Aside: Convection in tall, thin box (question from LE).

Recall:
- Lesson: From study of Rayleigh-Benard Convection, it is that boundary conditions matter!
- Recall case study was "short, wide" box
  - dismissal side
  - focus on stress-free no slip top/bottom

Robert 0
d

1. Increasing diffusive damping due high \( \lambda \)
2. Increasing diffusive damping due small vertical scales (\( \sim \) high \( \lambda_2 \)).
tall, thin box

Now:

- Ignore top/bottom boundary
- No-slip on side walls is key and only relevant case

\[ v_z \rightarrow \omega_j \quad \omega(z) = \omega(a) = 0 \]

- Long thin cell must fight no-slip b.c.'s on side walls
- Higher \( k_h \) will introduce high \( k_h \) damping.

- Expect high \( R_a \) due to side wall no-slip even at \( k_h \rightarrow k_h_{\text{min}} \)

- Curvature of \( R_a \) vs \( k_h \) curve TBD. Speculate rather weak curvature.

- As side surface area \( \propto \) top surface area expect top no-slip vs others free comparison not significant.
Introduction to Rotating Convection
- Intermezzo on Landau Theory

Basics of Waves

\[ \text{Ax} \] Convection + Rotation
- see lecture 6 for freezing-in law
- Taylor-Proudman Thm.
  - Key pt: For \( R^2 \) large enough, flow is two-dimensionalized

- Inertial Waves
- Rotating Convection

\[ \text{P} \]
- Inertial Waves \( \rightarrow \) (radial) buoyancy waves in rotating fluid

\( \Rightarrow \) recall Problem 4, set 1:

Fluid rotates at \( \Omega \neq \frac{1}{2} \), \( \omega_0 = 0 \)
- (Convenience), \( k_1, k_2 \neq 0 \)

\[ \omega^2 = k_2 \left( \frac{4 \Omega^2}{k_2^2 + k_1^2} \right) / k_2^2 + k_1^2 \]

and, with radial b.c., eigenvalue.
Quick downshifting:

- From vorticity equation (T-P thm.)
  \[
  \frac{\partial \omega}{\partial t} + \nabla \cdot (\omega + 2\Omega) = 2\Omega \times \frac{\partial}{\partial x}
  \]
  \[
  + \frac{\partial}{\partial y}
  \]

So

\[
\frac{\partial}{\partial t} \omega_z = 2\Omega \partial_x \partial_z
\]

And, from EOM

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \times 2\Omega) = -\mathbf{\nabla} p + \mathbf{u} \times 2\Omega
\]

\[
\nabla \times \mathbf{u} = 0
\]

\[
\nabla \cdot (\nabla \times \mathbf{u}) - \nabla \times \frac{\partial^2 \mathbf{u}}{\partial t} = 0
\]

\[
+ \nabla \times \mathbf{u} \times \sum \mathbf{u} \times 2\Omega
\]

\[
\Omega - \nabla \times \frac{\partial^2 \mathbf{u}}{\partial t}
\]

\[
\nabla \times (2\Omega \partial_x^2 \mathbf{u}) - \mathbf{u} \cdot \nabla 2\Omega
\]
\[ - \mathbf{\nabla} \cdot \mathbf{V}_a = 2\Omega \nabla^2 \mathbf{\hat{u}}_a \]

\[ \partial_t \mathbf{\hat{u}}_a = 2\Omega \nabla^2 \mathbf{\hat{u}}_a \]

\[ \omega^2 = k_{\perp}^2 + \frac{l_2^2}{(k_{\perp}^2 + k_z^2)} \]

- Physical is rotating flow vortex fiber don't like being bent \((k_z \approx 0)\)
  - imposes energy penalty for motions with finite \(k_{\perp}\)
  - definite contribution to \(\mathbf{\omega}\)
- Rough picture is one of gyroscopic restoring force (conservation \(L_z\))

- analogous to Alfvén wave and field line bending in MHD
  \[ \omega^2 = k_{\parallel} V_A \]
- aside: \( \text{backward wave: } u_0 < 0 \)
- \( w = 0 \) finite \( k_\perp \) model
  (sheet layers)

\[ \Rightarrow \]

\( \text{as convection necessitates cellular overturning (i.e., finite } k_\perp \text{)} \)

\[ \Rightarrow \]

\( \text{low } k_\perp \text{ (i.e., min) motonic favored} \)

