Turbulence Theory

— An Introduction —

F. Basics of Fluid Turbulence

Characteristics of Fluid Turbulence:
- Broad range of spatial-temporal scales excited
- Decay of large scale energy; need "turbulence input/stirring to maintain stationarity"—Thekey

- Energy input dissipates as heat (to maintain stationarity) \( \rightarrow \) Viscosity
- Irreversible mixing occurs \( \rightarrow \) Irreversibility
- Intermittency manifested
  - i.e. spatially coherent structures (i.e. vortices)
  - Temporal bursts maximum probe trace

- Self-similarity/scale-similarity:
  - Turbulence looks the same on all scales, except the very largest (stirring) and the very smallest (dissipation)
  - Caveat: Intermittency — memory of large scales on small
Navier-Stokes Equation - Describes Fluid

\[ \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{v}) = -\nabla p + \nu \nabla^2 \mathbf{v} \]

\( \mathbf{v} \) - velocity field
\( \nabla \cdot (\mathbf{v} \mathbf{v}) \) - advection/pressure/viscous strain
\( \nabla^2 \mathbf{v} \) - diffusion/momentum

\[ \nabla \cdot \mathbf{v} = 0 \] - incompressibility

Note: Pressure determined from incompressibility
\[ \nabla \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\nabla p + \nu \nabla^2 (\mathbf{v} \cdot \mathbf{v}) \]

\[ \nabla^2 \mathbf{p} = -\frac{\partial \mathbf{v} \cdot \mathbf{v}}{\partial t} \]

\[ \mathbf{p} = -\nabla \left[ \mathbf{v} \cdot \nabla \mathbf{v} \right] \]

\[ = \int_{x' \neq x} \frac{\partial^2 \nabla V(x') \cdot \nabla V(x)}{|x - x'|} \]

More generally, can eliminate \( \mathbf{p} \)

\[ \partial_t \mathbf{v}_i + (\mathbf{v}_j - \partial_i \mathbf{v} \cdot \mathbf{v}) \partial_j (\mathbf{v}_i \mathbf{v}_i) = \nu \nabla^2 \mathbf{v}_i \]
Key Parameter: Reynolds #

\[ \text{Re} = \frac{1}{\nu} \frac{\sqrt{V}}{\ln R_d} \]

\[ \sim \frac{V(L)}{L} \]

- Re usually referenced to largest scale

\[ L = L_{\text{max}} \]
\[ V(L) = \text{largest scale velocity} \]

- Re always referenced to a particular scale

\[ L_{\text{max}}, V = \left( \frac{\langle (u' v')^2 \rangle}{\langle u'^2 \rangle} \right)^{-1/2} \]

(Taylor Scale)

- \( \text{Re} \gg 1 \) in turbulent pipe flow

\[ \text{Re} \sim 10^6 - 10^8 \text{ etc.} \]

- i.e. planetary boundary layer: \( h_{bu} \sim 4 \text{ km} \)
\[ \sim 10^5 \text{ cm} \]
\[ h_{bs} \sim 1 \text{ km} \]

- \( \text{Re} \): measure of ratio of inertial mixing of momentum to collisional mixing
(ii) Experimental Laws of Fully Developed Turbulence

Much/most of turbulence theory is empirically motivated. Experimental results preceded sophisticated theoretical analyses...

The experimental facts:

4) $\frac{2}{3}$ Law (Mundane)

In a turbulent flow with $Re \gg 1$,

$$\langle dV(e)^2 \rangle \sim e^{3/2}, \quad \text{where } e \text{ is the velocity increment between two scales separated by distance } l$$

\[ dV(e) = |V(x+e) - V(x)| \]

\[ S_2(l) = \langle dV(e)^2 \rangle \sim e^{3/2} \quad \Rightarrow \quad \text{Fundamental scaling relation} \]

2nd order structure function

\[ S_2(l) \sim \frac{1}{Re} l^{3/2} \]

\[ \text{slope } \frac{3}{2} \]

\( \text{dissipation range } l(l) \)
By law of Finite Energy Dissipation (Profound)

If in an experiment on turbulent flow all the control parameters are kept the same, except the viscosity, which is lowered as much as possible, the energy dissipation per unit mass $\frac{dE}{dt}$ decreases on a way consistent with a finite limit.

