3

Quasi-linear Theory - A Mean Field Theory of Plasma Transport

3.1 The Why and What of Quasilinear Theory

In the first part of the previous chapter, we discussed fluctuations and relaxation in a *stable* plasma, *close to equilibrium.* Now we embark on the principal discussion of this book, which deals with the far more difficult, but also more interesting, problem of understanding the dynamics of a *turbulent plasma*, *far from equilibrium.* The first topic in plasma turbulence we address is *quasi-linear theory.*

Plasma turbulence is usually thought to result from the nonlinear evolution of a spectrum of unstable collective modes. A collective instability is an excitation and a process whereby some available potential energy stored in the initial distribution function (either in its velocity space structure or in the gradients of the parameters which define the local Maxwellian, such as, $n(x)$, $T(x)$, etc) is converted to fluctuating collective electromagnetic fields and kinetic energy. A simple example of this process, familiar to all, is

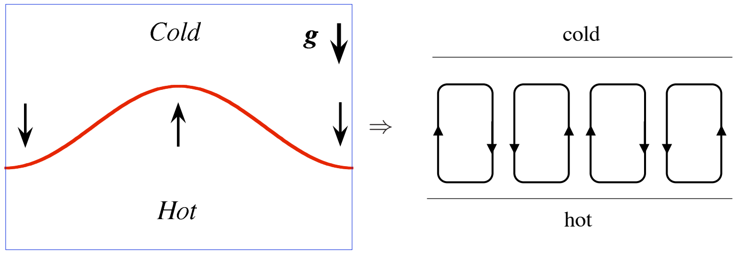


Fig. 3.1. Cartoon showing the evolution of a super-critical gradient to convective instability and convection rolls to turbulence and turbulent mixing of the temperature gradient.

Rayleigh - Benard (R-B) convection, the mechanism whereby hot air rises on time scales faster than that determined by molecular diffusion. The starting point is, unstably stratified air, which contains gravitational potential energy. R-B convection taps this available 'free energy', converting some of it to convection rolls. The convection rolls, in turn, relax the vertical temperature gradient $\mathrm{d}T/\mathrm{d}z$ which drives the instability (i.e. in R-B convection, $\mathrm{d}T/\mathrm{d}z < (\mathrm{d}T/\mathrm{d}z)_{\mathrm{crit}}$). Thus, they exhaust the available free energy and so eliminate the drive of the R-B instability. A cartoon schematic of this process is given in Fig.3.1. Examples of paradigmatic velocity space instabilities are the bump-on-tail (BOT) instability and the current-driven ion acoustic (CDIA) instability. In the BOT, the free energy is the kinetic energy of the 'bump' or weak beam population situated on the tail of the Maxwellian. The presence of the bump implies an interval of velocity for which $\partial\langle f\rangle/\partial v > 0$, so that waves resonant in that interval are unstable. The unstable spectrum will grow at the expense of the free energy in the bump, thus decelerating it and 'filling in' the distribution, so that $\partial\langle f\rangle/\partial v \leq 0$, everywhere. To conserve total momentum, heating of the bulk distribution must occur. A cartoon of this evolutionary process is given in Fig.3.2(a). In the case of the CDIA shown in Fig3.2(b), the current carried by the electrons can produce a region of positive $\partial\langle f\rangle/\partial v$ sufficient to overcome the effects of ion Landau damping, thus triggering instability. The turbulent electric fields will act to reduce $\partial\langle f\rangle/\partial v$ by reducing the shift in, or 'slowing down', the electron distribution function. Again, conservation of momentum requires some bulk heating and some momentum transfer to the ions. *In all cases, the instability-driven turbulence acts to expend the available free energy, thus driving the system back toward a stable or marginally stable state, and extinguishing the instability.* Since this evolution occurs on a time scale which is necessarily longer than the characteristic times of the waves, we may say that $\langle f\rangle = \langle f(v,t)\rangle$, so that $\langle f\rangle$ evolves *on slow* time scales. *Quasi-linear theory is concerned with describing the slow evolution of* $\langle f\rangle$ *and its relaxation back to a marginally stable state.* Quasi-linear theory is, in some sense, the *simplest possible* theory of plasma turbulence and instability saturation, since it is limited solely to determining how $\langle f\rangle$ relaxes. While the methodology of quasilinear theory is broadly applicable, our discussion will focus first on its applications to problems in Vlasov plasma turbulence, and later consider more complicated applications.

In quasi-linear theory, the mean field evolution of $\langle f\rangle$ is taken to be slow, so that

$$\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t} \ll \gamma_k.$$

Thus, the growth rate γ_k is computed using the instantaneous value of $\langle f\rangle$, which evolves more slowly than the waves do. So

$$\gamma_k = \gamma_k[\langle f(v,t)\rangle]$$

is determined by plugging $\langle f\rangle$ at the time of interest into the linear dielectric

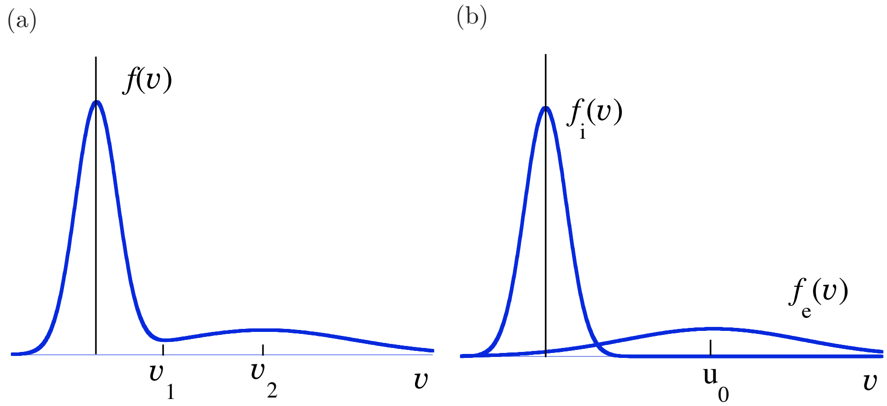


Fig. 3.2. (a)Sketch of the distribution function for the bump-on-tail instability. Phase velocities ω/k such that $v_1 < \omega/k < v_2$ are resonant where $\partial\langle f\rangle/\partial v > 0$, so instability occurs.(b)Sketch of the distribution function for the current driven ion acoustic instability. Here the electron distribution function has centroid $u_0 \neq 0$, and so carries a net current. Phase velocities $v_{t_i} < \omega/k < u$ may be unstable, if electron growth exceeds ion Landau damping.

function

$$\epsilon(k,\omega) = 1 + \sum_j \frac{\omega_{\mathrm{p}j}^2}{k} \int \mathrm{d}v\, \frac{\partial\langle f_j\rangle/\partial v}{\omega - kv}, \tag{3.1}$$

and then computing $\omega_k,\ \gamma_k$ via $\epsilon(k,\omega) = 0$. The equation for $\langle f\rangle$ is obtained by averaging the Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\,\frac{\partial f}{\partial v} = 0, \tag{3.2a}$$

and using the separation $f = \langle f\rangle + \delta f$, so

$$\frac{\partial\langle f\rangle}{\partial t} = -\frac{\partial}{\partial v}\left\langle \frac{q}{m}E\delta f\right\rangle. \tag{3.2b}$$

Note that Eq.(3.2b) constitutes the first of the Vlasov hierarchy, which couples the evolution of the first moment to the second moment, the evolution of the second moment to the third moment, etc. Quasi-linear theory *truncates* this hierarchy by simply approximating the fluctuating distribution function f by the *linear coherent response* f_k^{c} to E_k, i.e.,

$$\delta f_k = f_k^{\mathrm{c}} = -i\frac{q}{m}\frac{E_k \partial\langle f\rangle/\partial v}{\omega - kv}. \tag{3.2c}$$

Plugging f_k^{c} into Eq.(3.2b) gives the quasilinear equation for $\langle f\rangle$ evolution

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v}D(v)\frac{\partial\langle f\rangle}{\partial v} \tag{3.3a}$$

$$D(v) = \mathrm{Re}\sum_k \frac{q^2}{m^2}|E_k|^2\frac{i}{\omega_k - kv + i\,|\gamma_k|}. \tag{3.3b}$$

Thus, quasi-linear theory is a straightforward application of mean field theory methodology to the problem of $\langle f\rangle$ evolution. Note that all noise and mode-mode coupling effects are neglected, so *all fluctuations are assumed to be eigenmodes which satisfy* $\omega = \omega(k)$. Other parts of f, i.e., the incoherent part $\tilde{f}$ in Eq.(2.1), has impact on the relaxation. This effect is discussed in Chapter 8. The other issue is a truncation of δf at the linear response. The roles of nonlinear terms mode coupling, etc. will be explained in subsequent chapters. In this chapter, the ω-subscript is superfluous and hereafter dropped. In the language of critical phenomena, quasilinear theory is concerned with the evolution of the order parameter in a phase of broken symmetry, not with noise driven fluctuations while criticality is approached from below.

For completeness, then, we now write the full set of equations used in the quasi-linear description of Vlasov turbulence. These are the linear dispersion relation

$$\epsilon(k,\omega) = 0, \tag{3.4a}$$

the equation for the evolution of the electric field energy, which is just

$$\frac{\partial}{\partial t}|E_k|^2 = 2\gamma_k|E_k|^2, \tag{3.4b}$$

and the equations for $\langle f \rangle$ and $D(v)$, i.e.

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial \langle f \rangle}{\partial v}, \tag{3.4c}$$

$$D(v) = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{|\gamma_k|}{(\omega - kv)^2 + \gamma_k^2}. \tag{3.4d}$$

Note that the absolute value (i.e. $|\gamma_k|$) is required by causality. Since $D \sim |\gamma|$, negative diffusion is precluded, even if the modes are linearly damped. This is physically plausible, since damped waves of finite amplitude are quite capable of scattering particles and driving diffusion and relaxation. Equations (3.4a-3.4d) constitute the famous "quasi-linear equations", first derived by Vedenov, Velikov and Sagdeev and by Drummand and Pine's in the early 1960's. The quasilinear theory is implemented by solving equations (3.4a-3.4d) to describe the relaxation of $\langle f \rangle$ to a state where all $\gamma_k \leqq 0$. The concomitant evolution and saturation level of $|E_k|^2$ can also be calculated. Figure 3.3 gives a flow chart description of how Eq.(3.4a-3.4d) might actually be solved iteratively, to obtain an $\langle f \rangle$ which is everywhere marginal or submarginal.

At first glance, the quasi-linear theory seems easy, even trivial, and so bound to fail. Yet, quasi-linear theory is often amazingly successful! The key question of *why* this is so is still a subject of research after over 40 years. Indeed, the depth and subtlety of the quasi-linear theory begin to reveal themselves after a few minutes of contemplating Eq.(3.4a-3.4d). Some observations and questions one might raise include, but are not limited to,

i) The quasi-linear equation for $\langle f \rangle$ Eq.(3.4c) has the form of a diffusion equation. So, what is the origin of irreversibility, inherent to any concept of diffusion, in quasi-linear theory? Can Eq.(3.4c) be derived using Fokker-Planck theory?

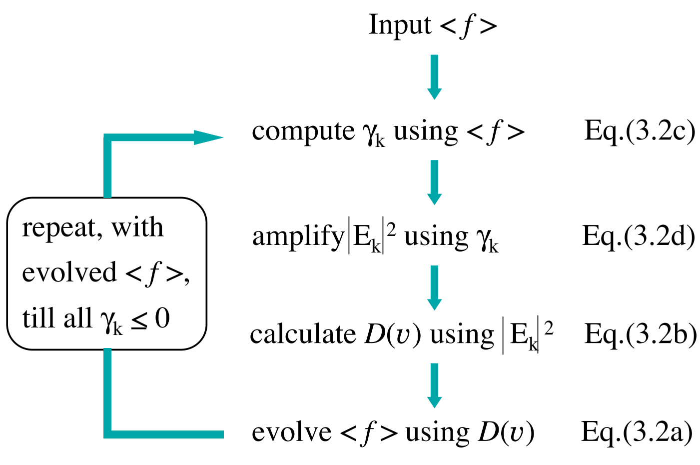


Fig. 3.3. Schematic for implementation of quasilinear theory

ii) $D(v)$, as given by Eq.(3.4d), varies rapidly with v, as for resonant particles with $\omega \sim kv$,

$$D(v) = \sum_k \frac{q^2}{m^2}|E_k|^2 \pi\delta(\omega - kv)$$

while for non-resonant particles with $\omega \gg kv$,

$$D(v) = \sum_k \frac{q^2}{m^2}|E_k|^2 \frac{|\gamma_k|}{\omega^2}.$$

What is the physics of this distinction between resonant and non-resonant diffusion? What does non-resonant diffusion *mean*, in physical terms?

iii) When and under what conditions does quasi-linear theory apply or break down? What criteria must be satisfied?

iv) How does quasi-linear theory balance the energy and momentum budgets for fields and particles?

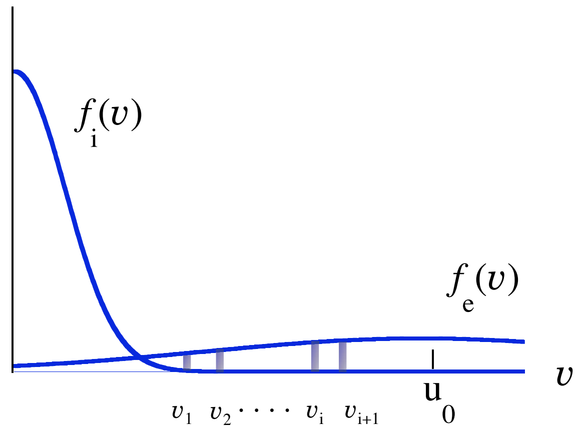


Fig. 3.4. Possible excitations of unstable CDIA modes, resonating to electrons.

v) How does a spectrum of unstable waves drive $\langle f \rangle$ to evolve toward a marginal state, with $\gamma_k = 0$ for all k.

These questions are addressed in the remainder of this chapter. Applications to some simple examples, such as the BOT and CDIA instabilities, are discussed as well.