\( \Rightarrow \)

\( \text{tracks T-P thm. conclusion of 3D-rotation of the flow} \)

\( \Rightarrow \)

\( \text{suggests that rotation is (strong) stabilizing effect on convection.} \)

Also suggests that another dimensionless number enters comparing rotation to viscous dissipation

\( \Rightarrow \) naturally:

\[ \text{Taylor Number} \]

\[ T_a = 4 \frac{\bar{\delta}^2}{d^4} \left( \frac{\text{Re}}{\text{Re}_c} \right)^2 \]

\( d \equiv \text{box scale} \)
- Taylor number captures natural competition between rotation and viscous diffusion.
- Ta joins Ra, Pr as key parameter in convective stability theory.
- \( Ra_{crit} = Ra_{crit} \left( Pr, Ta \right) \) is now stability threshold parameter.
N.B. Ra, Ta both involve \( \Delta T \) but are distinct - \( \Delta T / d \) vs \( \Delta T \).

Can combine stationary convection and (short) wave calculations to obtain basic equations:

\[
\begin{align*}
\frac{\partial \Theta}{\partial t} &= \Theta W + k \nabla^2 \Theta \\
\frac{\partial}{\partial t} \nabla^2 W &= 9 \alpha \nabla^4 \Theta + r \nabla^2 \nabla^2 W - 2 \Omega \nabla^2 \omega_2 \\
\Theta + \omega_2 &= -2 \Omega \nabla^2 W \\
\text{(dervive)}
\end{align*}
\]

notation \( \omega \) before.
For 
mixed

For 

- Favor
- Taylor columns (thin) and
- Stabilizing effect of rotation evident

Far

- One can further specify $R_a$ as minimum ($\alpha = k_n h$)
\[ \text{Ray} = \text{Ray} (T_a) \]  
\( \text{Taylor} \# \) dependence

For all cases \{ no slip, stress free \}

\[ \ln \text{Ray} \]

\[ \text{Ray} \]

\[ \ln T_a \]

demonstrates stabilizing effect rotation.

- can develop variational principle for exchange of stabilities core. Need also treat over-stable limit.

- Cultural Aside: Magneto-convective

Dynamo-generated magnetic fields can feed back on convection. A particular example is sunspots (i.e. dark - lower \( T \)) - convection (weekend), which are
N.B.1. Comment on dissipation effects.

Associated with strong magnetic fields

\[ \text{Magnetocconductivity} \]

- Similar to rotating problem by with 'bending' element due to \( B \)
  \[ \Rightarrow \text{i.e. energy density} \]
  - Alfvén wave replaces (magnetic) wave
  - EXPULSION CAN OCCUR etc.

Enough linear stability theory!

[Underlined: Intermediate]

Commentary: Landau Equation/Law
  \[ \Rightarrow \text{to date, linear stability} \]

\[ \Rightarrow \text{Nonlinear evolution}\]
  \[ \Rightarrow \text{Difficult problem especially for turbulence...} \]

\[ \Rightarrow \text{Seek characterize weekly nonlinear evolution i.e. Far Flow shear/tilt instability (stabilized by viscosity)} \]
then if $Re < Re_{crit}$ critical Reynolds number for instability. Equivalently, for convection, $Ra_{crit}$ exists. Then if

$$Re = Re_{crit} + \delta Re \quad \frac{\delta Re}{Re} \ll 1$$
$$Ra = Ra_{crit} + \delta Ra \quad \frac{\delta Ra}{Ra} \ll 1$$

Can one represent dynamics of some general form? Especially near marginal point?

Analogy: → Ginzburg–Landau theory

Leverage: → Symmetry !

So, if consider $N-S-E$ retaining nonlinear terms: $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}'$

$$\frac{\partial \mathbf{V}'}{\partial t} + (\mathbf{V}_0 \cdot \nabla) \mathbf{V}_0 + \mathbf{V}_0 \cdot (\nabla \mathbf{V})$$

$$\nabla \cdot \mathbf{V}' = \mathbf{V}' \cdot \nabla \mathbf{V}_0 + \mathbf{V}_0 \cdot \nabla \mathbf{V}_0$$

NL 

"..."
- For $0 < R < R_{cut}$, we might expect few or no modes relevant just above marginality.

\[ V_i = \psi_i(x) e^{(\pm ic_i t + \omega_i t)} \]

In general, we develop oscillating correction terms.