- What means 'Energy Dissipation Rate'?

\[ \frac{dE}{dt} = \frac{1}{2} C_D \rho S U^2 \]

\[ \text{drag} \]

\[ \text{face surface area} \]

\[ \text{c.e.} \]

\[ \frac{1}{2} \]

\[ p = \rho s(U^2) U \rightarrow \text{momentum transfer} \]

\[ M = \rho S U^2 \rightarrow \text{mass} \]

\[ V = u \]

if assume air momentum completely transferred to can

\[ \frac{dP_{can}}{dt} = \frac{dE}{dt} = \rho s U^2 \]
\[ C_0(Re) = \text{drag coefficient (slowly varying function of Re, depends on shape, etc.)} \]

\[ F_d = \frac{C_0 \rho S U^2}{2} \]

Now, power dissipated by drag force
\[ P_d = F_d U \]
\[ \Rightarrow P_d = \frac{C_0 \rho S U^3}{2} \]

Energy dissipation rate \[ E = \frac{P_d}{\text{Mass}} \]
\[ \text{(Re, Volume)} \]
\[ = \frac{C_0 U^3}{2} \frac{U}{L} \]

Also, \[ NS \Rightarrow \sigma_t \langle u'^2 \rangle \sim -\langle u' v' v'^2 \rangle \]

Why should we care?

Note, energy budget:
\[ \frac{\partial}{\partial t} \langle V_i \rangle + \nabla_j \nabla_i - \nabla_i \nabla_j V_i = -\delta_{ij} \rho \]
\[ \frac{\partial}{\partial t} \langle V_i^2 \rangle + \frac{\partial}{\partial x_j} \langle V_i V_j \rangle - \nabla \cdot \langle V_i \nabla V_i \rangle = -\nabla \cdot \langle V_i \alpha \rangle \]

\[ \langle \rangle \equiv \text{ensemble (fast space-time avg.)} \]

\[ \frac{\partial}{\partial t} \langle V_i^2 \rangle + \frac{\partial}{\partial x_j} \langle V_i V_j \rangle - \nabla \cdot \langle V_i \nabla V_i \rangle = \langle \partial_i \nabla \cdot \alpha \rangle \]

\[ \text{surface terms} = \langle \partial_i \nabla \cdot \rho \rangle \quad \text{upon \textit{IBP}} \]

\[ \frac{\partial}{\partial t} \langle V_i^2 \rangle = -\nabla \langle D V_i^2 \rangle \]

but \[ \varepsilon = -\frac{\partial}{\partial t} \langle V_i^2 \rangle \]  

\[ \varepsilon = \lambda \langle |D V_i|^2 \rangle \]

\[ \text{experiments suggest that} \epsilon \text{ finite as} \quad r \to 0 \]

\[ \quad \Rightarrow \text{remarkable} \]

\[ \Rightarrow \text{suggests that extremely large} \quad D V \]

\[ \text{forms as} \quad r \to 0 \quad \text{singularity vortex sheets} \]

\[ \Rightarrow \text{singular velocity gradients formed limit of weak viscosity} \]

\[ \{ \text{Heart of turbulence problem is grappling with} \}

\[ \text{singularity (especially its degree) of velocity gradients} \]
Dissipation Law

N.B.: Singularity formation is at the heart of why turbulence is a "hard" problem.

Re: Dissipation Law:

\[ \varepsilon \sim \frac{U^3}{L} \sim \frac{U^2}{(L/U)} \]

\[ \sim \frac{K.E. \text{ per Mass}}{Circulation Time} \]

i.e.- in 1 macro circulation time, a finite fraction of (more) kinetic energy is dissipated by viscosity,

\[ \Rightarrow \text{dissipation time scale is } (L/U) \]

V. Kolmogorov's Hypotheses and their Predictions/Implications. → Karman Theory of Turbulence

In the limit of Re → ∞, all possible symmetry of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.
What means?  

