Lecture 1 - Buckingham Pi Theorem

- Systematics of Dimensional Analysis

- Context: Drag

- Here consider systematics of dimensional analysis: scaling arguments.

- Why scaling?

  - Real problems - nonlinear
  - Many degrees freedom

  - Need estimation (guided to computation)
    - Computers are poor at asymptotics
    - But asymptotic Fundamentals

- Common issues:
  - Concept in
    - Self-similarity - problems where phenomena look the same over broad range of scales
    - E.g., heat wave, turbulence

- Scaling relations constitute "the answer"
→ emergent scale ⇒ i.e. nonlinear dynamics defines new scale⇒ e.g. Reynolds scale i.e. usually lin vs nonlinear balance.

→ new physics ⇒ it happened!

→ boundary layers ⇒ transition between scaling regimes different

⇒ no slip b.c. \( \frac{\partial u}{\partial y} = 0 \) sur.

⇒ potential flow far from body

⇒ boundary layer near body viscosity important

⇒ different scalings which must be matched.

⇒ Basic Question:

⇒ How formulate systematic of dimensional analysis? Limitations?

⇒ When might it fail?
Discuss in context of familiar example:

Drag on sphere in Fluid? Body

\[ \vec{F}_d \]

\[ \vec{V} \]

Preliminaries

- How is drag quantified, i.e., sure kept?
  - Drag coefficient speed
  - \[ C_D = \frac{F_d}{\frac{1}{2} \rho V^2 A} \]
    - Coefficient
    - Surface area
    - \[ C_D = C_D(Re, Ma, shape...) \]

"Idea of \( C_D \):

\[ P \approx \rho V^2 A (At) \]

Change in momentum pressure - force/area

but \( C_D \approx \frac{P}{At} \) of ball
What really is drag?

Consider frame where:

- Body fixed
- Fluid moving

Drag = rate of removal of momentum from moving fluid by immersed body

Removal → Transmission in boundary surface layer

→ No-slip condition / viscosity leaves wake → water acts to fill in dead region
\[ \frac{d}{dt} \mathbf{F} \sim \rho A \mathbf{u}^2 \]
\[ \sim C_D \rho A \mathbf{u}^2 \]

- \( C_D \) captures additional physics
- Why additional physics? \( \rightarrow \) see \( \mathbf{3} \)
- (Why not formula valid?)

\( \Rightarrow \)
- drag \( \Rightarrow \) wake
- ahead behind asymmetry

\( \Delta \) wake

- wake \( \Rightarrow \) irreversibility
- see big contrast

- wake is consequence of
- no-slip b.c.

\[ \partial_\tau \mathbf{u}_t (\mathbf{x}) = 0 \]

- (a potential flow) \( \nabla^2 \phi = 0 \)

\( \mathbf{u}_t \) can be large \( \Rightarrow \) no wake

\[ \Delta \mathbf{u}_1 \ni \Delta \mathbf{u}_2 \quad \Delta \mathbf{u} \ni \Delta \mathbf{u}_2 \text{ definition} \quad \text{same upstream downstream} \]
Ideal Fluid

Euler eq.
Reversible

Viscous Fluid
NS Eq.
Irreversible

vs -> clearly different
- wake and thus drag are due to 
  \[ \text{viscosity} \]
  \[ \text{irreversibility} \]
  \[ \text{verticality} \]
  \[ \text{highlight role of B.L. around object} \]

A.B. highlights role of boundary layers → drag can be independent of Re, but viscosity needed to satisfy b.c.

- additional physics → model

Model → Navier-Stokes Equation

\[ \rho \left( \partial_t + \mathbf{V} \cdot \nabla \right) \mathbf{V} - \nabla P = \mu \nabla^2 \mathbf{V} \]

advection, linear, 2nd order

nonlinear \[ \nu = 0 \text{ is singular} \]
Compressible NS Eqn:

\( \frac{\partial u}{\partial t} = -\nabla p + \rho \nabla \cdot \nabla u + (y + u) \frac{\partial u}{\partial x} \)
\[ S = \text{const}, \quad ds = 0 \]

\[ \Rightarrow \frac{dP}{ho} = 
\text{Vol} - \text{IdS} = d\omega \]

- Mostly
  - \( \rho \cdot \nabla = 0 \) \text{ determines pressure}
  - \( \rho \cdot \nabla = 0 \) \text{ need eqn. state.}

\text{Key Parameters}

\[ \frac{\rho \cdot \omega}{\sigma} = \text{Reynolds}\ # \]

\[ \Rightarrow \text{Re} \sim \frac{\nu \cdot L}{\omega} \]

expresses ratio of nonlinear inertial term to linear, diffusive term.