3.2 Foundations, Applicability and Limitations of Quasi-linear Theory

3.2.1 Irreversibility

We first address the issue of irreversibility. Generally, quasi-linear theory is applied in the content of a broad spectrum of unstable waves. Of course, one important question is "How broad is 'broad'?". In the case of the CDIA system, the unstable spectrum is sketched in Fig.3.4. Note that, as for any realistic system, k is quantized, so the phase velocities $v_{\mathrm{ph},i} =$

$\omega(k_i)/k_i$ are quantized, as well. Particle motion in such a wave field is entirely deterministic, according to Newton's laws, so that

$$m\frac{d^2x}{dt^2} = \sum_m qE_m \cos(kx_m - \omega_m t) \tag{3.5a}$$

and if $v \sim \omega_i/k_i$, one resonance dominates:

$$m\frac{d^2x}{dt^2} \simeq qE_i \cos(k_i x + (k_i v - \omega_i)t). \tag{3.5b}$$

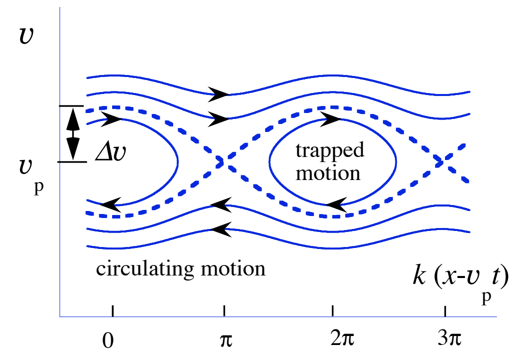
Hence, each resonant velocity defines a phase space island, shown in Fig. 3.5. The phase space island is defined by a separatrix of width $\Delta v \sim (q\phi_m/m)^{1/2}$, which divides the trajectories into two classes, namely trapped and circulating. In the case with multiple resonances *where the separatrices of neighboring phase space islands overlap*, the separatrices are destroyed, so that the particle motion becomes *stochastic*, and the particle can wander or 'hop' in velocity, from resonance to resonance. In this case, the *motion is non-integrable* and, in fact, *chaotic*. A simple criterion for the onset of chaos and stochasticity is the Chrikov overlap criterion

$$\frac{1}{2}(\Delta v_i + \Delta v_{i\pm1}) > |v_{\mathrm{ph},i} - v_{\mathrm{ph},i\pm1}|. \tag{3.6}$$

Here Δv is the separatrix width, so that the LHS of Eq.(3.6) is a measure of the excursion in v due to libration, while the RHS is the distance in velocity between adjacent resonances. If as shown in Fig.3.6(a), LHS $\ll$ RHS, separatrix integrity is preserved and the motion is integrable. If, on the other hand, LHS $\gg$ RHS, as shown in Fig.3.6(b), individual separatrices are destroyed and particle orbit stochasticity results.

It is well known that stochastic Hamiltonian motion in velocity may be described by a Fokker-Planck equation, which (in 1D) can be further simplified to a diffusion equation by using a stochastic variant of Liouville's theorem,

(a)



(b)

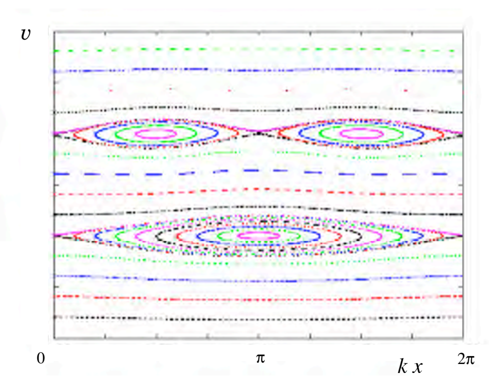


Fig. 3.5. (a) Structure of wave-particle resonance in phase space. The separatrix width is $\Delta v \sim (q\phi_i/m)^{1/2}$. Particles inside the separatrix (region of libration) undergo periodic motion on iso-energy contours and so are said to be trapped. Particles outside the separatrix circulate. (b) For several waves with distinct phase velocities, multiple resonance islands can co-exist and interact. [Courtesy of Prof. A. Fukuyama].

because the phase space flow is incompressible on account of the underlying Hamiltonian equations of motion. The resulting equation is identical to

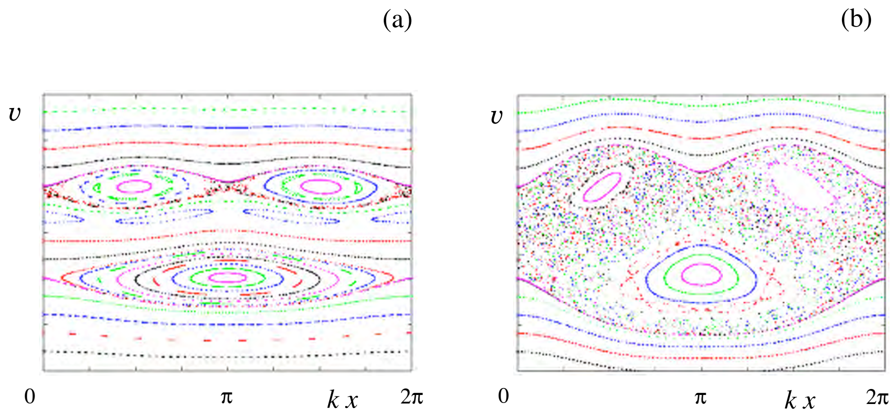


Fig. 3.6. Multiple separated resonances. Two waves (common in amplitude) with different frequencies (ω and $\omega + \Delta\omega$) coexist. When the amplitude is below the threshold, particles may be trapped in the vicinity of an individual resonance, but cannot interact with multiple resonances (a) $\omega_b^2/\Delta\omega^2 = 0.025$. When the amplitude is above the threshold, particle can stochastically wander or hop from resonance to resonance. This produces diffusion in velocity (b) $\omega_b^2/\Delta\omega^2 = 0.1$. [Courtesy of Prof. A. Fukuyama].

the resonant diffusion equation obtained in quasi-linear theory. Thus, we see that *the fundamental origin of the irreversibility presumed by the quasi-linear theory is the stochasticity of resonant particle trajectories*. While research on the question of the precise wave amplitude necessary for stochasticity is still ongoing, *the Chirikov overlap criterion (Eq.(3.6)) is a good 'working rule', and so constitutes a necessary condition for the applicability of the quasi-linear theory of resonant diffusion*. Note that, in contrast to the presentations given in older texts, no assumption of "Random wave phases", or "Random phase approximation", is necessary, apriori. Particle orbit stochasticity is the ultimate underpinning of the quasi-linear diffusion equation.

3.2.2 Linear response

At this point, the alert reader may be wandering about the use of linearized trajectories (i.e. unperturbed orbits) in proceeding from Eq.(3.5a) to Eq.(3.5b). Of course, linearization of δf occurs in the derivation of the quasi-linear theory, as well. This question brings us to a second important issue, namely that of the spectral auto-correlation time. The configuration of the electric field $E(x,t)$ which a particle actually "sees" at any particular x, t is a pattern formed by the superposition of the various modes in the spectrum, as depicted by the cartoon in Fig.3.2.2(a). For an evolving spectrum of (usually) dispersive waves, this pattern will persist for some lifetime τ_{L}. The pattern lifetime τ_{L} should be compared to the 'bounce time' of a particle in the pattern. Here the bounce time is simply the time required for a particle to reverse direction and return to the close proximity of its starting point. Two outcomes of the comparison are possible. These are

i) $\tau_{\mathrm{L}} \ll \tau_{\mathrm{b}}$	$\rightarrow$	field pattern changes prior to particle bouncing,
(Fig.3.2.2(b))		so that trajectory linearization is *valid*.
ii) $\tau_{\mathrm{b}} \ll \tau_{\mathrm{L}}$	$\rightarrow$	the particle bounces prior to a change in the field
(Fig.3.2.2(c))		pattern. In this case, *trapping* can occur,
		so linearized theory *fails*.

Not surprisingly, quasi-linear theory is valid when $\tau_{\mathrm{L}} \ll \tau_{\mathrm{b}}$, so that unperturbed orbits are a good approximation. The question which remains is how to relate our conceptual notations of τ_{L}, τ_{b} to actual physical quantities which characterize the wave spectrum.

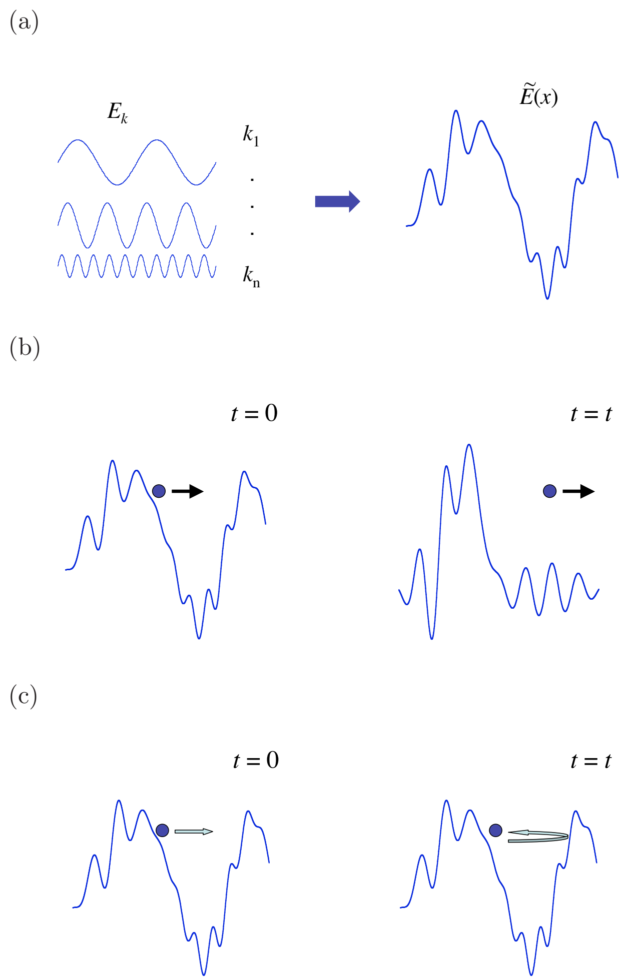


Fig. 3.7. (a) Cartoon of instantaneous pattern of electric field which a particle actually sees. The pattern has an effective duration time of τ_{ac}. (b) Cartoon showing that for $\tau_{\mathrm{ac}} < \tau_{\mathrm{b}}$, the E-field pattern a particle sees will change before the particle bounces, thus validating the use of unperturbed orbits. (c) Cartoon showing that for $\tau_{\mathrm{b}} < \tau_{\mathrm{ac}}$, the particle will bounce within a field pattern before the pattern changes. In this case, trapping occurs and the use of unperturbed orbits is not valid.

3.2.3 Characteristic time-scales in resonance processes

The *key point* for determining the value of $\tau_{\rm L}$ is the *realization that wave dispersion is what limits the pattern lifetime, τ_L*. Note the total electric field may be written (as before) as

$$E(x,t) = \sum_k E_k e^{i(kx-\omega t)}$$

or as

$$= \sum_k E_k \exp[i(k[x - v_{\rm ph}(k)t])]$$

where $v_{\rm ph}(k) = \omega(k)/k$. The pattern or packet dispersal speed is $\Delta(\omega_k/k)$, the net spread in the phase velocities in the packet. The net dispersal rate, i.e. the inverse time for a wave-packet to disperse one wavelength, then is just

$$\begin{aligned} 1/\tau_L &= k|\Delta(\omega_k/k)| \\ &= k\left|\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)\right| \\ &= |\,(v_{\rm g}(k) - v_{\rm ph}(k))\,\Delta k|. \end{aligned} \tag{3.7}$$

Equation(3.7) relates the pattern lifetime to Δk, the spectral width in k, and the net dispersion in velocity, which is just the difference between the phase ($v_{\rm ph}$) and group ($v_{\rm g}$) velocities. That is, the resonant particle, which has the velocity v_p, feels the difference of phase and group speeds, owing to the change of phase by wave dispersion (See Fig.3.8.) Note that regardless of Δk, $\tau_{\rm L} \to \infty$ for non-dispersive waves. In this case, the pattern coherence time must necessarily be set by wave steepening and breaking, or some other strongly nonlinear effect, which is outside the scope of quasilinear theory. Thus, we conclude that the applicability of quasi-linear theory is limited to $\langle f \rangle$ evolution in the presence of a sufficiently broad spectrum of dispersive waves. Interestingly, despite the large volume of research on the validity of quasilinear theory, this seemingly obvious point has received very little attention. Of course, the quantitative validity of quasi-linear theory requires that $1/\tau_{\rm b} < 1/\tau_{\rm L}$, so using

$$\frac{1}{\tau_{\rm b}} \simeq k\sqrt{\frac{q\phi_{\rm res}}{m}} \tag{3.8}$$

gives an upper bound on the bounce frequency $\sim 1/\tau_{\rm b}$ that is

$$\sqrt{\frac{q\phi_{\rm res}}{m}} < |v_{\rm g} - v_{\rm ph}|. \tag{3.9}$$

Here $\phi_{\rm res}$ is the potential of the waves in resonance with the particle. *Equation* (3.9) *gives an important upper bound on amplitude for the validity of quasilinear theory. Both* Eq.(3.6) and Eq.(3.9) must be satisfied for applicability of the quasi-linear equations.

One can isolate the range where both the Eq.(3.6) and (3.9) are satisfied. In the argument deriving Eq.(3.6), one considers the case that the neighbouring modes k_j and k_{j+1} have a similar amplitude. We also use an evaluation $\omega_{j+1} = \omega_j + (k_{j+1} - k_j)\partial\omega/\partial k$, where ω_j is the wave frequency for k_j. The phase velocity for the k_{j+1} mode, $v_{p,j+1}$, is given as

$$v_{p.j+1} \simeq v_{p,j} + (v_{p,j} - v_{g.j})\,(k_{j+1} - k_j)\,k_j^{-1}.$$

Thus, Eq.(3.6) is rewritten as

$$\sqrt{\frac{e\phi}{m}} \geq |v_{p,j} - v_{g,j}|\left(k_{j+1}k_j^{-1} - 1\right). \tag{3.10}$$

Combining Eqs.(3.9) and (3.10), the range of validity, for the quasi-linear theory, is given as

$$\frac{|v_p - v_g|}{kL} \leq \sqrt{\frac{e\phi}{m}} \leq |v_p - v_g|\,, \tag{3.11}$$

where the difference $k_{j+1} - k_j$ is given by L^{-1} (L : the system size). Therefore, the validity of the quasi-linear theory also requires that the wave length must be much shorter than the system size.

