PHYSICS 210A : STATISTICAL PHYSICS
FINAL EXAM

(1) Provide clear, accurate, and brief answers for each of the following:

(a) A particle in $d = 3$ dimensions has the dispersion $\varepsilon(\boldsymbol{k}) = \varepsilon_0 \exp(ka)$. Find the density of states per unit volume $g(\varepsilon)$. Sketch your result. `[4 points]`

(b) What is the Maxwell construction? `[4 points]`

(c) For the free energy density $f = \frac{1}{2}am^2 - \frac{1}{3}ym^3 + \frac{1}{4}bm^4$, what does it mean to say that 'a first order transition preempts the second order transition'? `[4 points]`

(d) A system of noninteracting bosons has a power law dispersion $\varepsilon(\boldsymbol{k}) = A\, k^\sigma$. What is the condition on the power σ and the dimension d of space such that Bose condensation will occur at some finite temperature? `[4 points]`

(e) Sketch what the radial distribution function $g(r)$ looks like for a simple fluid like liquid argon. Identify any relevant length scales, as well as the limiting value for $g(r \to \infty)$. `[4 points]`

(f) ν moles of ideal gaseous argon at an initial temperature T_A and volume $V_\mathrm{A} = 1.0\,\mathrm{L}$ undergo an adiabatic free expansion to an intermediate state of volume $V_\mathrm{B} = 2.0\,\mathrm{L}$. After coming to equilibrium, this process is followed by a reversible adiabatic expansion to a final state of volume $V_\mathrm{C} = 3.0\,\mathrm{L}$. Let S_A denote the initial entropy of the gas. Find the temperatures $T_\mathrm{B,C}$ and the entropies $S_\mathrm{B,C}$. Then repeat the calculation assuming the first expansion (from A to B) is a reversible adiabatic expansion and the second (from B to C) an adiabatic free expansion. `[4 points]`

(g) Explain how the Maxwell-Boltzmann limit results, starting from the expression for $\Omega_\mathrm{BE/FD}(T, V, \mu)$. `[4 points]`

(h) For the one-dimensional spin-1 Ising model $\hat{H} = -J \sum_n S_n\, S_{n+1}$, where each $S_n \{-1, 0, 1\}$, write down the transfer matrix. `[4 points]`

(2) The density of states per unit volume for a particle in three space dimensions is

$$g(\varepsilon) = \frac{\varepsilon\,(\varepsilon^2 + \Delta^2)}{\Omega\,\Delta^4}\,\Theta(\varepsilon) \quad .$$

(a) What are the dimensions of the constant Ω? `[6 points]`

(b) Find the single particle dispersion $\varepsilon(\boldsymbol{k})$. `[7 points]`

(c) Assuming the particles obey photon statistics find their density $n(T)$. `[7 points]`

(d) Assuming the particles are bosons, find the Bose condensation temperature $T_{\rm c}(n)$. [7 points]

(e) Assuming the particles are fermions, find the Fermi energy $\varepsilon_{\rm F}(n)$. [7 points]

(3) Consider the three-state ($\mathbb{Z}_3$) clock model, with Hamiltonian

$$H = -J\sum_{\langle ij\rangle} \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j \quad ,$$

where the interaction is between all unit vectors $\hat{\boldsymbol{n}}_i$ and $\hat{\boldsymbol{n}}_j$ lying on neighboring sites on a regular lattice of coordination number z. Each $\hat{\boldsymbol{n}}_i$ can take one of three possible values $\{\hat{\mathbf{e}}_1\,,\,\hat{\mathbf{e}}_2\,,\,\hat{\mathbf{e}}_3\}$, where

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{x}} \quad , \quad \hat{\mathbf{e}}_2 = -\tfrac{1}{2}\,\hat{\mathbf{x}} + \tfrac{\sqrt{3}}{2}\,\hat{\mathbf{y}} \quad , \quad \hat{\mathbf{e}}_3 = -\tfrac{1}{2}\,\hat{\mathbf{x}} - \tfrac{\sqrt{3}}{2}\,\hat{\mathbf{y}} \quad .$$

In service of analyzing this model, consider the variational density matrix $\varrho_N(\hat{\boldsymbol{n}}_1,\ldots,\hat{\boldsymbol{n}}_N) = \prod_i \varrho_1(\hat{\boldsymbol{n}}_i)$, where the single site variational density matrix is

$$\varrho_1(\hat{\boldsymbol{n}}) = \frac{1+2u}{3}\,\delta_{\hat{\boldsymbol{n}},\hat{\mathbf{e}}_1} + \frac{1-u}{3}\,\delta_{\hat{\boldsymbol{n}},\hat{\mathbf{e}}_2} + \frac{1-u}{3}\,\delta_{\hat{\boldsymbol{n}},\hat{\mathbf{e}}_3} \quad ,$$

wjere u is the variational parameter.

(a) What is the allowed range for u? Show that the density matrix is appropriately normalized. [4 points]

(b) Find the variational energy $E(u) = \mathsf{Tr}\,(\varrho_N H)$. [6 points]

(c) Find the entropy $S(u) = -k_{\rm B}\,\mathsf{Tr}\,(\varrho_N \ln \varrho_N)$. [6 points]

(d) Adimensionalize by defining $f = F/NzJ$ and $\theta = k_{\rm B}T/zJ$ and find the dimensionless free energy density $f(u,\theta)$. Do you expect a first or second order transition? Why? [6 points]

(e) Find the self-consistent mean field equation for u. [6 points]

(f) Analyze the model keeping only terms up to order u^4 in $f(u,\theta)$. Find the location of the phase transition and remark on whether it is first or second order. [6 points]

The following low order Taylor expansion may prove useful:

$$(1+\varepsilon)\ln(1+\varepsilon) = \varepsilon + \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{6}\varepsilon^3 + \tfrac{1}{12}\varepsilon^4 + \mathcal{O}(\varepsilon^5) \quad .$$

(4) Write a well-defined expression for the greatest possible number expressible using only five symbols. *Examples:* $1+2+3\,,\,10^{100}\,,\,\Gamma(99)$. [50 quatloos extra credit]