle \bar{v}_r \delta f^2 \rangle - \langle \delta f C(\delta f) \rangle = -\langle \bar{v}_r \delta f \rangle \langle f \rangle' \]

\[ \Rightarrow \int \sqrt{E} dE \frac{1}{\langle f \rangle'} \{ \partial_t \langle \delta f^2 \rangle + \partial_r \langle \bar{v}_r \delta f^2 \rangle - \langle \delta f C(\delta f) \rangle \} = -\langle \bar{v}_r \delta n_i \rangle \]

GK Poisson + Taylor + Flow ↔ Vorticity Flux Enters!

\[ \delta \phi - \nabla^2 \delta \phi = \frac{2}{n_{eq}} \int \sqrt{E} dE \delta f_i = \delta n_i \Rightarrow \langle \delta n_i \bar{v}_r \rangle = -\langle \bar{v}_r \nabla^2 \delta \phi \rangle = \partial_t \langle V_\theta \rangle + \nu \langle V_\theta \rangle \]

yields...
C-D Thm. for Darmet Model \( (KPD \equiv \int \sqrt{E} dE \langle \delta f^2 \rangle / \langle f \rangle') \)

\[
\partial_t \{KPD + \langle V_\theta \rangle \} = -\nu \langle V_\theta \rangle - \int dE \sqrt{E} \left[ \frac{1}{\langle f \rangle'} \left\{ \partial_r \langle \tilde{v}_r \delta f^2 \rangle + \langle \delta f C(\delta f) \rangle \right\} \right]
\]

\( KPD \equiv \int dE \sqrt{E} \langle \delta f^2 \rangle / \langle f \rangle' \), Kinetic ‘Phasetrophy’ Density

In non-resonant limit:
\[
\delta f_k = -\tilde{v}_{rk} \langle f \rangle' / (-i \omega_k), \quad KPD \sim \int \sqrt{E} dE \langle \tilde{v}_r^2 \rangle_k \langle f \rangle' / \omega_k^2 \sim -k_0 \epsilon / \omega_k
\]

→ corresponds to kinetic pseudomomentum

→ reduces to wave momentum in small amplitude limit, \( P_k = kN_k \),
\[
N_k = (\partial \epsilon / \partial \omega)|_{\omega_k}(|E_k|^2 / 8\pi)
\]

Non-Acceleration: Absent KPD/spreading or collisional dissipation,
\textbf{cannot} accelerate or maintain Z.F. with stationary KPD

→ Momentum Freezing-in Law for ZF and QP gas!!
Kinetic ‘Phasetrophy’ Density - What Does it Mean?

- c.f. Antonov Energy Principle for collisionless Self-Gravitating Matter (Stellar Dynamics, $F'_0 = \partial F_0 / \partial E$)

$$\delta W = \left[ \int d^3x d^3v \frac{\delta f^2}{|F'_0|} \right] - G \int d^3x d^3x' d^3v d^3v' \frac{\delta f(x, v) \delta f(x', v')}{|x - x'|}$$

→ KPD corresponds to fluctuation dynamic pressure
→ opposes self-gravity in usual Jean’s balance

- Formulate as response to external force
- Appears in Kruskal-Oberman Kinetic Energy Principle
Energetics → Flux Drive

- recall for kinetic energy principle → calculate response to external force \( \nabla \phi_{\text{ext}} \)

- \( \therefore \) for flux drive → calculate phasetropy response to applied heat flux

\[
Q = -\chi_{\text{neo}} \nabla \langle T \rangle + \langle \tilde{V}_r \tilde{T} \rangle \\
= -\chi_{\text{neo}} \nabla \langle T \rangle + \partial_t \left( \int dE \sqrt{EE} \frac{\langle \delta f^2 \rangle}{\langle f \rangle'} \right) \\
+ \int dE \frac{\sqrt{EE}}{\langle f \rangle'} \left( \partial_r \langle \tilde{V}_r \delta f^2 \rangle + \langle \delta f C(\delta f) \rangle \right)
\]

- identifies \( \int dE \sqrt{EE} \langle \delta f^2 \rangle / \langle f \rangle' \sim T_i \langle \tilde{q}^2 \rangle / \nu_* - \langle \delta f^2 \rangle \) moment - as central to \( Q \) balance

