d.) Fluctuations in Plasma and the Test Particle Model

Contents

(i) Basic Ideas – Equilibrium Fluctuations

→ plasma:

- $1/n\lambda_D^3 \ll 1$ ⇒ many particles in Debye sphere
- $k_B T \gg e^2/r$ ⇒ thermal energy dominates electrostatic energy

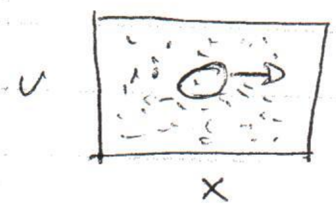
Diluteness!

→ Equilibrium Fluctuations : Balances

→ absorption vs. emission

(equivalent) → fluctuation vs. dissipation

as in Brownian motion.

i.e. 

(1) emission: discrete particle in Vlasov fluid emits waves?

$$\underline{\nabla} \cdot \underline{D} = \underline{\nabla} \cdot \underline{\underline{\epsilon}}\, \underline{E} = 4\pi n_0 q\, \delta(\underline{x} - x(t))$$

i.e. → a la' boat wake on water

→ Cerenkov emission

∴ discrete emission ⟺ fluctuation

i.e. – particle kinetic energy coupled to wave energy

– Cerenkov emission ⟺ particle slowing down

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② Cerenkov emitted waves damp via
→ Landau damping
→ in/on Vlasov fluid (i.e. Landau damping for $\nu \to 0$)

i.e.

emitted waves damp → $\left\{ \begin{array}{l} \text{absorption} \\ \text{dissipation} \end{array} \right\}$ > heats particles

So, conceptual picture of thermal equilibrium fluctuations is detailed balance of:

→ Cerenkov emission of waves from individual discrete particles

→ absorption of waves via Landau damping on Vlasov fluid

NB: ① Here, assume periodic B.C.'s → no radiative damping, outgoing waves, etc.

(Note in this picture, each particle plays a dual role (i.e. "double agent") : $\left\{ \begin{array}{l} \text{emitter} \\ \text{absorber} \end{array} \right.$

② In general, take damping length finite

i.e. $l_{damp} \sim \omega / \ldots$

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As:

- "emitter": a discrete particle moving along some specified (unperturbed) orbit

  i.e.  an identifiable 'pea' in a 'pea soup' composed of other peas.

- "absorber": an element of the Vlasov fluid responding to, and handling damping emission from (other) discrete particles

  i.e.  a "crushed pea" element of the "pea soup" of the Vlasov fluid.

so

→ equilibrium plasma = soup/gas of 'dressed' test particles
↑
Vlasov fluid screening

⇒ "test particle model"

→ $\boxed{\text{"every pea in the soup acts like soup for all the other peas"....}}$

Important to understand carefully!

Note: Useful Analogy

	Brownian Motion	Equilibrium Plasma
Fluctuation	$\langle \tilde{v}^2 \rangle_\omega$	$\langle \tilde{E}^2 \rangle_{k,\omega}$
Exciton	ω-mode	$k, \omega(k) \Rightarrow$ plasma waves
Emission/Source	$\langle \tilde{f}^2 \rangle$ – fluid thermal force	particle discreteness
Absorption/Damping	$\beta \leftrightarrow$ Stokes Drag on Particle	$\epsilon_{IM} \leftrightarrow$ Landau Damping of Collective Modes

n.b.: "Brownian Motion"

$$\frac{d\tilde{v}}{dt} + \beta \tilde{v} = \frac{\tilde{f}}{m}$$

ii.) Test Particle Model – Fluctuation Spectrum

→ As noted before, basic idea is that:

- each particle both a 'discrete emitter' and participant in Vlasov fluid screening cloud
- fluctuations weak → unperturbed orbits valid.

