Qualitative Physics Using Dimensional Analysis

R. Bhaskar and Anil Nigam
IBM Thomas J. Watson Research Center, P.O. Box 704, Yorktown Heights, NY 10598, USA

ABSTRACT

In this paper we use dimensional analysis as a method for solving problems in qualitative physics. We pose and solve some of the qualitative reasoning problems discussed in the literature, in the context of devices such as the pressure regulator and the heat exchanger. Using dimensional analysis, such devices or systems can be reasoned about without explicit knowledge of the physical laws that govern the operation of such devices. Instead, the method requires knowledge of the relevant physical variables and their dimensional representation. Our main thesis can be stated as follows: the dimensional representations of physical variables encode a significant amount of physical knowledge; dimensionless numbers provide a representation of the physical processes, and they can be obtained without direct, explicit knowledge of the underlying laws of physics. Then, a variety of partial derivatives can be computed and used to characterize the behavior of the system. These partials, along with some simple heuristics, can be used to reason qualitatively about the behavior of devices and systems. Here we present the techniques for dimensional analysis and develop the representation and reasoning machinery.

1. Introduction

In this paper we use dimensional analysis as a method for solving problems in qualitative physics. We pose and solve some of the qualitative reasoning problems discussed in the literature, such as the pressure regulator and the heat exchanger. Using dimensional analysis, such devices or systems can be reasoned about without *explicit* knowledge of the physical laws that govern the operation of these devices. Instead, the method requires knowledge of the relevant physical variables and their dimensional representation.[1]

The physical reasoning problem has usually required a representational apparatus that can deal with the vast amount of physical knowledge that is used in reasoning tasks. The programs of both naive physics and qualitative physics

[1] We do not make any serious distinctions between physical laws and physical equations, which we assume are simply convenient representations of laws. Of course, physical laws are not the same as physical knowledge. We make no claim that dimensional analysis works without physical knowledge.

Artificial Intelligence **45** (1990) 73–111

assume that the amount of knowledge needed for even simple physical tasks is quite large [17, p. 1], and as a consequence the representational apparatus that is needed for even simple physics problems has proved to be quite complex [9, 14, 24]. In this paper, we shall take a different approach: we will exploit the fact that the information contained in conventional physical representations of variables has not only a numerical component but also a symbolic component. The numerical component is well-known; it is simply the numerical value of some variable measured along some system of units. A force may be in poundals (lb ft/s^2) or a pressure may be in pascals (kg/(m/s^2)). To reason about the physical variables that the numbers represent, qualitative physics assigns qualitative values to them, either directly, or indirectly by using domain knowledge to specify and constrain the values the variable may take in a particular physical context.

The symbolic component is less familiar, at least in qualitative physics. *It is simply the dimensional representation of physical variables.*[2] For example, in the most familiar dimensional notation, learned in high-school or college physics, force is usually represented as MLT^{-2}. Such a dimensional representation of a variable is subject to a set of laws, and all physical laws are in fact constrained by these dimensional principles. The most familiar of these laws is the principle of dimensional homogeneity. There is in fact a substantial literature on the subject, most of which, however, is not very recent [3–5, 26, 31, 32, 34, 37]. The most well-known and widely used result in dimensional analysis is Buckingham's Π-theorem, stated and proved by Buckingham in 1914 [4]. This theorem identifies the number of independent *dimensionless numbers* that can characterize a given physical situation.

Traditionally, dimensional analysis has been used to derive formulas in college physics, and for purposes of modeling and similitude in engineering. Since then, it has been applied to a fairly diverse set of problems in engineering (see, e.g., [26, 37]). The technique's greatest successes have been in fluid mechanics, where it has generally been used for problems in modeling and similitude. One of the most familiar dimensionless numbers used in such modeling is the Reynolds' number, which determines whether a flow is laminar or turbulent.[3] Dimensionless numbers and dimensional analysis have been used a great deal in both old-fashioned fluid mechanics, as well as in newer physics, such as plasma physics and astrophysics [6, 7, 36].