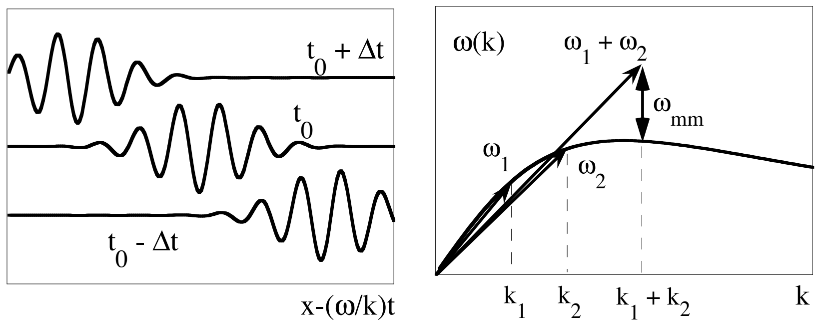


Fig. 3.8. Illustration of finite interaction times. Left: Wave packet in the frame of the resonant particles which are moving at the phase velocity ω/k. When the group velocity $\partial\omega/\partial k$ is different from the phase velocity (here, the case of $\partial\omega/\partial k < \omega/k$ is shown), a wave packet passes by the resonant particle. Therefore, the interaction time is limited. Right: mis-match of the frequency in the case where modes with k_1, k_2 and $k_1 + k_2$ are nonlinearly coupling.

3.2.4 Two-point and two-time correlations

In order to place the discussion given here on a more solid foundation, we now consider the two-point, two-time correlation $\langle E(x_1, t_1)E(x_2, t_2)\rangle$ along the particle orbit. Here the brackets refer to a space-time average. The goal here is to rigorously demonstrate the equivalence between the heuristic packet dispersal rate given in Eq.(3.7) and the actual spectral auto-correlation rate, *as seen by a resonant particle.* Now for homogeneous, stationary turbulence, the field correlation function simplifies to:

$$\langle E(x_1, t_1)E(x_2, t_2)\rangle = C(x_-, t_-), \tag{3.12}$$

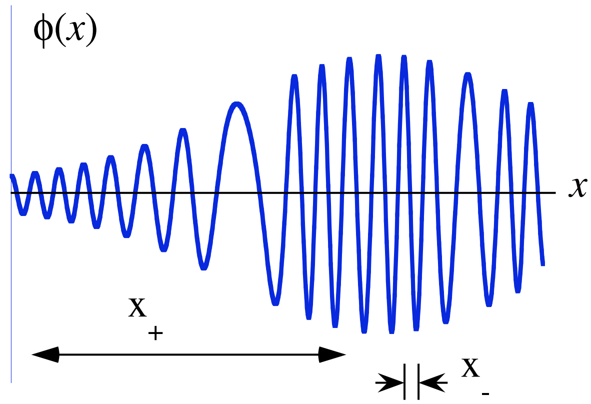


Fig. 3.9. Small scale variable and large scale variable for fluctuations.

where

$$x_1 = x_+ + x_-$$
$$x_2 = x_+ - x_- \tag{3.13a}$$

and

$$t_1 = t_+ + t_-$$
$$t_2 = t_+ - t_- \tag{3.13b}$$

The variables (x_-, t_-) denote the wave phase and (x_+, t_+) describe the slow variation of the envelope, as is illustrated in Fig.3.9 schematically. Upon taking the average over x_+, t_+, a short calculation then gives

$$C(x_-, t_-) = \sum_k |E_k|^2 \exp[i(kx_- - \omega_k t_-)]. \tag{3.14}$$

Evaluating x_- along unperturbed orbits, so that

$$x_- = x_{0-} + vt_-, \tag{3.15}$$

and assuming, for convenience, a continuous spectrum of the form

$$|E_k|^2 = \frac{E_0^2}{\Delta k}\left[\left(\frac{k-k_0}{\Delta k}\right)^2 + 1\right]^{-1} \tag{3.16}$$

then allows us to write the correlation function $C(x_-, t_-)$ in the simple, explicit form

$$C(x_-, \tau) = \int \frac{dk}{\Delta k}\ \frac{E_0^2\ e^{ikx_{0-}}\ e^{i(kv-\omega_k)t_-}}{\left[\left(\frac{k-k_0}{\Delta k}\right)^2 + 1\right]}. \tag{3.17}$$

Here $|E_0|^2$ is the spectral intensity, Δk is the spectral width, and k_0 is the centroid of the spectral distribution. Expanding $kv - \omega_k$ as

$$kv - \omega_k \simeq k_0 v - \omega_{k0} + \Delta(kv - \omega_k)(k - k_0) + \cdots ,$$

the integral in Eq.(3.17) can now easily be performed by residues, yielding

$$C(x_-, \tau) = 2\pi E_0^2\ e^{ikx_{0\text{-}}} e^{i(k_0 v - \omega_{k_0})\tau} \times \exp\left[-\Delta|kv - \omega_k|\tau - |\Delta k| x_{0\text{-}}\right]. \tag{3.18}$$

As is illustrated in Fig.3.10, Eq.(3.18) is an explicit result for the two point correlation, constructed using a model spectrum. Equation (3.18) reveals that correlations decay in time according to

$$C(x_-, \tau) \sim \exp[-\Delta|kv - \omega_k|\tau] \tag{3.19}$$

that is by frequency dispersion $\Delta(\omega_k)$ *and* its interplay with particle streaming, via $\Delta(kv)\tau$. Note that it is, in fact, the *width of the Doppler-shifted frequency which sets the spectral auto-correlation time, τ_{ac}*. Now,

$$1/\tau_{\mathrm{ac}} = |\Delta(kv - \omega_k)| = |(v - v_{\mathrm{gr}})\Delta k|, \tag{3.20a}$$

so, *for resonant particles* with $v = \omega/k = v_{\mathrm{ph}}$,

$$1/\tau_{\mathrm{ac}} = |(v_{\mathrm{ph}} - v_{\mathrm{gr}})\Delta k|, \tag{3.20b}$$

which is identical to the heuristic estimate of the pattern lifetime given in Eq.(3.7). Thus, we indeed have demonstrated that the dispersion in the

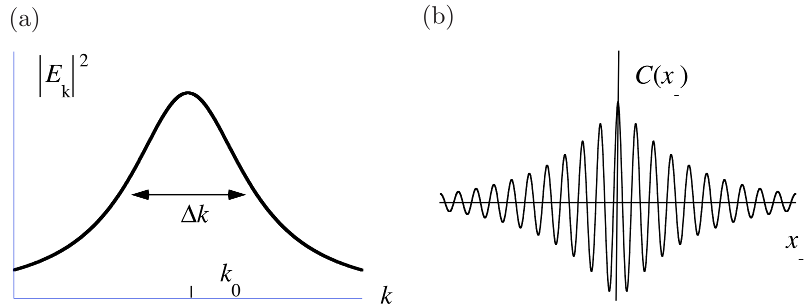


Fig. 3.10. An example of the power spectrum of electric field fluctuation, which is characterized by the peak and width of the wave number (a). Correlation function is given in (b).

Table 3.1.

$1/\tau_{\rm ac} = \lvert(v_{\rm gr} - v_{\rm ph})\Delta k\rvert$	the auto-correlation time or lifetime of the electric field pattern, as sensed by resonant particles.
$\gamma_{\boldsymbol{k}}$	the wave growth or damping rate, as determined by the linear dispersion relation.
$1/\tau_{\rm b} = k\left(\frac{q\phi_{\rm res}}{m}\right)^{1/2}$	the 'bounce' or 'trapping time' for resonant particles in the total packet potential.
$1/\tau_{\rm relax} = \frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t}$	the rate of slow relaxation of the average distribution function.

Doppler shifted frequency as 'seen' by a resonant particle (moving along an unperturbed orbit) sets the spectral auto-correlation time and thus the lifetime of the field pattern which the particle senses.

We now summarize this discussion by reviewing the basic time-scales characteristic of quasi-linear theory, and the relationships between them which are necessary for the applicability of quasi-linear theory. The basic temporal rates (i.e. inverse time scales $\sim 1/\tau$) are summarized in Table 3.1. As discussed above, several conditions must be satisfied for quasi-linear theory to be relevant. These are:

$$1/\tau_{\mathrm{b}} < 1/\tau_{\mathrm{ac}} \tag{3.21a}$$

for the use of unperturbed orbits (linear response theory) to be valid,

$$1/\tau_{\mathrm{relax}} \ll 1/\tau_{\mathrm{ac}}, \gamma_k \tag{3.21b}$$

for the closure of the $\langle f \rangle$ equation to be meaningful,

$$1/\tau_{\mathrm{relax}} < \gamma_k < 1/\tau_{\mathrm{ac}} \tag{3.21c}$$

for the quasi-linear equations to be applicable.

Of course, the irreversibility of resonant quasi-linear diffusion follows from the stochasticity of particle orbits, which in turn requires that the Chirikov overlap criterion (Eq.(3.6)) be met. (see Fig.3.11.) In retrospect, we see that applicability of the 'trivial' quasi-linear theory naively follows from several rather precise and sometimes even subtle conditions!

3.2.5 Short note on entropy production

At this conclusion of our discussion of the origin of irreversibility in quasi-linear theory, it is appropriate to briefly comment on entropy. The Vlasov equation leaves entropy invariant, since entropy

$$S = \int \mathrm{d}v \int \mathrm{d}x \, s(f),$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0$$

in a Vlasov plasma. The quasi-linear equation involves a *coarse graining*, as it describes the evolution of $\langle f \rangle$, not f. Hence, it should be no surprise that quasi-linear relaxation can produce entropy, since such entropy production

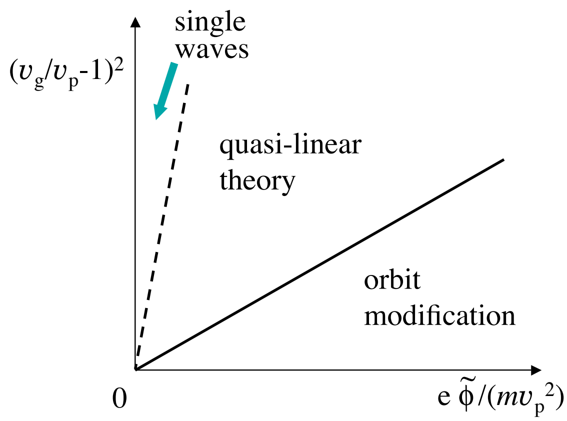


Fig. 3.11. Range of applicability for the quasi-linear theory, Eq.(3.11). Amplitude is normalized by the particle energy at phase velocity on the horizontal axis. Vertical axis shows the magnitude of dispersion, i.e., the difference between the group velocity and phase velocity.

is intrinsic to phenomena such as resonant particle heating, etc, which occur in the course of the evolution and saturation of plasma turbulence. Remember that the irreversible quasi-linear evolution of $\langle f \rangle$ requires the onset of chaos, Eq.(3.6). A deeper connection between resonant quasi-linear diffusion and entropy production enters via the requirement that particle orbits be stochastic. Strictly "stochastic" means that at least one positive Lypunov exists so the KS (Kolmogorov-Sinai) entropy is positive, i.e. $h > 0$. Any definition of dynamical entropy entails the definition of some partition of phase space, which also constitutes a coarse graining. We see that coarse graining, and thus entropy production, are intrinsic to the foundations of quasi-linear theory.

3.3 Energy and Momentum Balance in Quasi-linear Theory

3.3.1 Various energy densities

It is no surprise that energy and momentum conservation are non-trivial concerns, since the basic quasilinear equation for $D(v)$, Eq.(3.4d), makes a clear distinction between resonant and non-resonant particles. Resonant particles, for which

$$D_{\rm R}(v) = \sum_k \frac{q^2}{m^2} |E_k|^2 \pi \delta(\omega - kv), \tag{3.22a}$$

exchange energy with waves irreversibly, via Landau resonance. Note that the resonant diffusion coefficient does not depend on the wave growth rate. Non-resonant particles, for which

$$D_{\rm NR}(v) \simeq \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{|\gamma_k|}{(\omega - kv)^2}, \tag{3.22b}$$

support the wave by oscillating in it. Their motion is *reversible*, and their quiver velocities increase or decrease with the wave amplitude. Hence, $D_{\rm NR}(v)$ is explicitly proportional to $|\gamma_k|$, in contrast to $D_{\rm R}$. It is interesting to note that for $\omega \gg kv$, the non-resonant diffusion reduces to

$$\begin{aligned} D_{\rm NR} &= \sum_k \frac{q^2}{m^2} \left|\tilde{E}_k\right|^2 \frac{|\gamma_k|}{\omega^2} \\ &= \left(\frac{1}{n_0 m}\right) \left|\frac{\partial}{\partial t} E_p\right| \end{aligned} \tag{3.23a}$$

where $E_{\rm p}$ is the ponderomotive (or quiver) energy density

$$E_{\rm p} = \sum_k \frac{1}{2} \frac{n_0 q^2}{m^2} \frac{\left|\tilde{E}_k\right|^2}{\omega_k^2}. \tag{3.23b}$$

This observation illustrates that non-resonant diffusion is simply due to reversible quivering of particles in the wave field. Thus, *non-resonant diffusion cannot produce entropy.* Indeed, to understand non-resonant diffusion and energetics in quasi-linear theory, it is important to keep in mind that the familiar quantity, the total wave energy density W

$$W = \frac{\partial}{\partial \omega_k}(\omega\epsilon)\bigg|_{\omega_k} \frac{|E_k|^2}{8\pi}, \tag{3.24}$$

contains contributions from *both* the electric field energy density (E^{ef})

$$E^{\mathrm{ef}} = |E_k|^2/8\pi, \tag{3.25a}$$

and the non-resonant particle kinetic energy density $E_{\mathrm{kin}}^{\mathrm{nr}}$. This point is illustrated by considering simple Langmuir oscillations of amplitude E_0 with $\epsilon = 1-\omega_{\mathrm{p}}^2/\omega^2$, for which $E^{\mathrm{ef}} = |E_0|^2/8\pi$ while $W = |E_0|^2/4\pi$. A short calculation reveals that the remaining contribution of $|E_0|^2/8\pi$ is simply the non-resonant particle kinetic energy density ($E_{\mathrm{kin}}^{\mathrm{nr}}$), which is equal in magnitude to the E^{ef} for Langmuir waves. This is easily seen, since $E_{\mathrm{kin}}^{\mathrm{nr}} = (1/2)nm\,|\tilde{v}|^2$ and $\tilde{v} = q\tilde{E}/\omega m$. Together these give $E_{\mathrm{kin}}^{\mathrm{nr}} = (1/8)\omega_{\mathrm{p}}^2 \left|\tilde{E}\right|^2/8\pi\omega^2$, so that for $\omega = \omega_{\mathrm{p}}$, the identity $E_{\mathrm{kin}}^{\mathrm{nr}} = E^{\mathrm{ef}}$ is clear. Indeed, the thrust of this discussion suggests that since quasi-linear theory divides the particles into *two* classes, namely resonant and non-resonant, there should be *two* ways of balancing the total energy budget. Below, we show that an energy conservation relation can be formulated either as a balance of

resonant particle kinetic energy density	$E_{\mathrm{kin}}^{\mathrm{res}}$	vs	total wave energy density	W

or of

particle kinetic energy density	E_{kin}	vs	electric field energy density	E^{ef}

Momentum balance exhibits similar duality.

