Kolmogorov: Stability of Planetary Orbits (Lecture, 1950eth)

KAM-theorem, QLT of plasma, Chaos and beyond

• Everything started with “Stability of Solar System”
• KAM theory
• Did it make Planetary motion more stable?
• Quasilinear Theory is opposite limit re:KAM (Landau Resonances vs. Planetary Resonances)
• Hamiltonian Chaos
\[ H = H_0(I) + \varepsilon V(I, \mathcal{O}) \]

\[ \omega(I) = \frac{\partial H_0}{\partial I} \]

\[ \Delta I \approx \varepsilon^2 \]

Strength of Planet/Planet interaction

Width of resonant region

Going beyond 2-body problem (adding planet/planet interaction)

- Newton
- Laplace
- Poincare
- Perturbation technique and extraction of secular effects
- Planetary resonances of higher orders
- KAM theorem
Philosophiæ Naturalis Principia Mathematica

1687

Newton's conjecture - Solar System is UNSTABLE; needs DIVINE INTERVENTIONS (how frequently?)
Pierre-Simon Laplace
1749-1827

Méchanique céleste

Exposition du système du monde

``Je n'avais pas besoin de cette hypothèse-là''
Where do we stay today?

- KAM – theorem is not exactly applicable to Solar System

- Search of secular effects with direct computer simulation

- Effects of multi-dimensionality (N>2, Arnold)
Modern view resulting from such computer simulation

- Endangered species (planets) identified: Pluto, Mercury; time scale for cataclysmic outcome 100 – 800 Mln years

- Should we be afraid?

- IAU and Pluto

- Solar System might have had one more planet(?)
IKI – Institute of Space Research, Moscow;

Image of the Nucleus of Halley’s Comet by Vega S/C camera

Orbits of comets might be unstable in much shorter time scale.
Back to KAM theorem and take Landau resonances instead planetary ones:

\[ m \frac{dV}{dt} = e \sum E_i \exp \left( i (\omega_i - k \nu) t \right) \]
\[ |V - \omega/k| \leq \left( \frac{e \phi}{m} \right)^{\frac{1}{2}} \]
Boris Chirikov and “Standard MAP”

Atomic Energy (in Russian), 6, 630 (1959)
\[ \left( \frac{e \varphi}{m} \right)^{\frac{1}{2}} \text{ much less than } \left( \frac{\omega}{k} \right)_{n+1} - \left( \frac{\omega}{k} \right)_n \]

This limit corresponds to KAM (Kolmogorov-Arnold-Mozer) case.

**KAM-Theorem:**

As applied to our case of Charged Particle – Wave Packet Interaction –

“Particle preserves its orbit “
\( \left( \frac{e \varphi}{m} \right)^{\frac{1}{2}} \) greater than \( \left( \frac{\omega}{k} \right)_{n+1} - \left( \frac{\omega}{k} \right)_n \)

That means - overlapping of neighboring resonances

Repercussions:

-"collectivization" of particles between neighboring waves;

-particles moving from one resonance to another – “random walk”? And if yes

- what is **Diffusion Coefficient** ?(in velocity space)
\[ m \frac{dV}{dt} = e \sum E_i \exp i(\omega_i - kv)t \]

\[ V = e/ \sum E_i \exp i(\omega - kv)t \exp i(\omega - kv) \]

\[ V \times \frac{dV}{dt} = \]

\[ e^2 \sum \sum EE^* \exp i(\omega_i - \omega_j - k_i v + k_j v)t \exp i(\omega - kv) \]

\[ V^2 \propto Dt \]
\[ D = \frac{\pi e^2}{m^2} \sum |E|^2 \delta(kv - \omega) \]

\[ \sum_k = \frac{1}{2\pi} \int dk \]

Quasilinear Theory is an example of Anti-KAM limiting case (1961, Salzburg conference)