- "Small scales" : \( b \ll l_{10} \)  
  \[ \text{Integral scale} \rightarrow \text{characteristic of production} \]

- Symmetries  
  First, symmetries of Navier–Stokes Eqn. ?.

a) Space translations  
   \( t \rightarrow t + \Delta \)  
   (no explicit \( \Delta \) dep.)

b) Time translation  
   \( t \rightarrow t + \tau \)  
   (no \( \tau \) dep.)

c) Galilean boosts  
   (no frame dep.)  
   \[ \begin{align*}
   \mathbf{x} & \rightarrow \mathbf{x} + \mathbf{v}t \\
   \mathbf{v} & \rightarrow \mathbf{v} + \mathbf{a}
   \end{align*} \]
   i.e.,  
   \[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \]

\[ \text{Cov} \Rightarrow \]

\[ \mathbf{A} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \nabla \mathbf{v} + \nabla p + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \]

d) Parity (left-right)  
   (no preferred direction)  
   \( x \rightarrow -x \), \( \mathbf{v} \rightarrow -\mathbf{v} \)

e) Rotation (no preferred direction)  
   \( \mathbf{r} \rightarrow R \mathbf{r} \)
   \( \mathbf{v} \rightarrow R \mathbf{v} \)
e.) Scaling (for $v \to 0$) $\Rightarrow$ critical: scale elimination and passing

$$\lambda, \lambda^v, f \to \lambda^{\lambda^v}, \lambda^v f$$

$$\frac{\partial \lambda^v}{\partial t} + v \cdot \nabla \lambda^v = -\nabla \rho$$

$$\lambda^v \to \lambda^{\lambda^v}$$
$$f \to \lambda^v f$$

$$\frac{\partial \lambda^v}{\partial t} + \lambda^{\lambda^v} \cdot \nabla \lambda^v = -\nabla \rho$$

$\lambda^v = 0$ from $\nabla \lambda^v = 0$

$\lambda^{\lambda^v} = \lambda^{\lambda^v}$

$b = a - 1$

$\Rightarrow -\lambda^{\lambda^v} = \lambda^b$

$\Rightarrow \lambda^{\lambda^v} = \lambda^b$

Now, turbulence onset $\Rightarrow$ symmetry breaking

i.e. $\lambda^{\lambda^v}$: shear breaks translational invariance.

ii. rigid body boundary flow, etc

iii. flushing toilet, etc.
However, fully developed turbulence tends to restore symmetry, except near boundaries on a small scale.

i.e. if \( \partial V(x, t) = V(x + \Delta x) - V(x) \)

\[ \partial V(x + \Delta x) = \partial V(x) \]

Similarly: isotropy, parity, (facilitates scaling approach)

H2 For \( Re \to \infty \) turbulence, at small scales and away from boundaries, the flow is self-similar at small scales, i.e., possesses a unique scaling exponent \( h \)

\[ \partial V(\xi, \lambda t) \to \lambda^h \partial V(\xi, t) \]

(\( \to \) addresses \( h \) law)

H3 With assumptions similar to H1, the turbulent flow has a finite, nonvanishing mean rate of dissipation \( \varepsilon \) per unit mass.

\( Re \to \infty \Rightarrow r \to 0 \) with \( v_0 = v_{rms}, f_0 \) fixed

\[ \varepsilon = v_0^3 / f_0 \]
Alternative (not necessary): Kolmogorov's Second Universality Assumption: In the limit of infinite Reynolds number, all the small-scale statistical properties are uniquely and universally determined by the scale $l$ and the mean energy dissipation rate $\varepsilon$.

\[ \frac{4\varepsilon}{5} l = \langle u'^3 \rangle \]

\( \varepsilon \)

Implications:

\[ -\langle d\varepsilon l e^2 \rangle = \varepsilon_2 \quad ? \]

\[ S_2 \sim \frac{L^2}{T^2} \quad \text{dimensionally} \]

Now, \( \varepsilon \sim \frac{L^2}{T^3} \)

\[ \Rightarrow \langle d\varepsilon l e^2 \rangle \sim \varepsilon^{2/3} l^{2/3} \Rightarrow \text{recovers } \frac{1}{3} \text{ law.} \]