3.3.2 Conservation of energy

To prove conservation of energy between resonant particles and waves, one must first determine the rate of change of total particle kinetic energy density E_{kin} by taking the energy moment of the Vlasov equation, i.e.

$$\begin{aligned}\frac{\partial}{\partial t}E_{\mathrm{kin}} &= \frac{\partial}{\partial t}\int \mathrm{d}v\,\frac{mv^2}{2}\langle f\rangle \\ &= \int \mathrm{d}v\,qv\langle \tilde{E}\delta f\rangle\end{aligned} \tag{3.26}$$

where Eq.(3.2b) is substituted and the partial integration is performed. Because we are studying the balance in the framework of the quasi-linear theory, δf is approximated by the linear Vlasov response, so Eq.(3.26) gives

$$\frac{\partial}{\partial t}E_{\mathrm{kin}} = -i\int \mathrm{d}v\,\frac{vq^2}{m}\sum_k |E_k|^2\left(\frac{P}{\omega-kv} - i\pi\delta\left(\omega-kv\right)\right)\frac{\partial\langle f\rangle}{\partial v} \tag{3.27}$$

where P indicates the principal part of the integral and the familiar Plemelj formula has been used to decompose the linear response into resonant and non-resonant pieces. Choosing the resonant piece, we can express the rate of charge of resonant particle kinetic energy as

$$\begin{aligned}\frac{\partial}{\partial t}E_{\mathrm{kin}}^{\mathrm{res}} &= -\int \mathrm{d}v\,\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\delta\left(\frac{\omega}{k}-v\right)\frac{\partial\langle f\rangle}{\partial v}\,|E_k|^2 \\ &= -\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\,\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}|E_k|^2\,.\end{aligned} \tag{3.28}$$

To relate Eq.(3.28) to the change in wave energy density (using Eq.(3.24)), we may straightforwardly write,

$$\begin{aligned}\frac{\partial W}{\partial t} &= \sum_k 2\gamma_k\,\frac{\partial}{\partial\omega}\left(\omega\epsilon\right)\bigg|_{\omega_k}\frac{|E_k|^2}{8\pi} \\ &= \sum_k 2\gamma_k\omega_k\,\frac{\partial\epsilon}{\partial\omega}\bigg|_{\omega_k}\frac{|E_k|^2}{8\pi}.\end{aligned} \tag{3.29}$$

Now, for γ_k:

$$\epsilon = 1 + \frac{\omega_{\mathrm{p}}^2}{k}\int \mathrm{d}v\,\frac{\partial\langle f\rangle}{\omega-kv}, \tag{3.30}$$

and

$$\epsilon_{\mathrm{r}}\left(\omega_k + i\gamma_k\right) + i\mathrm{Im}\ \epsilon = 0, \tag{3.31}$$

so

$$\gamma_k = -\frac{\mathrm{Im}\ \epsilon}{\left(\partial\epsilon_{\mathrm{r}}/\partial\omega\right)\big|_{\omega_k}}. \tag{3.32}$$

Substituting Eq.(3.32) into Eq.(3.29) gives:

$$\frac{\partial W}{\partial t} = -\sum \omega_k \mathrm{Im}\ \epsilon\left(k, \omega_k\right)\frac{|E_k|^2}{4\pi}. \tag{3.33}$$

However, from Eq.(3.30) we have:

$$\mathrm{Im}\ \epsilon\left(k, \omega_k\right) = -\frac{\pi\omega_{\mathrm{p}}^2}{|k|k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}. \tag{3.34}$$

Substituting Eq.(3.34) into Eq.(3.33) then gives:

$$\frac{\partial W}{\partial t} = \frac{\pi q^2}{m}\sum_k \frac{\omega_k}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2 \tag{3.35}$$

where the density dependence of $\langle f\rangle$ has been factored out, for convenience. Comparing Eq.(3.28) and Eq.(3.35), we see that, within the scope of quasi-linear theory, we have demonstrated that

$$\frac{\partial}{\partial t}\left(E_{\mathrm{kin}}^{\mathrm{res}} + W\right) = 0 \tag{3.36}$$

i.e. that energy is conserved between collective modes ("waves") and resonant particles. Equation (3.36) is the fundamental energy conservation relation for quasi-linear theory.

Several comments are in order here. First, the quasi-linear energy conservation relation proved above is just a special case of the more general Poynting theorem for plasma waves, which states that:

$$\frac{\partial W}{\partial t} + \boldsymbol{\nabla}\cdot S + Q = 0 \tag{3.37}$$

i.e. wave energy density W is conserved against wave radiation ($\nabla \cdot S$, where S is the wave energy density flux) and dissipation ($Q = \langle E \cdot J \rangle$), where E is the electric field and J is the current. For a homogeneous system $\nabla \cdot S = 0$, so the Poynting relation reduces to just $\partial W/\partial t + \langle E \cdot J \rangle = 0$. Computing the plasma current J_k using the linear response $\tilde{f}_k$ then yields an expression identical to Eq.(3.36). The physics here is a simple consequence of the fact that only resonant particles "see" a DC electric field, so only they can experience a time averaged $\langle E \cdot J \rangle$.

3.3.3 Role of quasi-particles and particles

A second element of this discussion reveals an alternative form of the energy theorem. As discussed above, the total wave energy density W may be decomposed into pieces corresponding to the field energy density (E^{ef}) and the non-resonant particle kinetic energy density ($E^{\text{nr}}_{\text{kin}}$). In these terms, the quasi-linear energy conservation theorem can be written as shown below. We have demonstrated explicitly Eq.(3.36) that:

$$\frac{\partial}{\partial t} W + \frac{\partial}{\partial t} E^{\text{res}}_{\text{kin}} = 0$$

but also have noted the physically motivated decomposition

$$W = E^{\text{ef}} + E^{\text{nr}}_{\text{kin}}$$

so we have

$$\frac{\partial}{\partial t}(E^{\text{ef}} + E^{\text{nr}}_{\text{kin}}) + \frac{\partial}{\partial t} E^{\text{res}}_{\text{kin}} = 0.$$

Then, a re-grouping gives:

$$\frac{\partial}{\partial t} E^{\text{ef}} + \frac{\partial}{\partial t}(E^{\text{res}}_{\text{kin}} + E^{\text{nr}}_{\text{kin}}) = 0$$

where $E_{\text{kin}}^{\text{res}} + E_{\text{kin}}^{\text{nr}} = E_{\text{kin}}$, the total particle kinetic energy density. This we arrive at an alternative form of the energy conservation theorem, namely that

$$\frac{\partial}{\partial t}(E^{\text{ef}} + E_{\text{kin}}) = 0 \tag{3.38}$$

i.e. electric field energy density E^{ef} is conserved against total particle kinetic energy density E_{kin} from Eq.(3.27) *without* the (Plemelj) decomposition of the response into resonant and non-resonant pieces. Returning to Eq.(3.27), we proceed as

$$\frac{\partial}{\partial t}E_{\text{kin}} = -\sum_k \int \mathrm{d}v\, \frac{\omega_{\text{p}}^2}{k}(kv)\frac{|E_k|^2}{4\pi}\left(\frac{1}{\omega - kv}\right)\frac{\partial\langle f\rangle}{\partial v}. \tag{3.39a}$$

Now, using Eq.(3.30) for $\epsilon(k, \omega)$ we can write:

$$\begin{aligned}\frac{\partial}{\partial t}E_{\text{kin}} &= -i\sum_k \frac{|E_k|^2}{4\pi}\int \mathrm{d}v\, \frac{\omega_{\text{p}}^2}{k}\,(kv - \omega + \omega)\,\frac{1}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v} \\ &= -i\sum_k \frac{|E_k|}{4\pi}\int \mathrm{d}v\, \frac{\omega_{\text{p}}^2}{k}\frac{\omega}{\omega - kv}\frac{\partial\langle f\rangle}{\partial v}\end{aligned} \tag{3.39b}$$

as energy is real. Since $\epsilon\,(k, \omega_k) = 0$, by definition of ω_k, we thus obtain

$$\begin{aligned}\frac{\partial}{\partial t}E_{\text{kin}} &= i\sum_k \frac{|E_k|^2}{4\pi}\omega_k \\ &= -\sum_k \frac{|E_k|^2}{8\pi}\,(2\gamma_k) = -\frac{\partial}{\partial t}E^{\text{ef}}.\end{aligned} \tag{3.39c}$$

Thus completes the explicit proof of the relation $\partial(E_{\text{kin}} + E^{\text{ef}})/\partial t = 0$. The energy conservation laws of quasi-linear theory are summarized in Table 3.2. As indicated in the table, the two forms of the quasi-linear energy conservation theorem are a consequence of the two possible conceptual models of a turbulent plasma, namely as an ensemble of either:

a) quasi-particles(waves) and resonant particles, for which $\partial(W + E_{\text{kin}}^{\text{res}})/\partial t = 0$, Eq.(3.36), is the appropriate conservation theorem, or

b) particles(both resonant and non-resonant) and electric fields, for which $\partial(E_{\text{kin}} + E^{\text{ef}})/\partial t = 0$, Eq.(3.38) is the appropriate conservation theorem.

This distinction is possible since non-resonant diffusion can be counted either as:

a) the sloshing of particles which support the wave energy density

or as

b) part of the total particle kinetic energy density.

While both views are viable and valid, we will adopt the former in this book, as it is both appealingly intuitive and physically useful.

Finally, we note in passing that it is straightforward to show that the sum of resonant particle momentum and wave momentum ($P_W = k\,(\partial\epsilon/\partial\omega)_k \left|E_k\right|^2/8\pi$) is conserved. The proof closely follows the corresponding one for energy, above. No corresponding relation exists for particles and fields, since, of course, purely electrostatic fields have no momentum. In this case, the total particle momentum density is simply a constant. In electromagnetic problems, where the presence of magnetic fields allows a non-zero field momentum density (proportional to the Poynting flux), exchange of momentum between particles and fields is possible, so a second momentum conservation theorem can be derived.

3.4 Applications of Quasi-linear Theory to Bump-on-Tail Instability

As a complement to the rather general and theoretical discussion thus for, we now discuss two applications of quasi-linear theory -first, to the classic prob-

Table 3.2. *Energy balance theorems for quasilinear theory*

Constituents	Particles	Resonant $(v = \omega/k) \to E_{\rm kin}^{\rm res}$ Non-resonant $(v \neq \omega/k) \to E_{\rm kin}^{\rm nr}$
	Fields	Electric Field Energy $E^{\rm ef}$ Waves, Collective Modes $\to$ Total Wave Energy Density (W)
Perspectives	Resonant Particles vs Waves balance Particles vs Fields balance	
Relations and Conservation Balances	$\frac{\partial}{\partial t}\left(E_{\rm kin}^{\rm res} + W\right) \equiv 0 \leftrightarrow$ resonant particles vs waves $\frac{\partial}{\partial t}\left(E_{\rm kin} + E^{\rm ef}\right) \equiv 0 \leftrightarrow$ total particles vs electric field	

lem of the *bump-on-tail* instability in one dimension and then to transport and relaxation driven by drift wave turbulence in a 3D magnetized plasma. We discuss these two relatively simple examples in considerable depth, as they constitute fundamental paradigms, upon which other applications are built.

3.4.1 Bump-on-tail instability

The bump-on-tail instability occurs in the region of positive phase velocities which appears when a gentle beam is driven at high velocities, on the tail of a Maxwellian. The classic configuration of the bump on tail is shown in Fig.3.2(a). Based upon our previous discussion, we can immediately write down the set of quasi-linear equations:

$$\epsilon\left(k, \omega_k\right) = 0, \tag{3.40a}$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}, \tag{3.40b}$$

$$D = D_{\rm R} + D_{\rm NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi\delta(\omega - kv) + \frac{|\gamma_k|}{\omega^2} \right\}, \tag{3.40c}$$

$$\frac{\partial}{\partial t}\left(\frac{|E_k|^2}{8\pi}\right) = 2\gamma_k \left(\frac{|E_k|^2}{8\pi}\right). \tag{3.40d}$$

Initially $\epsilon(k, \omega_k)$ should be calculated using the distribution shown in Fig.3.2(a). It is interesting to note that the structure of the bump-on-tail distribution enables us to clearly separate and isolate the regions of resonant and non-resonant diffusion and heating, etc. In particular, since $\partial \langle f \rangle / \partial v > 0$ for a velocity interval on the tail, waves will be resonantly excited in that interval and particles in that region will undergo resonant diffusion. Similarly, since bulk particles are not resonant but do support the underlying Langmuir wave, we can expect them to undergo non-resonant diffusion, which can alter their collective kinetic energy but not their entropy.

3.4.2 Zeldovich theorem

Before proceeding with the specific calculation for the bump-on-tail problem, it is useful to discuss the general structure of relaxation in a Vlasov plasma and to derive a general constraint on the evolution of the mean distribution function $\langle f \rangle$ and on its end state. This constraint is a variant of a theorem first proved by Ya. B. Zeldvich in the context of transport of magnetic potential in 2D MHD turbulence. Proceeding, then, the Boltzmann equation says that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta f) = -\frac{q}{m} E \frac{\partial \langle f \rangle}{\partial v} + C(\delta f) \tag{3.41}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{q}{m}E\frac{\partial}{\partial v}$$

i.e. that fluctuation phase space density is conserved up to collisions (denoted by $C(\delta f)$) and relaxation of the phase space density gradients. Of course, total phase space density is conserved along particle orbits, up to collisions, *only*. Multiplying Eq.(3.41) by δf and averaging then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\left\langle \delta f^2 \right\rangle = \int \mathrm{d}v\,\left[-\frac{q}{m}\left\langle E\delta f\right\rangle \frac{\partial\left\langle f\right\rangle}{\partial v} + \left\langle \delta f C\left(\delta f\right)\right\rangle\right]. \qquad (3.42)$$

Here, the average implies an integration over space (taken to be periodic), so $\langle\ \rangle = \int \mathrm{d}x$, as well as the explicit integral over velocity. Thus, $\langle \mathrm{d}/\mathrm{d}t\rangle \rightarrow \partial/\partial t$. Furthermore, it is useful for physical transparency to represent $C(\delta f)$ using a Crook approximation $C(\delta f) = -\nu(\delta f)$, so that Eq.(3.42) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\left\langle \delta f^2 \right\rangle = \int \mathrm{d}v\,\left[-\frac{q}{m}\left\langle E\delta f\right\rangle \frac{\partial\left\langle f\right\rangle}{\partial v} - \nu\left\langle \delta f\right\rangle\right].$$

Ignoring collisions for the moment, Eq.(3.42) simply states the relation between mean square fluctuation level and the relaxation of the mean distribution fluctuation embodied by the Vlasov equation, i.e.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0. \qquad (3.43\mathrm{a})$$

and

$$f = \langle f\rangle + \delta f \qquad (3.43\mathrm{b})$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle f\rangle + \delta f\right)^2 = 0. \qquad (3.43\mathrm{c})$$

Averaging them gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\langle\delta f^2\rangle = -\int \mathrm{d}v\,\langle f\rangle\frac{\partial\langle f\rangle}{\partial t} = \int \mathrm{d}v\,\langle f\rangle\frac{\partial}{\partial v}\left\langle\frac{q}{m}E\delta f\right\rangle \tag{3.43d}$$

since, of course,

$$\frac{\partial\langle f\rangle}{\partial t} = -\frac{\partial}{\partial v}\left\langle\frac{q}{m}E\delta f\right\rangle. \tag{3.43e}$$

The content of the relation between the LHS and RHS of Eq.(3.43d) is obvious-relaxation of $\langle f\rangle$ drives $\langle\delta f^2\rangle$.