- cannot support heat flux in stationary state, absent collisions and/or phasetropy spreading/mixing
Flux Drive, cont’d

Observe:

\[ \partial_t \{KPD - \langle V_\theta \rangle \} = \]

\[ Q = -\chi_{neo} \nabla \langle T \rangle + \ldots \]

\{ \text{define coupled equations} \}

\{ \text{for } \langle \delta f^2 \rangle / \langle f \rangle', \text{ its moments, flow} \}

\text{fixed } Q \leftrightarrow \text{ closure}

\text{profiles, via Poisson + mean field equation}

\therefore \text{dynamics described by moments of kinetic phasetrophy distribution! } \langle \delta f^2 \rangle \rightarrow \text{emerges as fundamental}

\text{resembles quasi-particle gas dynamics, i.e.}

\text{Q.P. momentum } k_\theta N \rightarrow \langle \delta f^2 \rangle / \langle f \rangle'

\text{Q.P. energy } \omega_k N \rightarrow E \langle \delta f^2 \rangle / \langle f \rangle'

\{ \text{coupled hierarchy} \}

\text{NO a priori, tie to linear instability dynamics } \rightarrow \text{suitable to describe granulations, structure, etc}
Partial Summary: What Did We Get?

- C-D Thms. for HW and Darmet Model

$$\partial_t \{ WAD + \langle V_\theta \rangle \} = -\langle \tilde{V}_r \tilde{n} \rangle - \frac{1}{\langle q \rangle'} \left\{ \partial_r \langle \tilde{V}_r \delta q^2 \rangle + \mu \langle (\nabla \delta q)^2 \rangle \right\} - \nu \langle V_\theta \rangle$$

$$\partial_t \{ KPD + \langle V_\theta \rangle \} = -\int dE \sqrt{E} \left[ \frac{1}{\langle f \rangle'} \left\{ \partial_r \langle \tilde{v}_r \delta f^2 \rangle + \langle \delta f C(\delta f) \rangle \right\} \right] - \nu \langle V_\theta \rangle$$

- $\left\{ \begin{array}{l}
  WAD = \langle \delta q^2 \rangle / \langle q \rangle' \propto -k_\theta N_k \\
  KPD = \int dE \sqrt{E} \langle \delta f^2 \rangle / \langle f \rangle' \propto -k_\theta N_k
\end{array} \right.$ in non-resonant limit

- Spreading, $\partial_r \langle \tilde{v}_r \delta q^2 \rangle, \partial_r \langle \tilde{v}_r \delta f^2 \rangle \Leftrightarrow$ ZF momentum Evolution

- $\delta q \propto \langle q \rangle', \delta f \propto \langle f \rangle'$ in non-resonant limit:

  What of Resonant Limit? WAD, KPD not well-defined??
Single Structure Evolution in Phase Space with ZF

- consider localized $\delta f$ in phase space, ‘hole,’ ‘blob’ (Dupree, B & B) $\rightarrow$ strongly resonant limit

\[
\delta f_i = \delta f_i \left( \frac{x - x_0}{\Delta x}, \frac{E - E_0}{\Delta E} \right)
\]

- Structure Growth, Dupree ’82: $\partial_t \int dv \delta f_i^2 = -2\langle \tilde{V}_r \tilde{n}_i \rangle \frac{\partial \langle f \rangle}{\partial x} |_0$

- Key: net dipole moment $\int dx \sum_{\alpha} q_{\alpha} n_{\alpha}(x)x$ invariant
  $\rightarrow$ include polarization contribution

- Structure Growth + net dipole invariance + Taylor $\Rightarrow$

\[
\frac{\partial}{\partial t} \left\{ \int dE \frac{\delta f_i^2}{2\langle f \rangle}' |_0 + \langle V_\theta \rangle \right\} = -\nu \langle V_\theta \rangle - \langle \tilde{V}_r \tilde{n}_e \rangle
\]

phase space blob/hole can’t avoid Z.F. coupling due flux of polarization charge
Some Observations

i) $\delta f_i$ structure evolution and C-D theorem for HW

$$
\frac{d}{dt} \left\{ \frac{1}{2\langle f \rangle} \int dv \delta f_i^2 + \langle V_{\theta} \rangle \right\} = -\nu \langle V_{\theta} \rangle - \langle \tilde{V}_r \tilde{n}_e \rangle
$$

$$
\partial_t \{(WAD) + \langle V_{\theta} \rangle \} = -\nu \langle V_{\theta} \rangle - \langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle
$$