\text{Re} < 1 \rightarrow \text{viscous flow - Stokes}

\text{Re} > 1 \rightarrow \text{laminar flow - Blasius}

\text{Re} > 10^5 - 10^6 \rightarrow \text{turbulent flow.}
see especially basic trends

\[ C_d \sim \frac{1}{Re} \]

\[ F_d \sim \frac{1}{Re} \]

\[ \delta_{sw} + \nu \frac{d\delta}{dt} - \frac{\nu}{Re} \]

large Re \rightarrow turbulent

\[ C_d \sim Re^{0.5} \rightarrow \text{notable why?} \]

How recover scaling?
Wake structure:

Re < 10
creeping flow

10 < Re ≤ 40
attached vortices

40 < Re < 200,000
vortex tail

Re > 200,000
turbulent wake
Buckingham's Pi Theorem

- E. X. First

- How do dimensional analysis systematically?

1. Identify physically relevant variables

2. Identity variables of these

\[ \rightarrow \text{required answer should depend on relevant variables and be listed among.} \]

So, for drag on sphere:

- Steady
- Temp.
- Dim

\[ R \rightarrow \text{Length} \quad L \]

\[ V \rightarrow \text{Velocity} \quad L/T \]

\[ \nu \rightarrow \text{Viscosity} \quad L^2/T \quad \text{Water} \rightarrow 10^{-3} \text{m}^2/\text{s} \]

\[ \rho \rightarrow \text{Fluid density} \quad M/L^3 \]
\[ F_d \rightarrow \text{drag force} \quad F_d \sim ML^2/T^2 \]

2. Count under dimensions

\[ \Rightarrow M, L, T \]

so \( n = 5 \) quantities

\( r = 3 \) independent

\[ \Rightarrow \]

3. \( n - r = 2 \) dimensionless ratios possible

\[ 1 \text{ involves } F_d, \]

\[ \Rightarrow \]

\[ \Pi \text{ theorem} \]

\[ n \rightarrow \text{indep. quantities} \]

\[ r \rightarrow \text{indep. dimensions} \]

\[ \therefore \quad n - r \text{ dimensionless ratios} \]
\[ \Pi_c = \Phi(\Pi_1, \Pi_2, \ldots, \Pi_i, \Pi_{i+1}, \ldots, \Pi_n) \]

i.e. can express dimensionless ratios in terms each other.

Practical Matters:

→ One \( \Pi \) involves answer

→ Other \( \Pi \)'s involve relevant dimensionless parameters in model.

N.B. here - model useful.

→ Use insight # values (i.e. experiment) to eliminate some \( \Pi \) variables

(i.e. \( \ll a \), or \( \gg d \))

back to sphere example:

→ \( 2 = \Pi \) variables.

where \( \Pi_1 = \Phi(\Pi_2) \)
\[ \pi_1 = \frac{F_d}{\rho_0 A^2 v^2} \rightarrow \text{involves} \] \[
\pi_2 = Re = \frac{vR}{v} \rightarrow \text{suggested as natural} \]

and \( \pi_3 \) theorem \( \rightarrow \)

\[ \pi_1 = \pi_1 (\pi_2) = f(\pi_2) \]

\[ \boxed{\text{St theorem result}} \]

- basic result
- plausible that drag depends on \( Re \)
- not a unique answer

Now, key step:

- explore asymptotic behaviour, i.e.

\[ Re \ll 1, \quad Re \gg 1? \]
$Re \ll 1$

- have $F_d \sim \frac{\rho R^2 V^2}{2} F(Re)$

- $Re \ll 1 \implies$ expect $F_d \sim V$
  
  i.e. proportional to velocity

so

$F(Re) \sim \frac{1}{Re}$, for $Re \ll 1$

$F_d \sim \frac{\rho R^2 V^2}{2} \frac{1}{RV}$

$F_d \sim \frac{[\rho RV]}{RV} V \sim nRV$

- corresponds to Stokes' (in) flow drag

- general result:

Equation: 

$-\nabla P = -\nabla P$
Consider flat plate moving head on:

\[ F_t \sim 6\pi MLV \]
\[ \eta = \frac{c_0v}{\rho} \]

Show Stokes scaling applies:

\[ \eta = \frac{C_0v}{\rho} \]

\[ F_t \sim PA - PLw \]

For \( P \):

\[ \nu \frac{D^2V}{Dx^2} = -\frac{\nabla P}{\rho} \]

What is the scale? Smaller will dominate gradients:

\[ \frac{\rho}{\nu} \sim \frac{V}{w} \]
So \[ p \sim \frac{\rho r v}{W} \]

\[ F_d \sim pA \sim \rho r v (\frac{y}{y}) \]

\[ \sim \left[ \text{only} \right] v \]

Show for edge-on incidence - HW!

Now, for \( Re \gg 1 \), expect\[ F_d \sim r^0 \]

but

\[ F_d \sim \rho v^2 R^2 f(Re) \]

\[ \sim \rho v^2 R^2 \left( \frac{Re}{Re^0} \right) \rightarrow \rho v^2 R^2 \]

so \( Re \rightarrow \infty \)

\[ F_d \sim \rho v^2 R^2 \]
Note:

- $F(Re) \sim Re^\alpha$ unsure about small corrections to $a$

- amazing that $F \sim Re^\alpha$ at large $Re$?

No $v$ dependence!

yet drag requires viscosity $\nu$?

Why?

- on small scales or BL $v$ matters

i.e. $Re = Re_{[scale]}$

\[ \frac{vP}{\nu} \]

- but momentum transport to small scales independent of $\nu$ (i.e. inertial).
Ex 2

What are the scaling rules for motion of surface ship of length $L$, speed $V$? Consider:

- Drag scaling $D$ - assume due to surface water
- Model size dependence $\frac{1}{L}$
- Streamline

Physics: $\bigcirc$ ill with $g$

Surface ship - modest speed 10-15 m/sec.
- Radiates waves.

So drag mechanism is resistive
i.e. film radiation - induced slowing down of charge.

Now, with gravity:

$$\dot{u} + u \cdot \nabla u = -\frac{\nabla p}{\rho} - g \zeta + \nu \nabla^2 u$$

Parameters:
$V \rightarrow \frac{L}{T}$

$l \rightarrow L$

$\rho \rightarrow \frac{M}{L^3}$

$V \rightarrow \frac{L^3}{T}$

$F_d \rightarrow \frac{ML}{T^2}$

and

$g \rightarrow \frac{L}{T^2}$

$n = 6$

index: $r = 3$ \[M, L, T\]

$\Rightarrow 3 = \pi$ variables

as before:

$\Pi_1 = \frac{F_d}{\rho_0 L^2 V^2}$

$\Pi_2 = \frac{V L}{\nu} \equiv Re$
Need one more dimensionless ratio.

Now: obviously must involve $g$.

- expect $Re >> 1$.

\[
\begin{align*}
\frac{d_x v}{u} + u \cdot \frac{dv}{dx} &= \frac{-dp}{\rho} - g \frac{z^2}{2} + v \frac{d^2 v}{dx^2}
\end{align*}
\]

\[\Rightarrow\]

\[\frac{\Pi_3}{c} \sim \frac{\rho u}{\nu} \sim \frac{v^2}{\sqrt{g}} \quad \text{Frandel number}\]

\[\sim \frac{v^2}{\sqrt{g}} = \text{Fr}.
\]

Now:

\[\Pi_2 = \Pi_1 \left( \Pi_2, \Pi_3 \right) \]

\[\Rightarrow \quad \frac{F_l}{\rho \nu v^2} \sim f(\text{Re}, \text{Fr})
\]

\[= \sim \left( \frac{\nu L}{V}, \frac{v^2}{\ell g} \right)\]
Now:
\[- \text{Re} \to \infty, \quad F_d \propto \text{Re} \rightarrow \infty \]

\[F_d = c l^3 v^2 \left( \frac{V}{g} \right)^x\]

\[\Rightarrow \] expect \( F_d \) must increase with \( g \)

\[\text{c.e. wave drag increases with } g.\]

\[F_d \propto c l^3 v^2 \left( \frac{V}{g} \right)^{-1}\]

so \( x = -1 \)

\[F_d = c l^3 v^2 \left( \frac{V}{g} \right)^{-1}\]

\[\boxed{F_d \sim \rho c l^3 g}\]
$F = p$ \( \sim \) \( \rho \cdot v \cdot a \) 

and scaling \( \sim (\text{size})^3 \) 