Till now, the calculation has been formal, reflecting only the conservative symplectic structure of the Vlasov-Boltzmann equation, Equation (3.42) is a structure relating fluctuation growth to transport ($\sim \frac{q}{m}\langle E\delta f\rangle$) and collisional damping. To make contact with quasi-linear theory, we close Eq.(3.42) by taking $\delta f \to f^c$, the coherent linear response, in $\langle E\delta f\rangle$. (The role of incoherent part $\tilde{f}$ in $\delta f, \delta f = f^c + \tilde{f}$, is explained in Chapter 8.) This gives the *Zeldovich relation*

$$\frac{\partial}{\partial t}\int \mathrm{d}v\,\langle\delta f^2\rangle = \int \mathrm{d}v\,D\left(\frac{\partial\langle f\rangle}{\partial v}\right)^2 - \int \mathrm{d}v\,\nu\langle\delta f^2\rangle \tag{3.44}$$

which connects fluctuation growth to relaxation and collisional damping. Here D is the quasi-linear diffusion coefficient, including both resonant and non-resonant contributions, i.e.

$$D = D_{\mathrm{R}} + D_{\mathrm{NR}}.$$

3.4.3 Stationary states

The point of this exercise becomes apparent when one asks about the nature of a stationary state, i.e. where $\partial\langle\delta f^2\rangle/\partial t = 0$, which one normally associates with instability saturation. In that case, Eq.(3.44) reduces to

$$\int \mathrm{d}v\, D_{\mathrm{R}} \left(\frac{\partial\langle f\rangle}{\partial v}\right)^2 = \int \mathrm{d}v\, \nu\langle\delta f^2\rangle \tag{3.45}$$

which states that fluctuation growth by resonant instability induced relaxation and transport *must* balance collisional damping in a stationary state. This is the Vlasov analogue of the production-dissipation balance generic to the mixing length theory and to turbulent cascades. Notice that *non-resonant diffusion necessarily vanishes at stationarity, since* $D_{NR} \sim |\gamma|$, *explicitly.* With the important *proviso that* δf *should not develop singular gradients, then Eq.*(3.45) *states that for a collisionless* ($\nu \to 0$), *stationary plasma*, $\int \mathrm{d}v\, D_{\mathrm{R}}\, (\partial\langle f\rangle/\partial v)^2$ *must vanish.* Hence either $\partial\langle f\rangle/\partial v \to 0$, so that the mean distribution function flattens (i.e. forms a *plateau*) at resonance, or $D_{\mathrm{R}} \to 0$, i.e. the saturated electric field spectrum decays and vanishes. These are the two possible end-states of quasi-linear relaxation. Notice also that Eq.(3.45) states that any deviation from the plateau or $D_{\mathrm{R}} = 0$ state must occur via the action of collisions, alone, and so the rate at which this deviation evolves must be proportional to the collision frequency.

3.4.4 Selection of stationary state

We now proceed to discuss which state (i.e., $D_R = 0$ or $\partial\langle f\rangle/\partial v = 0$) is *actually* selected by the system by explicitly calcutlating the time dependence of the resonant diffusivity.

To determine the time evolution of D_{R}, it is convenient to first re-write it as

$$D_{\mathrm{R}} = 16\pi^2 \frac{q^2}{m^2} \int_0^\infty \mathrm{d}k\, E^{\mathrm{ef}}(k)\delta(\omega - kv) \tag{3.46}$$

where $E^{\mathrm{ef}}(k) = |E_k|^2/8\pi$. Then we easily see that $\partial D_{\mathrm{R}}/\partial t$ is given by

$$\frac{\partial D_{\mathrm{R}}}{\partial t} = \frac{16\pi^2 q^2}{m^2 v}\left(2\gamma_{\omega_{\mathrm{pe}}/v}\right) E^{\mathrm{ef}}\left(\frac{\omega_{\mathrm{pe}}}{v}\right) \tag{3.47a}$$

where $\omega_k = \omega_{\mathrm{pe}}$. Since $\gamma_k = \gamma_{\omega_{\mathrm{pe}}/v} = \pi v^2 \omega_{\mathrm{pe}}\left(\partial\langle f\rangle/\partial v\right)$, a short calculation then yields

$$D_{\mathrm{R}}(v,t) = D_{\mathrm{R}}(v,0)\exp\left[\pi\omega_{\mathrm{pe}} v^2 \int_0^t \mathrm{d}t'\, \frac{\partial\langle f\rangle}{\partial v}\right]. \tag{3.47b}$$

Using Eq.(3.46) and the expression for γ_k, we also find that

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial v}\left[\frac{D_{\mathrm{R}}(v,t)}{\pi\omega_{\mathrm{pe}} v^2}\right] \tag{3.48a}$$

so

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left(\frac{D_{\mathrm{R}}(v,t) - D_{\mathrm{R}}(v,0)}{\pi\omega_{\mathrm{pe}} v^2}\right). \tag{3.48b}$$

Taken together, Eqs.(3.47b) and (3.48b) simply that quasi-linear saturation must occur via plateau formation. To see this, assume the contrary, i.e. that $D_{\mathrm{R}} \to 0$ as $t \to \infty$. In that case, Eq.(3.48b) states that

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D_{\mathrm{R}}(v,0)}{\pi\omega_{\mathrm{p}} v^2}\right]. \tag{3.49}$$

Since $D_{\mathrm{R}}(v,0) = 16\pi^2 q^2 E\left(\omega_{\mathrm{p}}/v, 0\right)\left(m^2 v\right)^{-1}$, it follows that

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\frac{2E^{\mathrm{ef}}\left(\omega_{\mathrm{p}}/v, 0\right)}{nmv^2/2} \tag{3.50}$$

so $\langle f(t)\rangle \cong \langle f(0)\rangle$, up to a small correction of O(initial fluctuation energy/ bump energy) $\times$ (n_{b}/n), where the bump density n_{b} satisfies $n_{\mathrm{b}}/n \ll 1$. Hence $\langle f(v,t)\rangle \cong \langle f(v,0)\rangle$ to excellent approximation. However, if $D_{\mathrm{R}} \to 0$ as $t \to \infty$, damped waves require $\partial\langle f\rangle/\partial v < 0$, so $\langle f(v,t)\rangle$ *cannot* equal $\langle f(v,0)\rangle$, and a contradiction has been established. Thus, the time asymptotic state which the system actually selects is one where a plateau forms for which $\partial\langle f\rangle/\partial v \xrightarrow[t\to\infty]{} 0$, in the region of resonance.

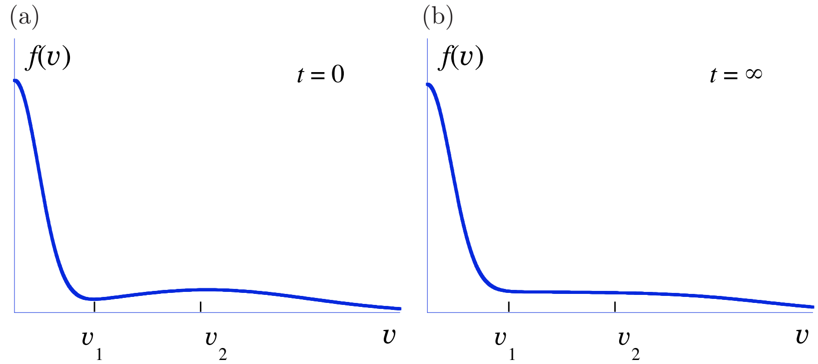


Fig. 3.12. The plateau formation process: initial state (a) and final state (b).

To calculate the actual plateau state, it is important to realize that *two* processes are at work, simultaneously. First, resonant particles will be stochastically scattered, so as to drive $\partial f/\partial v \rightarrow 0$ by *filling in* lower velocities. This evolution shown in Fig.3.12 is similar to the propagation of a front of δf from the bump to lower velocities which fall in between the bulk Maxwellian and the bump-on-tail. The end state of the plateau is shown in Fig.3.12(b). Second, the non-resonant bulk particles will experience a *one-sided heating* (for $v > 0$, only) as waves grow during the plateau formation process. It is important to realize that this heating is *fake heating* and does not correspond to an increase in bulk particle entropy, since it originates from non-resonant diffusion. The heating is one sided in order to conserve total momentum between bump-on-tail particles (which slow down) and bulk particles, which so must speed up.

To actually calculate the time-asymptotic distribution function and fluctuation saturation level, it is again convenient to separate the evolution into resonant and non-resonant components. The actual saturation level is most expeditiously calculated using the conservation relation $\partial(E_{\rm kin}^{\rm res}+W)/\partial t = 0$.

This allows us to equate the change in kinetic energy in the resonant velocity region with the change in the energy of waves in the corresponding region of k values. Thus

$$\Delta\left(\int_{v_1}^{v_2} \mathrm{d}v\, \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathrm{d}k\, E^{\mathrm{ef}}(k)\,. \tag{3.51}$$

Here v_1 and v_2 correspond to the lower and upper limits of the range of instability, and, using $k = \omega_\mathrm{p}/v$, $k_2 = \omega_\mathrm{p}/v_1$, $k_1 = \omega_\mathrm{p}/v_2$. The factor of 2 which appears on the RHS of Eq.(3.51) reflects the fact that non-resonant particle kinetic energy and field energy (E^{ef}) contribute equally to the total wave energy. Then, assuming the fields grow from infinitesimal levels, the total saturated field energy is then just

$$\int_{k_1}^{k_2} dk\, E^{\mathrm{ef}}(k) = -\frac{1}{2}\Delta\left(\int_{v_1}^{v_2} \mathrm{d}v\, \frac{mv^2}{2}\langle f\rangle\right). \tag{3.52}$$

To compute the RHS explicitly, a graphical, equal area construction is most convenient. Figure 3.13 illustrates this schematically. The idea is that resonant diffusion continues until the upper most of the two rectangles of equal area empties out, toward lower velocity, thus creating a flat spot or plateau between v_1 and v_2. The result of the construction and calculation outlined above gives the saturated field energy and the distortion of the tail.

To determine the change in the bulk distribution function, one must examine the non-resonant diffusion equation. This is

$$\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial t}D_{\mathrm{NR}}\frac{\partial\langle f\rangle}{\partial v} \cong \frac{8\pi q^2}{m^2}\int dk\, E^{\mathrm{ef}}(k)\,\frac{\gamma_k}{\omega_{\mathrm{pe}}^2}\frac{\partial^2\langle f\rangle}{\partial v^2}. \tag{3.53a}$$

Here $\gamma_k \geq 0$ for modes in the spectrum, so the absolute value is superfluous. Thus, using the definition of γ_k, we can write the diffusion equation as

$$\frac{\partial\langle f\rangle}{\partial t} = \left(\frac{1}{nm}\frac{\partial}{\partial t}\int dk\, E^{\mathrm{ef}}(k)\right)\frac{\partial^2\langle f\rangle}{\partial v^2}. \tag{3.53b}$$

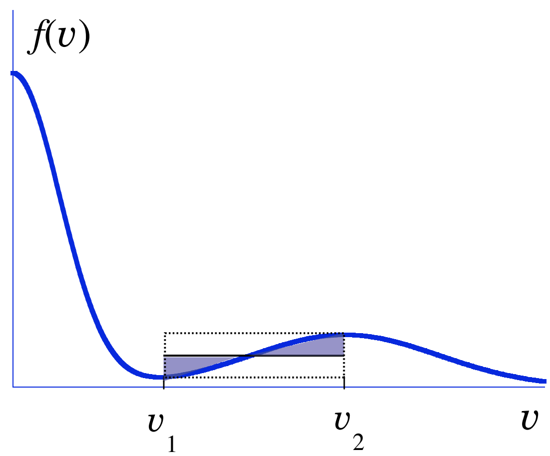


Fig. 3.13. Cartoon of initial and final (plateau) distribution function for resonant region in bump-on-tail instability. Note that quasi-linear diffusion has filled in the initial "hollow" and smoothed out the "bump" centered at v_2.

Now, defining

$$\tau(t) = \left(\frac{2}{n} \int \mathrm{d}k\, E^{\mathrm{ef}}(k,t) \right) \tag{3.54}$$

reduces Eq.(3.53b) to a simple diffusion equation

$$\frac{\partial \langle f \rangle}{\partial \tau} = \frac{1}{2m} \frac{\partial^2 \langle f \rangle}{\partial v^2} \tag{3.55}$$

with solution (taking the initial bulk distribution to be Maxwellian)

$$\langle f \rangle = \left[\frac{m}{2\pi \left(T + \tau(t) - \tau(0)\right)} \right]^{1/2} \exp\left[-\frac{mv^2/2}{\left(T + \tau(t) - \tau(0)\right)} \right]. \tag{3.56}$$

Hence, non-resonant particle of saturation undergo an apparent temperature increase

$$T \to T + \frac{2}{n} \int \mathrm{d}k \left[E^{\mathrm{ef}}(k, \infty) - E^{\mathrm{ef}}(k, 0) \right] \tag{3.57}$$

so that the bulk electrons appear to be heated by a net increase in field energy. Of course, as is explained in the begining of this subsection, this heating is *fake*, i.e. does not correspond to an increase in entropy, as it results from non-resonant diffusion. Furthermore it is one sided (i.e. occurs only for particles with $v > 0$), as a consequence of the need to conserve momentum with beam particles which are slowing down. This result may also be obtained using the conservation relation $\partial \left(E_{\mathrm{kin}} + E^{\mathrm{ef}}\right) / \partial t = 0$, and noting that since in this case $\partial \left(E_{\mathrm{kin}}^{\mathrm{res}} + 2E^{\mathrm{ef}}\right) / \partial t = 0$, we have

$$\frac{\partial}{\partial t} \left(E_{\mathrm{kin}}^{\mathrm{nr}} - E^{\mathrm{ef}} \right) = 0 \tag{3.58}$$

so $\Delta \left(E_{\mathrm{kin}}^{\mathrm{nr}}\right) = \Delta \left(E^{\mathrm{ef}}\right)$, consistent with Eq.(3.57). Note, however, that an explicit computation of $\Delta \left(E^{\mathrm{ef}}\right)$ requires an analysis of the distortion of the distribution function in the resonant region. This should not be surprising, since in the bump-on-tail instability, the non-resonant particle are in some sense ‘slaved’ to the resonant particles.