→ if consider stationary cons:

$$\delta f = \underset{\substack{\text{coherent}\\ \text{Vlasov}\\ \text{response}\\ \text{(screening)}}}{f^{c}} + \underset{\substack{\text{discrete}\\ \text{particle source}}}{\tilde{f}}$$

(Cald' Debye Calculation)

$$\delta f = \underbrace{\frac{\frac{|e|}{m}\,\tilde{E}_{k,\omega}\,\partial\langle f\rangle/\partial v}{-i(\omega-kv)}}_{\text{coherent response}} + \underbrace{|e|\,\delta(x-x(t))\,\delta(v-v(t))}_{\text{discreteness source}}$$

$$\nabla^2\phi = 4\pi n_0|e|\int dv\,\delta f = 4\pi n_0|e|\int dv\, f^{c} + 4\pi n_0|e|\int dv\,\tilde{f}$$

$$\Rightarrow \quad \vec{\phi}_{k,\omega} = \frac{4\pi n_0 |e|}{k^2} \int dv\, \tilde{f} \Big/ \epsilon(k,\omega)$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

using u.p.o. :

$$\int dv\, \tilde{f}_k = \int dx\, e^{-ikx} |e|\, \delta(x - x(t))$$

$$= |e|\, e^{-ikvt}$$

$\underline{\underline{\text{So}}}$

$$\vec{\phi}_k(t) = \underline{\underline{\epsilon}}^{-1}(k,t)\, \frac{4\pi n_0 |e|}{k^2}\, e^{-ikvt}$$

Note: strictly speaking, have :

$$\underline{\underline{\epsilon}}(k,t)\, \vec{\phi}_k(t) = \frac{4\pi n_0 |e|}{k^2}\, e^{-ikvt}$$

$\underline{\text{So}}$ $$\phi_k(t) = \underline{\underline{\epsilon}}^{-1}(k,t)\, \frac{4\pi n_0 |e|}{k^2}\, e^{-ikvt} + \phi_k^{\text{homog.}}\, e^{-i\omega_k t}$$

driven solution (discreteness)

homogeneous solution

$\omega_k = \omega_r(k) + i\omega_i(k)$ ⇒ eigenmode freq

Now, – time asymptotically ....

– for $\omega_i(k) < 0$ ⇒ collective modes damped ...

⇒ only discreteness driven solutions persist ....

Catch: → For $\omega_i \lesssim 0$ ⇒ i) need wait quite a long time.

ii) for sufficient source strength, amplification to nonlinearity occurs ...

n.b. moving toward, but not to, marginal stability ⇒ $\tau_{relax} \to \infty$

→ if unstable modes, require ultimate nonlinear damping to balance noise

i.e. $\epsilon_{IM} = \epsilon_{IM}(k, \omega, \langle \tilde{\phi}^2 \rangle)$ ....

"noise" = thermal + nonlinear, in that case

Proceeding, then test particle model $\Rightarrow$

$$\boxed{\langle \hat{\phi}^2\rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2}\right)^2 \int dv_1 \int dv_2 \, \frac{\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega}}{|\epsilon(k,\omega)|^2}}$$

∴ all content $\rightarrow \langle \tilde{f}^2\rangle_{k,\omega}$ (abbreviation)

$\rightarrow \epsilon(k,\omega)$

Now, for discreteness noise:

$$\tilde{f} = \frac{1}{n}\sum_{i=1}^{N} \delta(x - x_i(t))\,\delta(v - v_i(t))$$

$\rightarrow$

$$\left.\begin{array}{l} x_i(t) = x_{i0} + v_i t \\ v_i(t) = \text{const} \end{array}\right\} \text{u.p.o.}$$

Diluteness!

$\rightarrow$ assume (discrete) uncorrelated test particles, so:

$$\text{so} \quad \langle \ \rangle = n\int d\underline{x}_0 \int d\underline{v}_i \, \underbrace{\langle f(v_i)\rangle}_{\text{@ Maxwellian}}$$

i.e. simple avg. over equilibrium distribution

$(k_B T \gg e^2/\bar{r})$

so

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle = n\int dx_i \int dv_i \left[\left(\frac{1}{n}\sum_{i=1}^{N}\delta(\underline{x}_1-\underline{x}_i(t))\,\delta(\underline{v}_1-\underline{v}_i(t))\right) * \frac{1}{n}\sum_{j=1}^{N}\delta(\underline{x}_2-\underline{x}_j(t))\,\delta(\underline{v}_2-\underline{v}_j(t))\right]\langle f\rangle$$

$$= \frac{1}{n}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)\,\langle f\rangle$$

as avg. vanishes unless $\begin{pmatrix}\underline{x}_i\\ \underline{v}_i\end{pmatrix} = \begin{pmatrix}\underline{x}_j\\ \underline{v}_j\end{pmatrix}$

N.B. : Uncorrelated test particles can only correlate with themselves...

so

$$\boxed{\langle \tilde{f}(1)\tilde{f}(2)\rangle = \frac{\langle f\rangle}{n}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)}$$

Discreteness correlation

See... Pg. 71 for furthur details ...