Repercussions: Quasilinear Theory, Plateau Formation,

Beam + Plasma Instability Saturation etc.
Extention of Quasilinear approach to Instability: Velocity Anisotropy ("Cyclotron Instability" of Alfven waves)

\[ \omega + kv_z = \omega_B \] (Cyclotron resonance)

\[ \gamma \propto \int dv \left[ (1 - kv_{\perp}/\omega) \frac{\partial f}{\partial v_{\perp}} + kv_{\perp}/\omega \frac{\partial f}{\partial v_{\perp}} \right] \]

(Sagdeev&Shafranov,1960)
\( \hat{D}_{\text{QL}} f = \)

\[
\left( \frac{e}{M} \right)^2 \sum \left| E \right|^2 \delta (\omega - \omega_{\text{B}}) \left[ \left( 1 - k v_z / \omega \right) \frac{1}{\sqrt{1 - v_z / v}} v_z \right. + k v_z \frac{\partial}{\partial \omega} \]

\[ X \left[ \left( 1 - k v_z / \omega \right) \frac{\partial f}{\partial v_z} + k v_z \frac{\partial}{\partial \omega} f \right] \]

\[ \omega = \omega_{\text{B}} + k v_z \]

(Same paper at Salzburg, 1961)
\[ \hat{D}_{QL} f = \]

\[
\left( \frac{e}{M} \right)^2 \sum |E|^2 \delta(\omega - \omega_0) \left[ \frac{1}{(1 - kv_z/\omega)} \frac{1}{\sqrt{v}} \frac{\partial}{\partial v} v + kv_z \frac{\partial}{\partial v_z} \right] \]

\[ X \left[ (1 - kv_z/\omega) \frac{\partial f}{\partial v} + kv_z/\omega \frac{\partial f}{\partial v_z} \right] \]

\[
\frac{1}{\omega_B} \frac{\partial}{\partial \mathcal{G}} \left( \sum |B|^2 \delta(\omega - \omega_0) \frac{\partial}{\partial \mathcal{G}} f \right)
\]

(Kennel, Petchek, 1966)
Feedback on particles: Quasilinear Theory of Particles/Cyclotron Waves Interaction

If $\beta \gg 1$
Simplified approach

- Spatial Diffusion approximation is valid:
  - QL estimate of $V_{\text{eff}} \approx \omega_B \left( \frac{(\delta_B)^2}{B^2} \right)$
  - $L_{\text{eff}} \approx \frac{C}{V_{\text{eff}}}$
Magnetic field lines diffusion

Rosenbluth M.N., Sagdeev R.Z., Taylor J.B., Zaslavsky G.M.,
Nucl. Fusion, 6, 297 (1966)

\[
\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}
\]

Z plays role of time;

dB – role of wave amplitude
\begin{align*}
V' &= V \times \cos(Q) - \left( U + K \times \sin(V + 2 \times \pi \times F \times N) \right) \times \sin(Q) \\
U' &= V \times \sin(Q) + \left( U + K \times \sin(V + 2 \times \pi \times F \times N) \right) \times \cos(Q) \\
Q &= 2 \times \pi \div A
\end{align*}
\[ V^\prime = V \cos(Q) - (V + K \times \sin(V + 2 \times \pi \times \Phi \times n)) \times \sin(Q) \]

\[ V^\prime = V \times \sin(Q) + (V + K \times \sin(V + 2 \times \pi \times \Phi \times n)) \times \cos(Q) \]

\[ Q = 2 \times \Phi / n \]

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**PARAMETERS OF MAP**

- \( n \) \( \times 1.00000 \times 10^{00} \)
- \( A \) \( \times 1.11000 \times 10^{03} \)
- \( F \) \( \times 2.50000 \times 10^{00} \)

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**THE CHANGING OF ALL VALUES**

**IS FOLLOWED BY START MAP FROM THE BEGINNING**
<table>
<thead>
<tr>
<th>Map Types</th>
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<tbody>
<tr>
<td>WEB WAVE MAP</td>
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<tr>
<td>STANDARD</td>
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<td>STANDARD x N</td>
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“Minimal Chaos”