3.5 Application of Quasi-linear Theory to Drift Waves

3.5.1 Geometry and drift waves

A second, and very important application of quasilinear or mean field theory is to drift wave turbulence. A typical geometry is illustrated in Fig.3.14. It is well known that a slab of uniformly magnetized plasma (where $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$) which supports cross-field density and/or temperature gradients i.e. $n = n_0(x)$, $T = T_0(x)$, where n_0 and T_0 are the density and temperature

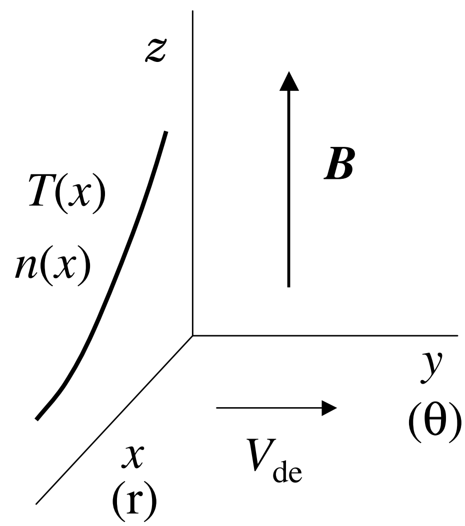


Fig. 3.14. Geometry of magnetized inhomogeneous plasma. The gradients and magnetic field are in the x-direction and z-direction, respectively. The electron diamagnetic drift velocity V_{de} is in the y-direction. Radial and poloidal directions (r, θ) are also illustrated.

profiles, which parameterize the local Maxwellian distribution function, is unstable to low frequency ($\omega < \omega_{ci}$) drift wave instabilities. Such "universal" instabilities, which can occur either in collisionless or collisional plasmas, tap expansion free energy stored in radial pressure gradients (i.e. $\partial p/\partial r$) via either collsionless (i.e. wave-particle resonance) or collisional dissipation. Indeed, the collisionless electron drift wave is perhaps the simplest kinetic low frequency instability of the myriad which are known to occur in inhomogeneous plasma. A short primer on the linear properties of drift waves may be found in Appendix. Here, we proceed to discuss the quasilinear dynamics of the collisionless, electron-driven drift instability.

In the collisionless electron drift instability, the ion response is hydrodynamic, while the electrons are described by the drift kinetic equation

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B_0} \boldsymbol{\nabla}\phi \times \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} f - \frac{|e|}{m_e} E_z \frac{\partial f}{\partial v_z} = 0. \tag{3.59}$$

Equation (3.59) simply states that phase space density f is conserved (i.e. $\mathrm{d}f/\mathrm{d}t = 0$) along the drift orbits

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= v_z, \\ \frac{\mathrm{d}v_z}{\mathrm{d}t} &= -\frac{|e|E_z}{m_e}, \\ \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= -\frac{c}{B_0}\boldsymbol{\nabla}\phi \times \hat{\boldsymbol{z}}. \end{aligned}$$

These orbits combine Vlasov-like dynamics along the magnetic field with $\boldsymbol{E} \times \boldsymbol{B}$ drift across the field. Note that the phase space flow for drift kinetic dynamics in a straight magnetic field is manifestly incompressible, since

$$\boldsymbol{\nabla}_\perp \cdot \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \right) = 0,$$

and parallel dynamics is Hamiltonian. As a consequence, Eq.(3.59) may be re-written as a continuity equation in phase space, i.e.

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z} v_z f + \boldsymbol{\nabla} \cdot (\boldsymbol{v}_\perp f) + \frac{\partial}{\partial v_z} a_z f = 0 \tag{3.60a}$$

where the perpendicular $\boldsymbol{E} \times \boldsymbol{B}$ flow velocity is

$$\boldsymbol{v}_\perp = -\frac{c}{B_0} \boldsymbol{\nabla}\phi \times \hat{\boldsymbol{z}} \tag{3.60b}$$

and the parallel acceleration a_z is

$$a_z = \frac{|e|}{m_e} \boldsymbol{\nabla}_z \phi. \tag{3.60c}$$

Assuming periodicity in the $\hat{\boldsymbol{z}}$-direction and gradients in $\hat{\boldsymbol{x}}$-direction, averaging Eqs.(3.60b) then yields the mean field equation for $\langle f \rangle$, i.e.

$$\frac{\partial}{\partial t} \langle f \rangle + \frac{\partial}{\partial x} \left\langle \tilde{v}_x \tilde{f} \right\rangle + \frac{\partial}{\partial v_z} \left\langle \tilde{a}_z \tilde{f} \right\rangle = 0. \tag{3.61}$$

In this example, we see that the quasinlinear dynamics are necessarily two dimensional, and evolve $\langle f \rangle$ in a reduced phase space of $\left(x, v_{\parallel}\right)$, which combines position space (r) and velocity space (v_z) evolution. Thus the quasilinear evolution involves both a radial flux of particles and energy, as well as heating in parallel velocity as in the 1D Vlasov example. An energy theorem may be derived by constructing the energy moment of Eq.(3.61), i.e. taking an weighted integral ($\int \mathrm{d}^3 v \, \left(m_e v^2/2\right) *$) of the drift kinetic equation. This gives

$$\frac{\partial}{\partial t} \langle E_{\mathrm{kin}} \rangle + \frac{\partial}{\partial r} Q_e - \langle E_z J_z \rangle = 0, \tag{3.62a}$$

where

$$\langle E_{\mathrm{kin}} \rangle = \int \mathrm{d}^3 v \, \frac{m_e v^2}{2} \langle f \rangle \tag{3.62b}$$

is the kinetic energy density,

$$Q_e = \int \mathrm{d}^3 v \, \left\langle \tilde{v}_r \frac{1}{2} m_e v^2 \tilde{f} \right\rangle \tag{3.62c}$$

is the fluctuation included energy flux and

$$\begin{aligned} \langle E_z J_z \rangle &= \int \mathrm{d}^3 v \, m_e v_z \left\langle \tilde{a}_z \tilde{f} \right\rangle \\ &= \left\langle \nabla_z \phi \int \mathrm{d}^3 v \, |e| v_z \tilde{f} \right\rangle \end{aligned} \tag{3.62d}$$

is the fluctuation-induced heating. Note that in drift kinetics, the *only* possible heating is parallel heating. In a related vein, the drift wave energy density W_{DW} satisfies a Poynting theorem of the form:

$$\frac{\partial}{\partial t} W_{\mathrm{DW}} + \frac{\partial}{\partial r} S_r = - \left\langle E_{\parallel} J_{\parallel} \right\rangle_{\mathrm{R}} \tag{3.63}$$

where S_r is the radial wave energy density flux and $\left\langle E_{\parallel} J_{\parallel} \right\rangle_{\mathrm{R}}$ is the heating by *resonant* particles. Equation (3.63) is seen to be the analogue of the wave energy vs. resonant particle energy balance we encountered in 1D, since we can use Eq.(3.62a) to write

$$\langle E_z J_z \rangle_{\mathrm{R}} = \left(\frac{\partial}{\partial t} \langle E_{\mathrm{kin}} \rangle + \frac{\partial}{\partial r} Q_e \right)_{\mathrm{R}}, \tag{3.64}$$

so that Eq.(3.63) then becomes

$$\frac{\partial}{\partial t} \left(W_{\mathrm{DW}} + \langle E_{\mathrm{kin}} \rangle_{\mathrm{R}} \right) + \frac{\partial}{\partial r} \left(Q_{e,\mathrm{R}} + S_r \right) = 0. \tag{3.65}$$

Likewise, an energy theorem for the evolution of particle plus field energy may be derived in a similar manner. Interestingly, Eq.(3.65) states that the volume-integrated wave-plus-resonant-particle energy is now conserved only up to losses due to transport and wave radiation through the boundary, i.e.

$$\frac{\partial}{\partial t} \int \mathrm{d}r \, \left(W_{\mathrm{DW}} + \langle E_{\mathrm{kin}} \rangle_{\mathrm{R}} \right) = - \left(Q_{e,\mathrm{R}} + S_r \right) \big|_{\mathrm{bndry}} . \tag{3.66}$$

In general, transport exceeds radiation, except where $\tilde{n}/n_0 \to 1$, as at the tokamak edge.

3.5.2 Quasi-linear equations for drift wave turbulence

To construct the explicit quasilinear equation for drift wave turbulence, we substitute the linear response $f^c_{\boldsymbol{k}}$ to $\phi_{\boldsymbol{k}}$ into Eq.(3.61), to obtain a mean field equation for $\langle f \rangle$. Unlike the 1D case, here $f^c_{\boldsymbol{k}}$ in deriven by both spatial and velocity gradients, so

$$f^c_{\boldsymbol{k}} = \frac{\phi_{\boldsymbol{k}}}{\omega - k_z v_z} L_k \langle f \rangle \tag{3.67a}$$

where L_k is the operator

$$L_k = -\frac{c}{B_0} k_\theta \frac{\partial}{\partial r} + \frac{|e|}{m_e} k_z \frac{\partial}{\partial v_z}. \tag{3.67b}$$

Here it is understood that $\omega = \omega(\boldsymbol{k})$ – i.e. all fluctuations are eigenmodes. Then, the quasilinear evolution equation for $\langle f \rangle$ can be written as

$$\frac{\partial}{\partial t}\langle f \rangle = \mathrm{Re}\sum_{\boldsymbol{k}} \boldsymbol{L}_{\boldsymbol{k}} |\phi_{\boldsymbol{k}}|^2 \left(\frac{i}{\omega - k_z v_z}\right) \boldsymbol{L}_{\boldsymbol{k}} \langle f \rangle \tag{3.68a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\langle f \rangle = & \frac{\partial}{\partial r} D_{r,r} \frac{\partial}{\partial r}\langle f \rangle + \frac{\partial}{\partial r} D_{r,v} \frac{\partial}{\partial v_z}\langle f \rangle \\ & + \frac{\partial}{\partial v_z} D_{v,r} \frac{\partial}{\partial r}\langle f \rangle + \frac{\partial}{\partial v_z} D_{v,v} \frac{\partial}{\partial v_z}\langle f \rangle\end{aligned} \tag{3.68b}$$

where the four diffusion coefficients describe radial diffusion, i.e.

$$D_{r,r} = \mathrm{Re}\sum_{\boldsymbol{k}} \frac{e^2}{B_0^2} k_\theta^2 |\phi_{\boldsymbol{k}}|^2 \frac{i}{\omega - k_z v_z} \tag{3.68c}$$

velocity diffusion, i.e.

$$D_{v,v} = \mathrm{Re}\sum_{\boldsymbol{k}} \frac{e^2}{B_0^2} k_z^2 |\phi_{\boldsymbol{k}}|^2 \frac{i}{\omega - k_z v_z} \tag{3.68d}$$

and two cross-terms

$$D_{r,v} = \mathrm{Re}\sum_{\boldsymbol{k}} \frac{c}{B_0} \frac{|e|}{m_e} k_\theta k_z |\phi_{\boldsymbol{k}}|^2 \frac{i}{\omega - k_z v_z} \tag{3.68e}$$

$$D_{r,v} = \mathrm{Re}\sum_{\boldsymbol{k}} \frac{c}{B_0} \frac{|e|}{m_e} k_\theta k_z |\phi_{\boldsymbol{k}}|^2 \frac{i}{\omega - k_z v_z}. \tag{3.68f}$$

In general, some spectral asymmetry i.e. $\langle k_\theta k_z \rangle \neq 0$ (where the bracket implies a spectral average) is required for $D_{r,v} \neq 0$ and $D_{v,r} \neq 0$. Equation (3.68b), then, is the quasilinear equation for $\langle f \rangle$ evolution by drift wave turbulence.

It is interesting to observe that the multi-dimensional structure of wave-particle resonance in, and the structure of the wave dispersion relation for, drift wave turbulence have some interesting implications for the auto-correlation time for stochastic scattering of particles by a turbulent fluctuation field. In general, for drift waves $\omega = \omega\left(k_\theta, k_\parallel\right)$, with stronger dependence of $k_\perp$. Then, modelling

$$|\phi_{\boldsymbol{k}}|^2 = |\phi_0|^2 \left(\frac{\Delta k_\theta}{(k_\theta - k_{\theta_0})^2 + \Delta k_\theta^2} \right) \left(\frac{\Delta k_z}{(k_z - k_{z_0})^2 + \Delta k_z^2} \right) \tag{3.69}$$

we see that

$$\begin{aligned} D_{r,r} &= \mathrm{Re} \int \mathrm{d}k_\theta \int \mathrm{d}k_\parallel \, \left|\phi\left(k_\theta, k_\parallel\right)\right|^2 \frac{c^2}{B_0^2} k_\theta^2 \left(\frac{i}{\omega - k_z v_z} \right) \\ &\cong \mathrm{Re} \, |\phi_0|^2 \frac{c^2}{B_0^2} k_{\theta_0}^2 \\ &\quad \times \left\{ i \left(\omega\left(k_{\theta_0}, k_{z_0}\right) + i \left| \frac{\mathrm{d}\omega}{\mathrm{d}k_\theta} \Delta k_\theta \right| \right.\right. \\ &\quad \left.\left. + i \left| \frac{\mathrm{d}\omega}{\mathrm{d}k_z} \Delta k_z \right| - k_{z_0} v_z - i \left|\Delta k_z v_z\right| \right)^{-1} \right\}. \end{aligned} \tag{3.70}$$

Hence, the effective pattern decorrelation rate for *resonant* particle in drift wave turbulence is

$$\frac{1}{\tau_{\mathrm{ac}}} = \left\{ \left| \left(\frac{\mathrm{d}\omega}{\mathrm{d}k_z} - \frac{\omega}{k_z} \right) \Delta k_z \right| + \left| \frac{\mathrm{d}\omega}{\mathrm{d}k_\theta} \right| |\Delta k_\theta| \right\}^{-1}. \tag{3.71}$$

The constast with 1D is striking. Since particles do not 'stream' in the θ direction, decorrelation due to poloidal propagation is stronger than that due to parallel propagation, which closely resembles the case of 1D. Usually, the effective turbulence field will decorrelate by simple poloidal propagation at v_{de}, and by parallel dispersion at the parallel phase velocity, since $d\omega/dk \cong 0$ for drift waves. Thus, quasilinear diffusion is, in some sense, more robust for 3D drift wave turbulence than for 1D Vlasov turbulence.