Also, implies \( h = \frac{1}{3} \Rightarrow \text{scaling exponent, etc.} \)

\( H_1, H_2, H_3 \) (2nd Universality Assumption) \( \Rightarrow H_4 \) phenomenology.
Phenomenology

Picture: (Richardson) Cascade / Eddy Mijtasis

\[ l_0 \]

\[ l_1 = \alpha l_0 \quad (\alpha < 1) \]

\[ l_n = \alpha^n l_0 \]

Key Idea: \( \text{Flux of energy in 'scale space',} \)

- From \( l_0 \) (integral scale) to \( l_d \) (dissipation scale)
- Energy flux is self-similar
- Symmetry restoration

Flux \( \Rightarrow \) energy dissipation \( \Rightarrow \) finite limit \( \Rightarrow \) no end point re-adjustment

K-Similarity \( \Rightarrow 2/3 \) law

\[ S_2 = C \left( \frac{l}{l_0} \right)^{2/3} \]

\[ l_0 \rightarrow \frac{u_l}{u} \]

\[ C \rightarrow Cx^2 \]
Ingredients in K41 Phenomenology:

- $l$: scale parameter / eddy scale
- $V(l)$: $V(l) \sim \left( \langle \omega^2 \rangle l \right)^{1/2}$

- eddy velocity $\omega^2 \sim (V(\Delta + l) - V(l)) \frac{\Delta}{l}$

- longitudinal velocity increment

- $V_0$: rms velocity fluctuation (large scale dominates)
  $V(l_0) \sim V_0$

- $T(l)$: eddy lifetime / turn-over rate
  - characteristic rate of transfer through scale $l$

Self-similarity: Energy throughput rate is scale invariant

- $E = \frac{V(l)^2}{l}$
  - energy balance / input
  - dissipation rate $\propto l$ scale
  - life-time $\propto l$

Now, $T(l)$?
\( \sim (l) \rightarrow \) "lifetime" of structure of scale \( l \)

\( \rightarrow \) i.e. time for structure to be distorted out of existence

Scales \( l' > > l \):

\( \rightarrow \) reduced eddies \( \approx \) apply Galilean boost, but don't affect lifetime.

\( \rightarrow \) symmetry under random Galilean transformations.

\( \rightarrow \) would also violate symmetry restoration.

Scales \( l' \ll l \):

\( \rightarrow \) irrelevant as very little energy/shear in such eddies/scales.

\( \Rightarrow \) scales which...

\[ \begin{align*}
3 & \sim 4 \quad \Rightarrow \quad 3 \\
2 & \quad \quad \quad \sim l / (\nu, \nu')
\end{align*} \]

\( \rightarrow \) \( Y(l) \sim \frac{1}{l^4} \); cascade local in scale space.

\[ \begin{align*}
l_0 & \quad l_1 \quad l_2 \quad l_3 \quad l_4 \quad \ldots
\end{align*} \]
\[ \varepsilon = \frac{V(\varepsilon)^3}{2} \]

\[ \Rightarrow \quad \frac{V(\varepsilon) \sim V(\varepsilon)}{V(\varepsilon)^3 \sim \varepsilon^{2/3}} \]

- Verifies \( 2/3 \) law
- For spectrum:

If \( E(k) = |\psi(k)|^2 \)

\[ s.t. \quad E = \int dk \ E(k) \quad \{ \text{ie absorbs density of states} \} \]

Then \( V(l) = \int \frac{k^2}{l} \ln E(k) \)

\[ \Rightarrow \quad V(l)^2 \sim \varepsilon^{2/3} = \varepsilon^{2/3} V(l)^2 \]

\[ \Rightarrow \quad E(k) = \varepsilon^{2/3} k^{-2/3} \]

\[ \text{Kolmogorov Spectrum} \]