Our task, in this paper, is to extend the scope of dimensional analysis and develop it as a method for qualitative reasoning about physical devices,

[2] Our recognition of the symbolic content of physical variables is not original; it has already made its appearance in AI as a tool for problem solving and discovery, through the seminal work of Mitch Kokar of Northeastern University [20–23]. We are grateful to Wlodek Zadrozny for bringing this work to our attention. Kokar's work is discussed in Section 5 of this paper.

[3] Another dimensionless number that is of enormous importance in physics is the strain on a physical element, traditionally defined as $\varepsilon x/x$, where x is some length of the physical element.

processes and systems. We carry out this extension by developing conceptual machinery for *reasoning* with dimensionless numbers, which we call *regimes*, using elementary notions about partial differentiation. This extension requires that the variables that enter into a particular physical problem, as well as their dimensional representations, be known. Using just such knowledge, we have been able to pose and solve some of the qualitative reasoning problems posed in the literature, such as the pressure regulator [9], the projectile, the spring, and the heat exchanger [42]. Our method, we will show, is especially useful for tackling qualitative reasoning problems of the following kinds:

(a) to resolve, under certain circumstances, some of the ambiguities inherent in reasoning with a $\{+, 0, -\}$ qualitative calculus;
(b) to provide a comparative qualitative representation for a physical process;
(c) to derive the causal structure of the device's behavior, given the inputs and the outputs of a device.

In general, we will develop dimensional analysis as a reasoning method that can be used along with any of the various methods that have already been proposed, rather than as an exclusionary alternative. Limit analysis, central to the qualitative simulation approach, is not addressed [24].

The rest of this paper is organized as follows: in the remainder of this section, we illustrate the use of dimensional reasoning with a familiar example. The example, a simple pendulum, is simply a memory aid, reminding the reader of the way the principle of dimensional homogeneity can be used to derive physical equations or laws.

Section 2 discusses the machinery of dimensional analysis, beginning with a discussion of the principle of dimensional homogeneity and then presents two theorems about dimensions, Bridgman's product theorem [3] and Buckingham's Π-theorem [4].[4] These theorems are used to develop a notation and then to show that the dimensionless numbers that are produced are in fact a representation of the physical processes going on in the device. Using these dimensionless numbers, we develop some machinery using partial derivatives; this machinery is the kernel from which we reason about physical devices qualitatively.

Section 3 presents a series of examples from the literature, including the pressure regulator of de Kleer and Brown [9], the heat exchanger of Weld [42] and the circuit analysis of Williams [44]. Each of these examples are problems that can be posed, understood and, within certain limits, solved with dimensional analysis; furthermore, the examples serve to illustrate different aspects of the strengths and weaknesses of dimensional analysis. Our first example in

[4] Both these theorems are fairly simple creatures; their proofs, which will be obvious to many, are included simply because the original literature may be a little hard to get hold of.

this section is the simple block and spring of Weld [42], and we use this to illustrate the simplest kind of dimensional reasoning, so-called "intra-regime analysis." In this example we also demonstrate that irrelevant variables will not affect the result of a dimensional analysis. The second example, the projectile, also first used in qualitative physics by Weld, illustrates the inverse problem, that of not knowing all the variables that are relevant to a particular situation. We show that this situation, a lack of awareness of relevant variables, can yield misleading results, but that there are some heuristic cues in a situation that can suggest that a problem has in fact been inadequately stated. This example also illustrates how an analysis can be extended to include variables that were not originally included in the analysis. We use the heat exchanger to demonstrate the use of so-called "inter-regime analysis," and the pressure regulator of de Kleer and Brown [9] to show how the causal structure of systems with feedback can be analyzed using what we term "inter-ensemble analysis." (Our use of all these terms is defined in Section 2.3.1). Finally we use an example from circuit analysis used by Williams [44] to show how dimensional analysis can be used for studying electrical systems. Taken together, these examples illustrate the class of problems we mentioned above, viz. resolve ambiguities (the spring), provide a compact qualitative representation of physical processes (via the projectile and the heat exchanger) and provide a causal understanding of feedback devices (via the pressure regulator).