Clear correspondence!

commonality: $f \leftrightarrow q$ conservation; Kelvin’s Theorem

$\rightarrow$ flow momentum + Generalized Pseudomomentum conserved!

ii) obtain stationary $\langle V_{\theta} \rangle$ for fixed KPD:

$$
\langle V_{\theta} \rangle = -\frac{1}{\nu} \langle \tilde{V}_r \tilde{n}_e \rangle = \frac{1}{\nu} \left( D[\delta f] \frac{\partial \langle n_e \rangle}{\partial x} \right)
$$

- $\delta f_i$ scattering off electrons scatters polarization charge and pumps Z.F.

- localized structure may excite larger scale flow
Zonal Flows and Phase Space Turbulence

- recover generic structure from Dupree-Lenard-Balescu theory
  \[ \partial_t \langle \delta g^2 \rangle + \frac{1}{T_{1,2}} \langle \delta g^2 \rangle = P_{1,2} \]
  dispersion \quad \text{production, } \partial \langle f \rangle / \partial t

- \[ \partial_t \langle f \rangle = -\partial_r \left[ -D_r \partial \langle f \rangle / \partial r + \mathcal{F}\langle f \rangle \right] \]
  diffusion \quad \text{dynamical friction}

- envelope coupling → Reynolds stress/vorticity flux
  contribution via screening in dynamical friction

- novel effect
  → beyond intensity damping, cross-phase mod.
  → Z.F. drag on clump granulation → Wake

- shearing → resonance: \[ \omega - \omega_D E - k_\theta \langle V_E \rangle' x \]
  → can maintain resonance with \((E, r)\) dual interchange
  → no trivial diffusion - drag cancellation
VI.) The Current Challenge:

Avalanches, ‘Non-locality’
and the Zonal Flows
⇒ the PV Staircase
Analogy with geophysics: the ‘$\mathbf{E} \times \mathbf{B}$ staircase’

\[ Q = -n\chi(r)\nabla T \Rightarrow Q = -\int \kappa(r, r')\nabla T(r') \, dr' \]

- ‘$\mathbf{E} \times \mathbf{B}$ staircase’ width $\equiv$ kernel width $\Delta$
- coherent, persistent, jet-like pattern $\Rightarrow$ the ‘$\mathbf{E} \times \mathbf{B}$ staircase’

Dif-Pradalier, Phys Rev E. 2010
The point:

\[ Q = - \int dr' \kappa(r, r') \nabla T(r') \]

- fit:
  \[ \kappa(r, r') \sim \frac{S^2}{(r - r')^2 + \Delta^2} \]
  \[ \rightarrow \text{some range in exponent} \]

- \( \Delta \gg \Delta_c \)
  \[ \text{i.e. } \Delta \sim \text{avalanche scale} \quad \Rightarrow \quad \Delta_c \sim \text{correlation scale} \]

- Staircase `steps' separated by \( \Delta! \)

N.B.
- The notion of a `staircase' is not new - especially in systems with natural periodicity (i.e. NL wave breaking ....)
- What IS new is the connection to stochastic avalanches, independent of geometry

\[ \rightarrow \text{What is process of self-organization linking avalanche scale to zonal pattern step?} \]

i.e.
How extend predator-prey feedback model to encompass both avalanche and zonal flow staircase?

Self-consistency is crucial!
A Possible Road Forward...

→ The idea:

Avalanches ↔ shocklets → Burgers turbulence, etc (cf: Hwa, Kardor, P.D., Hahm)

$$\delta p \sim V$$

→ staircase → pinned or punctuated profile jumps?

↔ pinning enforced by shear suppression → shear staircase (via feedback)

→ strategy:

- [avalanches + shear suppression] + [drift-zonal turbulence driven by near marginal gradient]

→ staircase?

- Test: Is staircase structure robust to changes in noise spectrum?