Details of TPM – Discreteness Correlation

1.) Discreteness Correlation Function

$$\begin{cases} 1 = x_1, v_1, t \\ 2 = x_2\ v_2\ t \end{cases}$$

$$\tilde{f}(1) = \frac{1}{n} \sum_{i=1}^{N} \delta(x_1 - x_i(t))\, \delta(v_1 - v_i(t))$$

$$\langle\ \rangle := \int dx_i \int dv_i\, n\langle f\rangle$$

$$\langle\ \rangle = \int dx_i \int dv_i\ \ n\langle f\rangle$$

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle =$$

$$= \int dx_i \int dv_i\ n \left(\frac{1}{n} \sum_{i=1}^{N} \delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{v}_1 - \underline{v}_i(t))\right) \left(\frac{1}{n} \ast \sum_{j=1}^{N} \delta(\underline{v}_2 - \underline{v}_j(t))\, \delta(\underline{x}_2 - \underline{x}_j(t))\right)$$

$$= \int dx_i \int dv_i\ \frac{\langle f\rangle}{n} \sum_{i,j=1}^{N} \left[\delta(\underline{x}_1 - \underline{x}_i(t))\ \delta(\underline{x}_2 - \underline{x}_j(t)) \ast \delta(\underline{v}_1 - \underline{v}_i(t))\, \delta(\underline{v}_2 - \underline{v}_j(t))\right]$$

only $\neq 0$ if arguments interchangeable

$$= \int dx_i \int dv_i\ \frac{\langle f\rangle}{n}\ \ \left[\delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{x}_i - \underline{x}_j)\, \delta(\underline{v}_1 - \underline{v}_2) \ast \delta(\underline{v}_i - \underline{v}_j)\right]$$

$$= \frac{\langle f\rangle}{n}\ \delta(\underline{x}_1 - \underline{x}_2)\, \delta(\underline{v}_1 - \underline{v}_2)$$

[particles uncorrelated unless same

- memory
- ordered

ergodic → ensemble, time

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle = \langle \tilde{f}(0)\tilde{f}(2-1)\rangle$$ stat homog.

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_k = \int e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle \, dx_-$$

2 'ahead' 1 ⇒

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k\omega} = \int_0^\infty d\tau\, e^{i\omega\tau}\, U(2|\tau) \int e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle \, dx_-$$

(2) forward in time

$$+ \int_0^\infty d\tau\, e^{i\omega\tau}\, U(1|-\tau) \int e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle \, dx$$

(1) backward

(2) ⇒ $x_2 \to x_2 + v_2\tau$

$$(2) = \int_0^\infty d\tau \int e^{-ik(x_2-x_1)}\, e^{i(\omega - kv_2)\tau} \langle \tilde{f}(1)\tilde{f}(2)\rangle \, dx_-$$

$$= \int_0^\infty d\tau\, e^{i(\omega-kv_2)\tau} \langle \tilde{f}\tilde{f}\rangle_k$$

$x_1 \to x_1 - v\tau$

$$(1) = \int_0^\infty d\tau\, e^{i(\omega-kv_1)\tau} \langle \tilde{f}\tilde{f}\rangle_k$$

$$(2) = \frac{-1}{i(\omega - kv_2)} \langle \tilde{f}\tilde{f}\rangle_k$$

$$(1) = \frac{-1}{\omega - kv_1} \langle \tilde{f}\tilde{f}\rangle_k$$

$$\langle \tilde{f}\tilde{f} \rangle = \frac{1}{n} \langle f \rangle \, \delta(x_-)\, \delta(v_-)$$

$$\langle \tilde{f}\tilde{f} \rangle_k = \frac{1}{n} \langle f \rangle \, \delta(\cancel{v}_-)$$

$\rightarrow$ real

$v_1 = v_2$

$$① + ② = \left(\frac{i}{\omega - kV} + \frac{i}{\omega - kV} \right) \langle \tilde{f}\tilde{f} \rangle_k$$

$$\mathrm{re}\,(① + ②) = 2\pi\, \delta(\omega - kV)\, \frac{\langle f \rangle}{n}\, \delta(v_-)$$