Section 4 discusses the representational role of dimensional analysis, while Section 5 discusses the theoretical relationship of dimensional analysis to other work in qualitative physics, and finally some possible applications of the technique (Section 6.1).

Before we proceed, a historical remark is in order. AI is often considered by critics to "re-invent the wheel," by ignoring studies and previous results in other domains. We don't agree with that view of AI, and we present dimensional analysis as an example of serious intellectual borrowing, engaging in a retrospective similar in spirit to that described by Bobrow [1].

1.1. The simple pendulum

Consider a simple pendulum of the familiar high-school kind, a more or less spherical bob at the end of a taut string, hanging freely from a ceiling or other horizontal support (see Fig. 1).

Consider the problem of determining the period of oscillation of the pendulum:

$$t = f(m, l, g, \theta)\,, \tag{1}$$

where the variables represent the physical quantities as shown in Fig. 1. The units of the left-hand side of the equation are time units (say seconds), and the

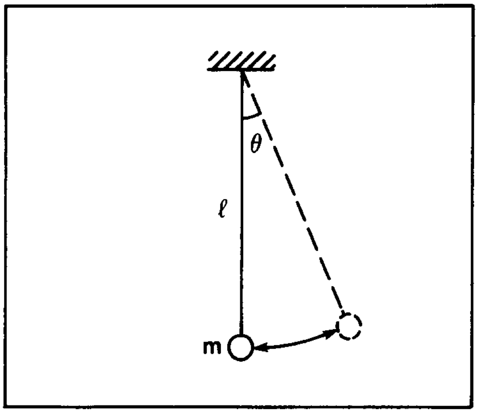

Fig. 1. A simple pendulum.

units of the quantities on the right-hand side are as follows:

- m mass units $[M]$ (say pound-mass),
- l length units $[L]$ (say feet),
- g acceleration units $[LT^{-2}]$ (say feet per second squared),
- θ no dimensions [ ].

By inspection, it is clear that mass units are not needed on the left-hand side of (1). Since only one variable in the right-hand side contains the dimension M, it can be safely omitted, so that we may rewrite (1) as:

$$t = f(l, g, \theta) . \tag{2}$$

Because the solution has the dimensions of time, l and g enter into the equation in such a way that the length dimension must cancel out, so that the solution must be of the form

$$t = f(l/g, \theta) . \tag{3}$$

θ has no dimensions and can be temporarily ignored, but l/g has the dimensions T^2, while the solution has the dimension T^1. Thus, we may rewrite (3) as

$$t = f((l/g)^{1/2}, \theta) . \tag{4}$$

Finally, since θ is dimensionless, it can only enter as a product, so that we may finally write the equation

$$t = \phi(\theta)\sqrt{l/g}\,. \tag{5}$$

The form of this expression will be familiar, from high-school physics; for small amplitudes, we actually know that:

$$\phi(\theta) = 2\pi\,. \tag{6}$$

This allows us to write the equation for the period of oscillation of a simple pendulum in standard form:

$$t = 2\pi\sqrt{l/g}\,. \tag{7}$$

From (7) we see that the mass m of the bob is irrelevant (because $\partial t/\partial m = 0$), and that the longer the length, the greater the period of the bob (since $\partial t/\partial l > 0$).

We have now used dimensions (or units as they are called in high-school physics) to derive the form of an equation. This example illustrates the traditional use of dimensional analysis in both physics [3] and in artificial intelligence [21].

2. Dimensional Analysis

2.1. Principle of dimensional homogeneity

Let

$$y = \sum_i a_i x_i$$

be a physical law or equation. Then all the $a_i x_i$ must have the same dimensions as y; if the a_i are dimensionless constants, then each of the x_i must have the same dimensions as y. This principle, which is usually referred to as *the principle of dimensional homogeneity* is assumed in all of physics, and is often not stated, even in elementary books (see, e.g., [13]). The principle appears to have been first stated by Fourier [3, p. 55].