3.5.3 Saturation via quasi-linear mechanism

We can obtain some interesting insights into the mechanisms of saturation of drift wave turbulence by considering the process of 2D plateau formation in the r, v_z phase space for $\langle f \rangle$. Initial contours of constant $\langle f \rangle$ are shown in Fig.3.15(a). Evolved level lines, i.e. contours of $\langle f \rangle$ for which $\partial \langle f \rangle / \partial t =$

0, and which thus define the plateau contours of $\langle f \rangle$ at saturation of the instability, are those of which $L_k \langle f \rangle = 0$. Taking $k_z = \omega_k / v_z$, we see that

$$L_k \langle f \rangle = \frac{k_\theta}{\Omega_e} \frac{\partial}{\partial r} \langle f \rangle + \frac{\omega_k}{v_z} \frac{\partial}{\partial v_z} \langle f \rangle = 0 \tag{3.72}$$

thus defines the structure of the "plateaued" distribution function. Constant $\langle f \rangle$ curves thus satisfy

$$\frac{k_\theta}{\omega_{ce}} \frac{\langle f \rangle}{\Delta x} + \frac{\omega}{v_z} \frac{\langle f \rangle}{\Delta v_z} = 0 \tag{3.73}$$

so the level curves of $\langle f \rangle$ are defined by

$$x - \frac{k_\theta v_z^2}{2\omega_k \omega_{ce}} = \text{const} \tag{3.74}$$

at saturation, The change in level contours is shown in Fig.3.15(b). Note then that any spatial transport which occurs due to the drift wave turbulence is inexorably tied to the concomitant parallel heating. This is no surprise, since the essence of drift wave instability involves a trade-off between relaxation of density and temperature gradients (which destabilize the waves) and Landau damping (which is stabilizing *but* which also provides the requisite dissipative response in $\langle f \rangle$ to produce instability). In particular, any particle displacement δx from its initial state must be accompanied by a heating (due to Landau damping) δv_z^2, which satisfies:

$$\delta x = \frac{k_\theta}{2\omega_k \omega_{ce}} \delta v_z^2. \tag{3.75}$$

Since $\omega_k \ll k_\parallel v_{\mathrm{Te}}$, the heating is small i.e. $\delta v_z^2 \sim \alpha v_{\mathrm{Te}}^2$ where $\alpha \ll 1$, so necessarily $\delta x \ll L_n$ – i.e. the maximum displacement is also small, and considerably smaller than the gradient scale length. Hence, the instability is quasi-linearly self-saturating at low levels and the resulting particle and/or heat transport is quite modest. To obtain significant steady state transport,

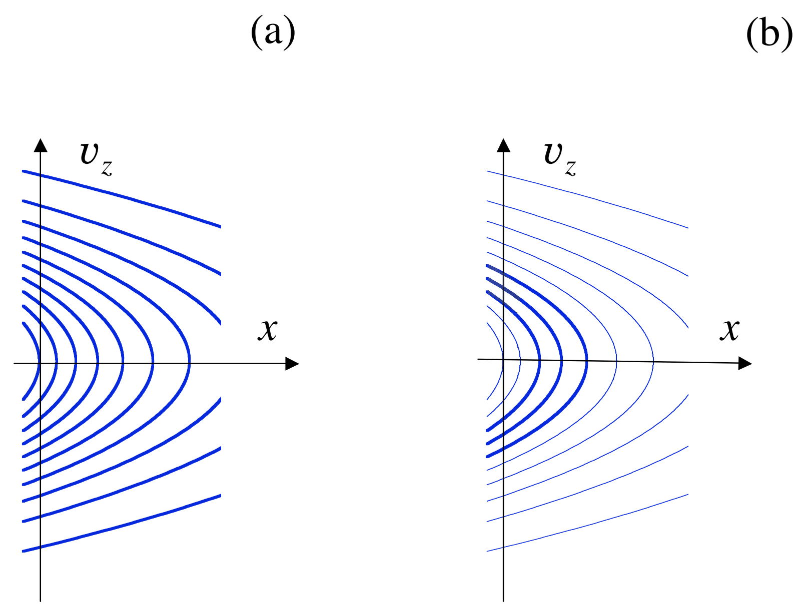


Fig. 3.15. Contour of the mean electron distribution function in the phase space (a). According to the resonance with drift waves, a flattening may occur, and the modification takes place in the level contours of mean distribution, as is illustrated by a shaded region (b).

the plateau must either be destroyed by collisions or the distribution must be externally "pumped" to maintain it as a Maxwellian.

3.6 Application of Quasi-linear Theory to Ion Mixing Mode

A third instructive example of qusilinear theory is that of particle transport due to the ion mixing mode. The ion mixing mode is a type of negative compressibility "ion temperature gradient driven mode" which is likely to occur in collisional plasmas, such as those at the tokamak edge. The mixing mode is driven by ∇T_i, but also transports particles and electron heat.

The example of the mixing mode is relevant since it is simple, clear and illustrates,

a) the application of quasilinear theory to a purely fluid-like, hydrodynamic instability

b) a possible origin of off-diagonal and even counter-gradient transport processes

The aim of this example is to calculate the particle flux induced by the mixing mode. The quasilinear density flux is simply $\langle \tilde{v}_r \tilde{n} \rangle$, so the task is to compute the density response to potential perturbation. In the mixing mode, electron inertia is negligible, so the parallel electron dynamics preserve pressure balance, i.e.

$$\boldsymbol{\nabla} \left(\tilde{p}_e - |e| n \phi + \alpha_T n \boldsymbol{\nabla}_{\parallel} \tilde{T}_e \right) = 0 \tag{3.76a}$$

where α_T is the coefficient for the electron thermal force, or equivalently

$$\boldsymbol{\nabla}_{\parallel} \left(\tilde{n}_e + n \tilde{T}_e - |e| n \phi \right) - \alpha_T n \boldsymbol{\nabla}_{\parallel} \tilde{T}_e = 0 \tag{3.76b}$$

$$\frac{\tilde{n}_e}{n} = \frac{|e|\phi}{T_e} - \frac{\tilde{T}_e}{T} \left(1 + \alpha_T \right). \tag{3.76c}$$

To calculate the electron temperature perturbation, we use the temperature evolution equation

$$\frac{3}{2} n \left(\frac{\partial \tilde{T}_e}{\partial t} + \tilde{v}_e \frac{\mathrm{d} \langle T_e \rangle}{\mathrm{d}x} \right) + n T_e \boldsymbol{\nabla}_{\parallel} \tilde{v}_{\parallel e} = \boldsymbol{\nabla}_{\parallel} n \chi_{\parallel} \boldsymbol{\nabla}_{\parallel} \tilde{T}_e \tag{3.77a}$$

and the continuity equation

$$\frac{\partial \tilde{n}}{\partial t} + \tilde{v}_e \frac{\mathrm{d} \langle n \rangle}{\mathrm{d}x} + n \boldsymbol{\nabla}_{\parallel} \tilde{v}_{\parallel e} = 0 \tag{3.77b}$$

to obtain, after a short calculation

$$\left(\frac{\tilde{T}_e}{T_e}\right)_{\boldsymbol{k}} = \frac{1}{3\omega/2 + i\chi_{\|}k_{\|}^2}\left\{\omega_{*e}\left(\frac{3}{2}\eta_e - 1\right)\frac{|e|\phi}{T_0} + \omega\frac{\tilde{n}}{n}\right\}. \tag{3.77c}$$

Here $\chi_{\|} = v_{\mathrm{Te}}^2 / \nu_e$ is the parallel thermal conductivity, ω_{*e} is the electron diamagnetic frequency and $\eta_e = \mathrm{d}\ln T_e / \mathrm{d}\ln n_e$ is the temperature gradient parameters. Equations (3.76c) and (3.77c) may then be combined (in the relevant limit of $\chi_{\|}k_{\|}^2 \gg \omega$) to yield the density perturbation

$$\begin{aligned}\left(\frac{\tilde{n}_e}{n}\right)_{\boldsymbol{k}} &= \frac{|e|\phi_{\boldsymbol{k}}}{T_e}\left\{1 + \frac{i(1+\alpha_T)}{\chi_{\|}k_{\|}^2}\left(\omega - \omega_{*e} + \frac{3}{2}\omega_{*e}\eta_e\right)\right\} \\ &\approx \frac{|e|\phi_{\boldsymbol{k}}}{T_e}\left\{1 + \frac{i(1+\alpha_T)}{\chi_{\|}k_{\|}^2}\left(-\omega_{*e} + \frac{3}{2}\omega_{*T_e}\right)\right\}\end{aligned} \tag{3.78}$$

since the mixing mode has $\omega \approx 0$. Here ω_{*T_e} is just the diamagnetic frequency computed with the electron temperature gradient. Thus, the mixing mode driven particle flux is:

$$\langle \tilde{v}_r \tilde{n}_e \rangle = -D\frac{\partial \langle \tilde{n} \rangle}{\partial x} + V\langle \tilde{n} \rangle \tag{3.79a}$$

where

$$D = (1+\alpha_T)\sum_{\boldsymbol{k}} \frac{c^2}{B_0^2}\frac{k_\theta^2 |\phi_{\boldsymbol{k}}|^2}{\chi_{\|}k_{\|}^2} \tag{3.79b}$$

$$V = \frac{3}{2}(1+\alpha_T)\sum_{\boldsymbol{k}} \frac{c^2}{B_0^2}\frac{k_\theta^2 |\phi_{\boldsymbol{k}}|^2}{\chi_{\|}k_{\|}^2}\frac{1}{\langle T \rangle}\frac{\mathrm{d}\langle T \rangle}{\mathrm{d}x}. \tag{3.79c}$$

Observe that in this example, the quasilinear particle flux consists of two pieces, the 'usual' Fichian diffusive flux down the density gradient ($-D\times \partial\langle n \rangle / \partial x$) *and* a convective contribution ($\sim V\langle n \rangle$). It is especially interesting to note that for normal temperature profiles (i.e. $\mathrm{d}\langle T \rangle / \mathrm{d}x < 0$), $V < 0$, so the convective flux is *inward*, and opposite to the diffusive flux! Note that for $|(1/\langle T \rangle)(\mathrm{d}\langle T \rangle/\mathrm{d}x)| > |(1/\langle n \rangle)(\mathrm{d}\langle n \rangle/\mathrm{d}x)|$, the *net* particle flux is consequently inward, and "up" the density gradient. This simple example is typical of a broad class of phenomena manifested in quasilinear theory which are classified as off-diagonal, gradient-driven fluxes. Off-diagonal inward flows are frequently referred to as a "pinch". The temperature gradient driven pinch described here is sometimes referred to as a thermo-electric pinch. Pinch effects are of great interest in the context of laboratory plasmas, since they offer a possible explanation of profiles which peak on axis, in spite of purely edge fueling. To this end, note that for $V > 0$, the particle flux vanishes for $(1/\langle n \rangle)(\mathrm{d}\langle n \rangle/\mathrm{d}x) = V/D$, thus tying the profile scale length to the inward convective velocity. The allusion to 'off-diagonal' refers, of course, to the Onsager matrix which relates the vector of fluxes to the vector of thermodynamic forces. While the diagonal elements of the quasilinear Onsager matrix are always positive, the off-diagonal elements can be negative, as in this case, and so can drive 'inward' or 'up-gradient' fluxes. Of course, the net entropy production *must* be positive since relaxation occurs. In the case of the ion mixing mode, which is $\boldsymbol{\nabla} T_i$-driven, the entropy produced by ion temperature profile relaxation must exceed the entropy 'destroyed' by the inward particle flux. This requires

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \mathrm{d}r \left\{ \chi_i \left(\frac{1}{\langle T \rangle} \frac{\partial \langle T \rangle}{\partial x} \right)^2 - \langle \tilde{v}_r \tilde{n}_e \rangle \frac{1}{\langle n \rangle^2} \frac{\mathrm{d}\langle n \rangle}{\mathrm{d}x} \right\} > 0, \tag{3.80}$$

where χ_i is the turbulent thermal duffusivity. In practice, satisfaction of this inequality is assured for the ion mixing mode by the ordering $\chi k_\parallel^2 \gg \omega_{\boldsymbol{k}}$, which guarantees that the effective correlation time in χ_i and the ion heat flux is longer than that in the particle flux. We remark that the up-gradient flux is similar to the phenomenon of chemotaxis.

3.7 Nonlinear Landau Damping

In this chapter, the quasi-linear response of the perturbation to the mean is explained. The perturbation technique, which is the fundamental element in the procedure, can be extended to higher orders. The nonlinear interactions, which include the higher order terms, are explained in the next chapters in detail. Before going into the systematic explanation of the interactions in turbulent fluctuations, we here briefly describe the perturbations to higher orders in fluctuation amplitude [****]. The method, which is based on the expansion and truncation of higher order terms, has limited applicability to turbulence. However, this method can illuminate one essential element in nonlinear interactions, i.e., the Landau resonance of a beat mode. This process is known as 'nonlinear Landau damping', and merits illustration before developing a systematic explanation of coupling in turbulence.

Consider the perturbed electric field (in one dimensional plasma here for the transparency of the arguments)

$$\frac{d}{dt}v = \frac{e}{m}E\left(x,t\right) = \frac{e}{m}\sum_{k}E_{k}\exp\left(ikx - i\omega t\right),$$

where the frequency ω is considered to satisfy the dispersion relation $\omega = \omega_k$. The turbulence is weak, and fluctuations are taken as the sum of linear eigenmodes. (The case of strong turbulence, in which broad band fluctuations are dominantly excited, is not property treated by these expansions and is explained in the following Chapters.) The issue is now to derive the higher order diffusion coefficient in the velocity space D owing to the fluctuating electric field, by which the mean distribution function evolves $\partial\left\langle f\right\rangle/\partial t = \partial/\partial v\left(D\partial\left\langle f\right\rangle/\partial v\right)$. The diffusion coefficient in the velocity space is given by the correlation of fluctuating accelerations, i.e.,

$$D = \int_0^\infty d\tau \left\langle F\left(t+\ \tau\right) F\left(t\right)\right\rangle, F\left(t\right) = \frac{e}{m} E\left(x\left(t\right), t\right). \tag{3.81}$$

In the method of perturbation expansions, the correlation $\langle F(t+\ \tau) F(t)\rangle$ is calculated by the successive expansion with respect to the amplitude of electric field perturbation.