Let \( \to \) :

\[ V_0 \sim \varepsilon^{1/3} l_0 \Rightarrow \frac{V_0^3}{l_0} = \varepsilon. \]
For dissipation scale:

\[ \ell_d \text{ occurs in } l \text{-scale when cascade terminated} \]

\[ \varepsilon \text{ is viscosity, scales itself } \rightarrow \operatorname{Re}(\varepsilon) \rightarrow 1 \]

\[ \frac{1}{\operatorname{Re}(\varepsilon)} \sim \frac{1}{7d} = \frac{\varepsilon}{l^2} \]

\[ \Rightarrow \quad \varepsilon^{1/3} l^{-2/3} = \frac{\varepsilon}{l^2} \]

\[ \ell_d^{4/3} = \varepsilon^{1/3} \Rightarrow \quad \ell_d = \varepsilon^{3/4} l^{1/4} \]

\[ \ell_d \equiv l, \text{ in Frisch} \]

Recall:

\[ \varepsilon = \rho \langle (\nabla \psi)^2 \rangle \]

\[ \Rightarrow \quad \psi \rightarrow 0 \Rightarrow \langle (\nabla \psi)^2 \rangle \text{ divergent} \]

\[ \langle (\nabla \psi)^2 \rangle = \int_{k_0}^{k_{\ell d}} k^2 \varepsilon^{2/3} k^{-5/3} \]

\[ = \int_{k_0}^{k_{\ell d}} k^{4/3} \varepsilon^{2/3} \]

\[ = \ell_d^{4/3} \varepsilon^{2/3} \]

\[ \Rightarrow \quad \ell_d = \varepsilon^{1/3} l^{2/3} = \varepsilon^{1/3} \frac{l^2}{l^2} \rightarrow \langle (\nabla \psi)^2 \rangle \text{ divergent} \]
Counting Degrees of Freedom

How big is the inertial range?

\[ n \sim \frac{lo}{l} \sim \frac{lo}{(\nu/e)^{1/4}} \]

Number of grid points to resolve range of scales in numerical simulation

\[ N \sim Re^{3/4} \]

Now, i.e. atmospheric boundary layer:

\[ l_0 \sim 1 \text{Km} \]
\[ l_1 \sim 1 \text{mm} \]

\[ N \sim 10^6 \Rightarrow N \sim 10 \]
\[ Re \sim 10^8 - 10^9 \]

\( \Rightarrow \) Subgrid scale modelling...

B: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.
Exercises:

1. Consider passive scalar with concentration C:

\[ \frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = -\nabla \cdot \mathbf{D} \]

\( \nabla \cdot \mathbf{D} \) is dissipation rate in H41 turbulence, \( \alpha \)

\[ \alpha = \frac{c_2}{c_s} \frac{\nu}{\lambda_0} \]

\( \Rightarrow a) \) Calculate H41 spectrum for C.
\( \Rightarrow b) \) What if \( \nu \ll \lambda_0 \)?

2. Consider incompressible turbulence with \( M = \frac{V_0}{c_s} \ll 1 \).

Show: \( \frac{ld}{L_{int}} \sim M^{-1} R_{e}^{1/4} \)

\( \Rightarrow \) validity of continuum hydrodynamics gets better at high \( R_{e} \).
Particle Separation / Richardson Law

Consider 2 particles (text) in $k-\varepsilon$ turbulence. Rate of separation?

$\rightarrow$ larger eddys advect both
$\rightarrow$ smaller eddys do nothing

$\Delta$ divergence controlled by eddys of scale $\lambda \sim |x_1-x_2|$. 

1. if $\lambda \approx |x_1-x_2|$

$$\frac{d\lambda}{dt} = \nu(\lambda) = \varepsilon^{1/3} \lambda^{1/3}$$

$$\lambda^{3/3} = \varepsilon^{1/3} t$$

2. $\lambda \sim \varepsilon^{1/2} + t^{3/2}$

Richardson's 3/2 Law

$\lambda \sim \varepsilon^{1/2} + t^{3/2}$

N.B.: Non-diffusive!

$\lambda^3 = \varepsilon f^3 \Rightarrow T_{sep} \sim \lambda^{2/3}/\varepsilon^{1/3}$

N.B. 2

Process is self-accelerating $\Rightarrow$ large eddys move faster.