- Often, spatial carrier

\[ V_i = \psi_i(x) e^{i\chi_i x} e^{-i\omega_i t} e^{i\chi i t} \]

- Envelope

Then for convenience:

\[ V_i = A_i (\psi_i, \psi_i^*) \]

- Amplitude

Linear growth.

\[ \frac{d}{dt} |A|^2 = 2 \delta, |A|^2 + O(A^3) + o(A^4) \]

etc.
Now: For $\gamma \gg Re \sim Re^t$

$\mathcal{W}r \rightarrow \text{finite}$

i.e. time scale separation between growth and oscillation, i.e. $\mathcal{W}r > \gamma$

Then $A^2 = \int_{T/2}^{T} \text{d}t$

\[ T = \text{period of oscillations} \]

Now: $\mathcal{V}_1 \cdot (\mathcal{V}_1^* \mathcal{V}_1) \rightarrow 0$ single mode

(multi-mode $\leftrightarrow$ resonant coupling)

So $O(A^3)$ contribution vanishes.

Now $O(A^4)$, i.e.

\[ \mathcal{V}_1 \cdot \left[ \begin{array}{c} \mathcal{V}_2^* \mathcal{V}_2 \\ \mathcal{V}_2 \end{array} \right] \]

but $\mathcal{V}_2 \sim \mathcal{V}_1$

\[ \Rightarrow O(A^4) = \mathcal{V}_1 \cdot \mathcal{V}_1^* \mathcal{V}_1 \mathcal{V}_1 \sim 2 \left[ A^1 \right]^4 \]
\[2|A|^2 = 2\xi |A|^2 - \varepsilon |A|^4\]

- Landau equation
  - obvious structural / conceptual similarity to Ginzburg - Landau theory.
  - physical / mode feedback on profile, i.e.
    \[\n    \mathbf{v} \cdot \nabla \mathbf{v} (x) = \mathbf{v} \cdot \nabla \left( \mathbf{v}_0 + \tau \mathbf{v} (x) \right) \n    \]
    mean profile (driving shear)
    \[|A|^4\]
    non-linearity acts to deplete free energy source.
    - predicts saturation at:
      \[|A|^2 \approx 2\xi / \lambda \approx (Re - Re_{crit})\]
      so \[|A| \sim (Re - Re_{crit})^{1/2}\] (stationary state)
\( \Lambda \) \{ super-critical \}

\( \text{Re} \)

→ calculation of \( \chi \) in detail using reductive perturbation theory
→ come back for 221 a c)

\( \Lambda \) spring → extensive discussion of weakly nonlinear convection rolls

but:

→ \( \chi \) need not be positive. In this case \( O(\lambda^4) \) contribution →
growth/destabilization. So:

→ need \( O(\lambda^6) \) to saturate

→ strong enough perturbations can grow even if linearly stable
In this case: Landau Eqn. becomes

\[ 2\lambda_1 |A|^2 = 2\alpha_1 |A|^2 + \kappa_1 |A|^4 - \beta_1 |A|^6 \]

\[ \Rightarrow -2\alpha_1 |A|^2 + \kappa_1 |A|^4 - \beta_1 |A|^6 \]

So, for stationary state:

\[ |A|^2 \left[ -2\alpha_1 + \kappa_1 |A|^2 - \beta_1 |A|^4 \right] = 0 \]

\[ |A|^2 = \frac{1}{2\beta} \left( \frac{\kappa_1 |A|^2 - 4(2\alpha_1)}{2\beta} \right)^{1/2} \]

\[ |A|^2 = 0 \]

(2 roots)

Subcritical (non-linear) instability / bifurcation possible if \( |A|^2 > \frac{2|\alpha|}{|\kappa|} \)

\( (\delta_i < 0) \) metastable

\[ |A| \]

unstable

\[ \delta_0 \quad x < 0 \]
\[ |A_1| \rightarrow \eta \] order parameter

supercritical bifurcation \( \Leftrightarrow \) 2nd order transition

subcritical bifurcation \( \Leftrightarrow \) 1st order transition

(both exhibit metastable state)

\[ \Delta \left( R e \right) \rightarrow \Delta (T-T_c) \] factor

- can also develop London theory for phase and amplitude

i.e. \( V \rightarrow A e^{i\phi} \)

\[ \Delta \text{ phase dynamics} \]

\( \rightarrow \) CGL system