[N.B.: staircase not linked to q resonances]
The Model:

- Profile deviation from criticality $\delta p$ (Hwa, Kardar; P.D., Hahm)

$$\partial_t \delta p + \partial_x \{ \alpha_0 f(V'_E) \delta p^2 - D \partial_x \delta p \} = \tilde{S} \quad \rightarrow \text{noise, variable spectrum}$$

- Intensity Evolution $\langle N \rangle$

$$\partial_t \langle N \rangle = \frac{\partial}{\partial k_r} D_{kr} \frac{\partial}{\partial k_r} \langle N \rangle + \gamma_0 \delta p f(V'_E) \langle N \rangle + C(N) \quad \rightarrow \text{modulated stress} \rightarrow \text{compute via WKE}$$

- Flow $\langle V_y \rangle$

$$\partial_t \langle V_y \rangle + \partial_x \langle \delta (\tilde{V}_x \tilde{V}_y) \rangle = -\mu \langle V_y \rangle$$

ZF scattering

profile deviation from marginal

→ drive ↔ avalanche

avalanching
diffusion (i.e. neo)

$\tilde{S}$

$f(V'_E) = \frac{1}{[1 + \sigma V'_E^2 / V_0^2]^\gamma}$
- For modulation:

\[
\partial_t \tilde{N} + v_{gr} \partial_x \tilde{N} + |\gamma (\delta p)| \tilde{N} = \frac{\partial}{\partial x} (k_0 \tilde{V}_E) \frac{\partial \langle N \rangle}{\partial k_r}
\]

Related?:

- coupled spatial, spectral avalanches: P.D., Malkov.; Kim, P.D.

- structure of PV flux?: (Hsu, P.D.)

\[
\langle \tilde{V}_y \tilde{u} \rangle = -D \partial_y \langle u \rangle \quad \text{v.s.} \quad \langle \tilde{V}_y \tilde{u} \rangle = D \partial_y \langle u \rangle + \mu \partial^3_y \langle u \rangle
\]

\downarrow \text{diffusion} \quad \quad \rightarrow \text{negative-diffusion} \quad \rightarrow \text{hyper-diffusion}

\Rightarrow \text{ZF as spinodal phenomena}
VII.) Open Issues and Plans
Some interesting problems:

a.) Specific Extensions - Theory:
  - Kinetic predator-prey models and fluctuation entropy, relation to flows (Kosuga, et. al.)
  - PV ‘cascade’ via non-local straining (Gurcan, et. al.)
  - C-D theorem for parallel flows (McDevitt, et. al.)
  - Models of turbulence spreading (A. Ulvestad, et. al.) → i.e. how shear induces wave packet propagation
  - β-plane MHD, drift-Alfven turbulence (S. Tobias, et. al.)

magnetic field inhibition of PV mixing?
b.) More general theoretical issues:

- Relative spreading: $E(r, t) \text{ vs } \Omega(r, t)$
- Is there a general principle?
  - “Minimum enstrophy” (Bretherton)
  - “Most probable state” (Lynden-Bell)
  - “PV homogenization” (Batchelor, ...)

N.B. All tacitly involve mixing of locally conserved PV.

- Macro-patterns, i.e. the staircase (Dif-Pradalier, et. al. 2010)
  - what is the self-organization principle linking avalanches and staircase?
c.) More practical matters:

• Extract information from phase lag, during slow ramp-up

• 0D → 1D : space – time evolution of turbulence profile
  → population density evolution, staircase

• Critical parameters re: transition → macro-micro connection
  – Relation to LRC → $E_{ZF}/E_{DW}$ ratio, etc. ⇒ quantitative result!?

  – Bursts and bistability

  – $1/\tau_{c,turb} \text{ vs } \omega(k) \text{ GAM vs } \langle V_E \rangle$ ’ GAM → NL GAM dynamics

  – Relation to ‘benevolent’ pedestal modes: WCM, QCM, EHO, …
• $E_r$ reduced ZF screening $\rightarrow$ bias $\rightarrow$ threshold reduction and control

• ‘Holistic’ studies $\rightarrow$ examine trade-offs in optimizing access to H-phase

• Is there a unique trigger mechanism or pathway to LH transition? Need there be? How fit in I-mode?
  – Dynamics of ITB transition: similarities, differences?
  – Slow back transitions?
  – Better understanding of resonant $q \Leftrightarrow$ ZF link $\rightarrow$ intensity profile ?!