N.B. 3

Non-diffusive.
Turbulent Pipe Flow

Till now → homogeneous flow in a periodic box
→ cascade in scale space (Kolmogorov)

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundary layer (Prandtl)

Consider turbulent pipe flow:

- linear → logarithmic U(x) profile
- logarithmic profile persists over a broad range of Re

\[ \text{Re} = \frac{2Ua}{v} \]
What is going on?

- Turbulent resistance, curve universal
- Froude number
- $h_0$ = $u_0^2 / ho g$
- Mean flow energy
- $\lambda = 2 \cdot \frac{A / l}{\rho u_0^2}$, resistance increases with increasing pressure drop/length $h$

Legend to chart:
- U: upper branch
- $u(x)$ = $n = n(x)$
- $u(x)$ = $u(x)$
- $u(x)$ = $u(x)$

Integral of $\tilde{f}$ with momentum flux to wall

- No slip boundary condition
→ momentum flux to wall ⇒ stress on the wall

⇒ wall stress must balance pressure drop for steady flow

**Wall Stress:** \[\rho U_h^2 l = \Delta p \pi a^2\]

\[A = 2\pi a l\]

\[\frac{\Delta l}{l} \quad \text{Force on wall} = \frac{\rho U_h^2 A_{wall}}{}\]

\[\Delta \Delta p \quad \text{(Pressure Drop)} A \text{Flow} = \text{Force on Fluid}\]

\[\text{Friction Velocity} \Rightarrow \rho U_h^2 (4\pi hl) = (\Delta p) \pi a^2\]

\[U_h = \left[\frac{(\Delta p/2\rho)(a/2l)}{}\right]^{1/2}\]

Friction Velocity
\( U_x \equiv \) friction velocity
\( \equiv \) "typical" velocity of turbulence in turbulent pipe

Deriving the inertial sublayer profile:

1. Dimensional reasoning

in pipe flow inertial sublayer have

\( 3 \) dimensional parameters \( \rho, \mu, \alpha \)

Key point: Assumption of scale invariance

on scale \( \mu_x = \frac{\mu}{U_x} < x < \alpha \)

\( \Rightarrow \) universality of logarithmic profile motivated

scale invariance assumption

Now, seek velocity gradient \( \frac{dl}{dx} \)

\( \frac{dl}{dx} \equiv U_x \cdot X, \rho \)
The simplest form for \( \frac{dU}{dx} \) is:

\[
\frac{dU}{dx} = \frac{U_*}{x}
\]

\[
\Rightarrow U = \frac{U_* \ln (X/X_0)}{K}
= \frac{U_* \ln x + \text{const.}}{K}
\]

\[\text{Logarithmic profile (consequence of scale invariance in pipe flow)}\]

\[K \approx 0.4 \quad \text{universal constant} \Rightarrow \text{Von-Kármán constant}\]

\[X_0 \quad \text{width of viscous sublayer} \approx \frac{V}{U_*}\]

2) Physical Reasoning

Stationary flow \(\Rightarrow\)

Momentum flux to well = pressure drop
\[ \langle \tilde{v}_x \tilde{v}_z \rangle = U^2 \]

Reynolds stress:
\[ \langle \tilde{v}_x \tilde{v}_z \rangle = \frac{\Gamma_p}{\rho} \]

\( \frac{\Gamma_p}{\rho} = U^2 \)

Now, to calculate \( \langle \tilde{v}_x \tilde{v}_z \rangle \):

- take velocity fluctuation as generated by mixing of \( U(x) \), so

\[ \sigma_z \sim \int \frac{\partial u}{\partial x} \]

"mixing length"

analogous to Chapman-Enskog expansion, i.e.

\[ \ell \rightarrow \ell_{\text{mix}} \]

\[ \alpha \rightarrow \alpha_{\text{th}} \]
Here, scale invariance $l \sim x$

so

$$\langle J_x \nabla_x \rangle = \frac{\langle V_x \rangle}{\partial x} \frac{\partial U}{\partial x}$$

$$\approx U_\infty \frac{\partial U}{\partial x}$$

$$Y_T = U_\infty x \rightarrow \text{"eddy viscosity"} \rightarrow \text{"turbulent viscosity"} \rightarrow \text{key concept}$$

rate of turbulent transport of momentum

then momentum balance $\Rightarrow$

$$U_\infty \frac{\partial U}{\partial x} = \frac{U_\infty^2}{x}$$

$\Rightarrow$

$$U = \frac{U_\infty \ln (x/x_o)}{H} \rightarrow \text{Logarithmic Profile}$$

$\Rightarrow \text{Law of the Wall}$
Some comments:

- as in h41; clear phenomenology critical to guiding the approximations...
- scale invariance

"Mixing length theory always works... provided you know the mixing length..."
- P. D.