The acceleration at time t, $F(t)$, depends on the location of particle $x(t)$, through the space dependence of the electric field $E(x(t), t)$. In the perturbation expansion method, the particle orbit is expanded as

$$x(t) = x_0(t) + x_1(t) + \cdot\ \cdot\ \cdot \tag{3.82a}$$

where $x_0(t)$ is the unperturbed orbit, $x_0(t) = x(0) + v(0)\,t$, and $x_1(t)$ is the first order correction of the orbit due to the electric field perturbation (as is illustrated in Fig.3.16). Associated with this, the net acceleration, which particles feel, is given by:

$$\begin{aligned} F\left(x\left(t\right), t\right) &= F\left(x_0\left(t\right) + x_1\left(t\right) + \cdot\ \cdot\ \cdot, t\right) \\ &= F\left(x_0\left(t\right), t\right) + \frac{\partial}{\partial x} F\left(x_0\left(t\right), t\right) x_1\left(t\right) + \cdot\ \cdot\ \cdot, \end{aligned} \tag{3.82b}$$

which can be rewritten as $F(t) = F_1(t) + F_2(t) + \cdot\ \cdot\ \cdot$ in a series of electric field amplitude. Note that the expansion (3.82a) is valid so long as the change of the orbit occurs in a time that is much shorter than the bounce time of particles in the potential trough, τ_b, i.e., so that

$$\tau_{ac} << \tau_b,$$

where τ_{ac} is the autocorrelation time that the resonant particles feel, $\tau_{ac}^{-1} = (\omega/k - \partial\omega/\partial k)\,\Delta k$, and Δk is the spectral width of $|E_k|^2$. If the bounce time is short, $\tau_{ac} > \tau_b$, the orbit is subject to trapping, and an expansion based on the unperturbed orbit is not valid.

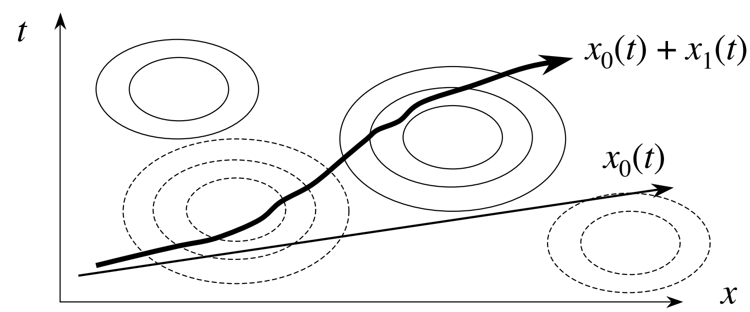


Fig. 3.16. Schematic drawing of the particle orbit in the presence of electric perturbations.

The second order term with respect to the electric field, $F_2(t)$, is

$$F_2 = \frac{e}{m} \sum_{k'} ik' x_1(t) E_{k'} \exp\left(ik'x(0)\right) \exp\left(i\left(k'v(0) - \omega'\right)t\right), \qquad (3.83)$$

and the third order term is calculated in a similar way. Noting the fact that

$$x_1(t) = -\frac{e}{m} \sum_{k} E_k \frac{\exp(ikx(0))}{(kv(0) - \omega)^2} \exp(i(kv(0) - \omega)t)$$

(where the upper limit of time integration is kept), the second order term F_2 is the sum of contributions of the beats $\exp\left[i\left((k \pm k')v(0) - (\omega \pm \omega')\right)t\right]$. These beat waves are virtual modes, driven by the nonlinear interaction of primary modes.

The contribution to the diffusion coefficient from the linear response has the phase (which particles feel) $\exp(i(kv(0) - \omega)t)$. Therefore, the quasi-linear contribution (which is the second order with respect to the electric field) comes from the resonance

$$kv(0) - \omega = 0, \qquad (3.84a)$$

while the phase of next order correction (the fourth order with respect to the electric field) is set by the resonance

$$\left(k \ \pm k'\right) v\left(0\right) - \left(\omega \ \pm \omega'\right) = 0. \tag{3.84b}$$

The successive expansion provides that the term, which is $2n$-th order with respect to the electric field, originates from resonances

$$\left(k_1 \quad \pm \cdots \pm k_n\right) v\left(0\right) - \left(\omega_1 \quad \pm \cdots \pm \omega_n\right) = 0. \tag{3.84c}$$

The 4th order term in the expansion of the total diffusion coefficient, $D = D_2 + D_4 + \ \cdot \ \cdot \ \cdot$, is given by

$$D_4 = \frac{e^4 \pi}{m^4} \sum_{k,k'} |E_k|^2 |E_{k'}|^2 \left(\frac{k - k'}{\left(kv - \omega\right)\left(k'v - \omega'\right)} \right)^2 \delta\left(\left(k \ - k'\right) v - \left(\omega \ - \omega'\right)\right), \tag{3.85}$$

where the label of particle velocity $v\left(0\right)$ is rewritten as v, and the resonance condition (3.84b) is give in terms of the delta-function. The resonance occurs for the particles, which have the velocity at the phase velocity of the beat wave

$$v = \frac{\omega \ - \omega'}{k \ - k'}. \tag{3.86}$$

This scattering process is known as nonlinear Landau damping. The change of kinetic energy $\overline{E}_{kin}$ associated with the relaxation of the mean distribution function at the 4th order of fluctuating field, $\frac{\partial}{\partial t}\overline{E}^{(2)}_{kin} = \int dv \frac{m}{2} v^2 \frac{\partial}{\partial v} D_4 \frac{\partial}{\partial v} \left\langle f \right\rangle$, gives the additional higher-order damping of wave energy.

This process is effective in connecting electrons and ions via wave excitations. Waves, which are excited by electrons, are often characterized by the phase velocity, $\omega/k \sim v_{T,e}$. For such cases, the phase velocity is too fast to interact with ions. When the wave dispersion is strong, the resonant velocity for the beat, $\left(\omega \ - \omega'\right)/\left(k \ - k'\right)$, can be much smaller than the phase velocity of primary waves, ω/k and ω'/k'. For the beat mode that satisfies the

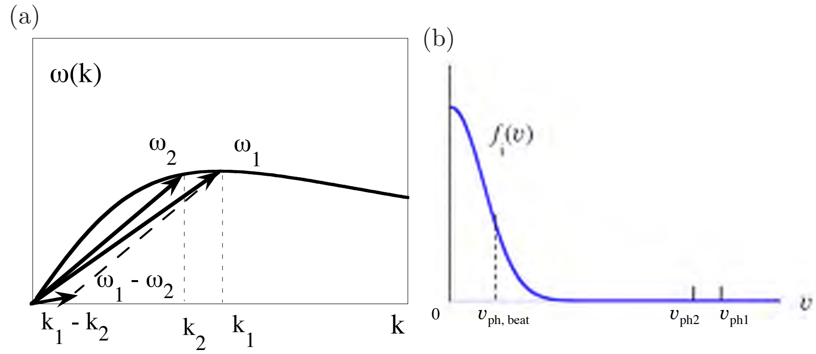


Fig. 3.17. Frequencies of the primary modes (k_1, ω_1, and k_2, ω_2) and the beat mode (a). Primary modes satisfy the dispersion but the beat mode does not. Phase velocities of primary modes and beat mode and the distribution function of ions (b).

condition $(\omega - \omega')\,/\,(k - k') \sim v_{T,i}$, strong coupling to ions occurs. Figure 3.17 shows schematically the case where beat mode can resonate with ions.

This nonlinear Landau damping is important in the case that waves have strong dispersion. Noting the resonance condition Eq.(3.86), the 4th order term (3.85) is rewritten as

$$D_4 = \frac{e^4\pi}{m^4}\sum_{k,k'}|E_k|^2|E_{k'}|^2\frac{(k-k')^6}{(k'\omega - k\omega')^4}\delta\left(\left(k - k'\right)v - \left(\omega - \omega'\right)\right). \quad (3.87)$$

This result shows that when dispersion is weak so that $\omega/k \simeq \omega'/k'$, the perturbation expansion is invalid. Compared to the first order term, D_2, the higher order term D_4 has a multiplicative factor, which is of the order of magnitude, $\sum_{k'}|E_{k'}|^2(k-k')^6(k'\omega - k\omega')^{-4} \propto \tau_{ac}^4\tau_b^{-4}$. This result shows that the perturbation theory has the expansion parameter $\tau_{ac}^2\tau_b^{-2}$. The expansion method requires $\tau_{ac} \ll \tau_b$, as was explained earlier in Chapter 3.

3.8 Kubo number and trapping

Fluctuations in plasmas can lead to random motion of plasma particles, which may lead to diffusive evolution of the mean distribution. The diffusivity is given by the step size in the jump of orbit and the rate of the change of orbit. The step size and the rate of change are determined by various elements in the fluctuation spectrum.

First, the turbulent fields have their own scale and rate, i.e., the autocorrelation length, λ_c, and autocorrelation time, τ_c. The spatial and temporal correlation functions, $C_s(\Delta r) = \left\langle \tilde{E}^2 \right\rangle^{-1} L^{-1} \int_0^L dr \tilde{E}(r) \tilde{E}(r + \Delta r)$ and $C_t(\Delta\tau) = \left\langle \tilde{E}^2 \right\rangle^{-1} T^{-1} \int_0^T dt \tilde{E}(t) \tilde{E}(t + \Delta\tau)$ (where L and T are much longer than characteristic scales of microscopic fluctuations), decay at the distances $\Delta r \sim \lambda_c$ and $\Delta\tau \sim \tau_c$. These correlation length and correlation time are those for 'Eulerian' correlations.

The diffusion is, in reality, determined by the step size (and correlation time) of particle motion, and *not* by those of fluctuating field. For the correlation length (and correlation time) of the particle orbit, the Lagrangian correlation is the key, and are not identical to those of fluctuating field. The Kubo number (sometimes referred to as the Strouhal number) is a key parameter that explains the relation between the Lagrangian correlation of particles and those of fluctuating field.

Let us consider the $\boldsymbol{E} \times \boldsymbol{B}$ motion of particle under the strong magnetic field and fluctuating radial electric field, $\tilde{\boldsymbol{E}}$. The equation of motion is written as

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}(t), t), \ \boldsymbol{v}(\boldsymbol{x}(t), t) = -\frac{1}{B}\tilde{\boldsymbol{E}} \times \boldsymbol{b}, \tag{3.88}$$

where $\boldsymbol{b}$ is a unit vector in the direction of strong magnetic field. The amplitude of perturbation is characterized by the average of the fluctuating velocity, $\tilde{V} = \sqrt{\langle v^2 \rangle}$. Thus, the fluctuating field is characterized by three parameters, i.e., amplitude $\tilde{V}$, (Eulerian) correlation length and time, λ_c and τ_c. Kubo number is the ratio of the correlation time τ_c to the eddy circumnavigation time by the $\boldsymbol{E} \times \boldsymbol{B}$ motion $\tau_{\text{cir}} = \lambda_c / \tilde{V}$, i.e.,

$$K = \frac{\tau_c}{\tau_{\text{cir}}} = \frac{\tau_c \tilde{V}}{\lambda_c}. \tag{3.89}$$

When the Kubo number is much smaller than unity, $K \ll 1$, the distance of the particle motion during the time period $0 < t < \tau_c$, $\tau_c \tilde{V}$, is much smaller than λ_c. Therefore, particle motion is modeled such that the step size and step time are given by $\tau_c \tilde{V}$ and τ_c. In contrast, for $K > 1$, the particle motion is decorrelated by moving the distance of decorrelation length λ_c, not by the decorrelation time of the field τ_c. In this limit, the fluctuation field stays (nearly) unchanged during the period of circumnavigation of particles in the trough of the perturbation potential. Note that $K \simeq 1$ loosely corresponds to the mixing length fluctuation level.

The transition of transport from the quasi-linear regime to the trapping regime is illustrated briefly here. The diffusion coefficient is given by the Lagrangian correlation of fluctuating velocity along the particle orbit

$$D = \int_0^t dt' \left\langle v_j \left(\boldsymbol{x} \left(t' \right), t' \right) v_j \left(\boldsymbol{x} \left(0 \right), 0 \right) \right\rangle, \tag{3.90}$$

where $j = x, y$ and coordinates (x, y) are taken perpendicular to the strong magnetic field. In the limit of small Kubo number, $K \ll 1$, where the decorrelation time of the field is very short, one has $\langle v_j (\boldsymbol{x}(t'), t') v_j (\boldsymbol{x}(0), 0) \rangle \sim \langle v_j (\boldsymbol{x}(0), t') v_j (\boldsymbol{x}(0), 0) \rangle \sim \tilde{V}^2 C_t (t')$ for the integrand of Eq.(8.90), and so one has

$$D \simeq \tilde{V}^2 \tau_c = \frac{\lambda_c^2}{\tau_c} K^2. \tag{3.91}$$

When the Kubo number becomes larger, the field, which particles feel, is decorrelated owing to the motion of particles comparable to (longer than) the decorrelation length of the field λ_c. Putting the circumnavigation time $\tau_{\rm cir} = \lambda_c/\tilde{V}$ into the step time, one has

$$D \simeq \tilde{V}^2 \tau_{\rm cir} = \tilde{V}\lambda_c = \frac{\lambda_c^2}{\tau_c} K. \tag{3.92}$$

In this case, the diffusivity is *linearly* proportional to the fluctuation field intensity, provided that λ_c and τ_c are prescribed. The limit of $K \gg 1$ is also explained in the literature. The unit $\lambda_c^2 \tau_c^{-1}$ in Eqs.(3.91) and (3.92) are considered to be the limit of complete trapping: in such a limit, particles are bound to the trough of potential, bouncing in space by the length λ_c, and the bounce motion is randomized by the time τ_c. In reality, detrapping time of particle out of the potential trough, $\tau_{\rm detrap}$, and the circumnavigation time τ_{cir} determines the step time (average duration time of coherent motion). The heuristic model for the step time is $\tau_{\rm detrap} + \tau_{\rm cir}$,

$$D \simeq \frac{\lambda_c^2}{\tau_{\rm detrap} + \tau_{\rm cir}} = \frac{\lambda_c^2}{\tau_c} K \frac{\tau_{\rm cir}}{\tau_{\rm detrap} + \tau_{\rm cir}}. \tag{3.93}$$

In the large amplitude limit (large K limit), the circumnavigation time $\tau_{\rm cir}$ becomes shorter than the detrapping time, $\tau_{\rm detrap}$. Thus, the ratio $\tau_{\rm cir}/(\tau_{\rm detrap} + \tau_{\rm cir})$ is a decreasing function of K, and may be fitted to $K^{-\gamma}$, where γ is a constant between 0 and 1. The theory based on a percolation in stochastic landscapes has provided an estimate $\gamma = 0.7$ [Isichenko]. The literature [Vlad] reports the result of numerical computations, showing that the power law fitting $K^{-\gamma}$ holds for the cases $K \gg 1$ while the exponent depends on the shape of the space-dependence on the Eulerian correlation function of the fluctuating field.