→ Blast Wave

Sedov–Taylor Blast Wave

- sudden release of energy
  \( E_0 \gg P_0 \cdot R \cdot t \)
  \( t = t_0 \)
  \( E \gg \text{Rayleigh } R^3 \)

For \( R < R_{\text{max}} \)

- inner scale
  \( p_{\text{inner}} \)

- outer scale
  \( p_{\text{outer}} \)

- blast expands as hemisphere

Blast wave "locks" until

the same between inner and outer structure at \( R_{\text{max}} \)

\( E/R_{\text{max}}^3 \sim P_{\text{amb}} \)

→ self-similar structure
For spherical blast, can write:

\[ \partial_r v_0 + v_r \partial_r v_0 = -\frac{dP}{\rho} \]
\[ \partial_t \rho + \frac{1}{r^2} (r^2 \rho v_r) = 0 \]
\[ \partial_r (r^2 \rho \phi) + v_r \partial_r (r^2 \rho \phi) = 0 \]

Seek \( p(t) \), but self-similarity

\[ p(t) = p(t) \rho(t) \rho(r/R(t)) \]

\( t \) blast radius \( \sim \) time \( \Rightarrow \) is re-scaling

\( \Rightarrow \) spatial structure self-similar relative to expanding radius

\( \Rightarrow \) structure re-scaled on time preserving ship.

Simple example: wave eqn:

\[ \partial^2 u = c^2 \partial^2_x u \] (1D mean wave)
\[ U = \frac{1}{2} \left[ F(x-c^t) + F(x+c^t) \right] + \frac{a}{2c} \int_{x-c^t}^{x+c^t} \varphi(s) \, ds \]

c.e. \( x^t \) enter as \( x \pm c^t \) only.

Now, \( \Pi \) thin approach: What is \( \Pi \) radius, in time?

\[ R \rightarrow \text{radius} \]
\[ L \]

\[ V \rightarrow \text{speed of front} \]
\[ \frac{L}{T} \]

\[ \rho \rightarrow \text{density} \]
\[ \frac{M}{L^3} \]

\[ p \rightarrow \text{pressure} \]
\[ \frac{ML^2}{T^2} \]

\[ E \rightarrow \text{energy} \]
\[ \frac{ML^2}{T^2} \]

\[ \Lambda = 5 \rightarrow 2 \quad \text{and} \]
\[ \Lambda = 3 \]

\[ \Pi_1 = \frac{R}{U} \rightarrow \text{molecular constant} \]

\[ \Pi_2 = \frac{E}{\rho R^2 U^2} \rightarrow \text{dimless ratio for energy} \]
So \[ T_1 = F(T_2) \]

i.e. \[ \frac{R}{v^+} = F \left( \frac{E}{\rho v^2 R^3} \right) \]

here:

- \[ F = \text{const} \sim 1 \quad \text{so} \]
  \[ R \sim v^+ \]

- \( \text{F} = \text{const and for energy balance} \)
  \[ E = \rho v^2 R^3 \]

So \[ v \sim \left[ \frac{E}{\rho R^3} \right]^{1/2} \]

and \[ R \sim \left[ \frac{E}{\rho R^3} \right]^{1/2} \]

\[ R^{5/2} \sim \left( \frac{E}{\rho} \right)^{1/2} \]
\[ R \sim \left( \frac{E}{E_0} \right)^{1/5} \]

Sedov-Taylor \quad B < 8.5

\text{wave radius}

M.B.:
- Could just put:

\[ E \sim \rho R^3 V^2 \sim \rho R^3 \frac{R^2}{t^2} \]

\[ P \sim \rho V^2 \sim \rho_0 R^2 / t^2 \]

\[ \sim \left( \frac{E}{E_0} \right)^{2/5} \frac{1}{\rho_0 t} \]

\( t \)

\text{dynamic pressure drops with time.}