- Why a single value of velocity, i.e. $U^* x$?

Consistent with mixing length hypothesis; velocity fluctuations generated by mixing of mean flow gradient, i.e.

$$\bar{u} = \frac{d u}{d x} \sim x \frac{d u}{d x}$$

$\sim x \frac{y^*}{x}$

abscissa at preferred scale.

Consistent... Assumption consistent with:
- logarithmic profile
- scale invariance
In viscous sublayer flow linear:

\[ \frac{\partial u}{\partial x} = u^2_x \]

\[ \therefore \quad u = \frac{u^2_x x}{v} \]

\[ \Rightarrow \text{note effect of turbulence is to:} \]

- Flatter profile
- Higher transport at fixed wall stress
- Reduce central velocity
- Limit \( Q \) (quality factor)
\[ x_0 = \frac{y}{U_t} \quad \text{so} \quad U_t = U_t \left( \frac{y}{x_0} \right) \]

Note: Flow in viscous sublayer is turbulent, but mixing there affected by dissipation rate scales \( \rightarrow \) linear profile.

New - turbulent dissipation? \[ \text{Consider NSE:} \]

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \frac{<V_x^2>}{2} \frac{\partial \mathbf{V}}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial <V_x>}{\partial x} \right) = -\nabla \rho + \nu \nabla ^2 \mathbf{V}
\]

\( \mathbf{V} \) and \( \mathbf{V} \cdot \nabla \mathbf{V} \) imply:

\[
\frac{\partial}{\partial t} \left( \frac{\partial <V_x>}{\partial x} \right) + \nabla \cdot \nabla \mathbf{V} + \frac{<V_x^2>}{2} \frac{\partial <V_x>}{\partial x} + \frac{\partial^2}{\partial x^2} \left( \frac{\partial <V_x>}{\partial x} \right) = -\left( \nabla \cdot \rho \right) - \nu (\nabla \mathbf{V})^2
\]

i.e. p


And:

\[ E = \left( \frac{u_x}{x} \right) \left( \frac{u_y}{y} \right) \]

Obviously, the y-axis scale is proporional to the input of the system.
For net energy budget:

\[ \frac{\partial E}{\partial t} = \frac{1}{\partial x} \left( \frac{\partial}{\partial x} \frac{\partial \left( \dot{u}_x \dot{v}_z \right)}{\partial x} \right) - \nu \frac{\partial^2}{\partial x^2} \left( \frac{\partial \left( \dot{u}_x \dot{v}_z \right)}{\partial x} \right) \]

- \( \frac{\partial}{\partial x} \left( \frac{\partial \dot{u}_x \dot{v}_z}{\partial x} \right) \) input to fluctuations by relaxation of mean shear flow (Reynolds work)

\[ \psi \]

- dissipation of fluctuation energy by viscosity

Can define:

\[ \psi = \frac{\partial}{\partial x} \frac{\partial \left( \dot{u}_x \dot{v}_z \right)}{\partial x} \]

- turbulent dissipation rate

and using mixing length theory:

\[ \dot{u}_x \dot{v}_z = U_t x \frac{\partial U}{\partial x} \]

\[ \Rightarrow \psi = (U_t x) \left( \frac{\partial U}{\partial x} \right)^2 = \gamma_t \left( \frac{\partial U}{\partial x} \right)^2 \]

- rate of "heating" by turbulent relaxation of mean flow
Now interesting to tabulate comparison between Pipe Flow and K41 Problem

<table>
<thead>
<tr>
<th>Pipe Flow (Prandtl)</th>
<th>K41 (Kolmogorov)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scales:</strong></td>
<td></td>
</tr>
<tr>
<td>( \alpha, \beta, \gamma, \nu, /u^+ )</td>
<td>( \log, \ln, L )</td>
</tr>
<tr>
<td><strong>Invariance:</strong></td>
<td></td>
</tr>
<tr>
<td>( x \rightarrow \text{real space} )</td>
<td>( L \rightarrow \text{scale space} )</td>
</tr>
<tr>
<td><strong>Inertial Sublayer</strong></td>
<td><strong>Inertial Range</strong></td>
</tr>
<tr>
<td><strong>Viscous Sublayer</strong></td>
<td><strong>Dissipation Range</strong></td>
</tr>
<tr>
<td><strong>Turbulence:</strong></td>
<td></td>
</tr>
<tr>
<td>( U = \gamma \frac{\partial \nu}{\partial x} )</td>
<td>( \epsilon = \frac{\nu \langle \epsilon \rangle^2}{\langle \nu \rangle} )</td>
</tr>
<tr>
<td><strong>Dynamics:</strong></td>
<td></td>
</tr>
<tr>
<td>eddy viscosity</td>
<td>turnover rate</td>
</tr>
<tr>
<td>( \gamma = \nu \langle \epsilon \rangle )</td>
<td>( 1/T(e) = \frac{\nu \langle \epsilon \rangle}{\langle \nu \rangle} )</td>
</tr>
<tr>
<td><strong>Result:</strong></td>
<td></td>
</tr>
<tr>
<td>( \frac{U}{H} )</td>
<td>( \nu \langle \epsilon \rangle )</td>
</tr>
<tr>
<td><strong>Universal profile</strong></td>
<td><strong>Universal spectral scaling</strong></td>
</tr>
<tr>
<td>( \gamma = \gamma_T )</td>
<td>( \nu \langle \epsilon \rangle / \epsilon = \nu \langle \epsilon \rangle^2 )</td>
</tr>
<tr>
<td>( x_0 = \gamma / u^+ )</td>
<td>( L_d = \nu^{3/4} / \epsilon^{1/4} )</td>
</tr>
</tbody>
</table>
Practical Issues

Resistance Law & Pipe Flows.

have: \( \frac{y}{\sqrt{u}} \leq x \leq \frac{y}{\sqrt{u}} \)

radius

can push to \( x = a \), with logarithmic accuracy

\[ U = \frac{U_e \ln \left( \frac{y+a}{y} \right)}{R} \]

but

\[ V_t = \frac{U_t}{R} = \left( \frac{a \Delta p}{2 \rho} \right)^{1/2} \]

\( \Rightarrow \) can re-write:

\[ U = \left( \frac{a \Delta p}{2 \rho \ell h^2} \right)^{1/2} \ln \left( \frac{a \left( \frac{a \Delta p}{2 \rho \ell h^2} \right)^{1/2}}{R} \right) \]

Convenient to define:

\[ x = \frac{2 a \Delta p / \ell}{\frac{1}{2} \epsilon U^2} \rightarrow \text{fraction factor} / \text{resistance coefficient} \]

\( \rightarrow \) flow \( k \ell \epsilon \)
Taking \( Re = 2aU/\nu \)

Can re-write friction law as:

\[
\frac{1}{\sqrt{f}} = 0.88 \ln (Re\sqrt{f}) - 8.5
\]

\( Re = 2aU/\nu \)

\( \lambda = \frac{2a \Delta p/\rho}{1/2 \rho U^2} \)

Good fit to pipe flow data.
Problems:

1a) A very strong explosion, with energy released $\Delta E$, creates a spherical blast wave in an atmosphere of pressure $P$, density $\rho$. Use dimensional analysis to derive the radius of the blast front as a function of time, i.e., $r(t)$? When does this scaling fail?

1b) A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate, assuming the hot plate drives a surface heat flux $Q$. (See Chapter 5, Landau).
Why Wave "kinetics"?

"A wave is never found alone but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions."

- Leonardo da Vinci
  Codice Atlantico, c.1500.
Leonardo on waves...
From Asian art...

The great wave at Kanagawa  

Hokusai