Having the same dimensions means the following: the exponents of the five basic dimensions that make up x_i and y (assuming, for simplicity, that the a_i are all dimensionless constants) must be the same.[5] This requirement about exponents leads to our first theorem.

[5] In this paper, we have assumed five basic independent dimensions, mass (M), length (L), time (T), temperature (θ) and electric potential (Φ). Alternate sets of dimensions are possible (such as force instead of mass), and there is also some question about temperature as a basic dimension. For some particularly dusty correspondence on this matter, see [5, 31, 32, 34]. Most problems do not require all five dimensions; instead three or four dimensions are often sufficient.

2.2. The early theorems

The early theory of dimensional analysis is based on two key results—the product theorem and Buckingham's Π-theorem. In this section we will state the theorems and discuss their significance. In the appendix we have included suitably annotated versions of the proofs obtained by Bridgman. More sophisticated proofs using group theory were later obtained by other researchers.

Product Theorem. *Let a secondary quantity be derived from measurements of primary quantities $\alpha, \beta, \gamma, \ldots$; assuming absolute significance of relative magnitudes, the value of the secondary quantity is derived as:*

$$C_1 \alpha^a \beta^b \gamma^c \cdots,$$

where $C_1, a, b, c, \ldots$ are constants [3, 12].

The assumption of absolute significance of relative magnitudes requires that the ratio of the numbers measuring two physical quantities (e.g. speeds of two different particles) must be independent of the units used to measure them.[6] The product theorem establishes the fact that dimensional representations must be multiplicative. If we assume that the primary quantities $\alpha, \beta, \gamma, \ldots$ are fundamental, then we can associate basic dimensions such as $[M]$, $[L]$, $[T], \ldots$ with them. These dimensions can be understood as corresponding to equivalence classes of units or as qualitative units. Now according to the product theorem the dimensional representation of the secondary quantity will have the form:

$$[M]^a [L]^b [T]^c \cdots.$$

Buckingham's Π-Theorem. *Given measurements of physical quantities $\alpha, \beta, \gamma, \ldots$ such that $\phi(\alpha, \beta, \gamma, \ldots) = 0$ is a complete equation, then its solution can be written in the form $F(\Pi_1, \Pi_2, \ldots \Pi_{n-r}) = 0$, where n is the number of arguments of ϕ, and r is the basic number of dimensions needed to express the variables $\alpha, \beta, \ldots$; for all i, Π_i is a dimensionless number* [3, 4, 26].

Buckingham's theorem rests on the requirement that implicit functions characterizing the physical situation, i.e. the physical laws, be complete. Buckingham's application of this theorem to different problems is the basis of the procedure for computing the dimensionless products Π. Using this theorem and the related procedure, we develop a method for analyzing physical devices, in Section 3. Before we do that we explore these theorems by laying out some machinery for partial differentiation of Π-expressions.

[6] This assumption is central to all physical measurement. Its justification or proof is outside the scope of this paper. For a comprehensive treatment, we refer the reader to Ellis [12].

2.3. The Π-calculus

The set of Πs that can be used to describe a particular physical situation are continuous functions of real variables. It is therefore possible to understand the behavior of the device by computing partials using the form of the Πs. In this section we present the mechanics of these partials.

2.3.1. *Notation and terminology*

A dimensionless product Π has the following form:

$$\Pi_i = y_i \times (x_1^{\alpha_{i1}} \cdots x_r^{\alpha_{ir}})$$

where $\{x_1, \ldots, x_r\}$ are the repeating variables, $\{y_1, \ldots, y_{n-r}\}$ are the performance variables and $\{\alpha_{ij} \mid 1 \leq i \leq n-r,\ 1 \leq j \leq r\}$ are the exponents.

We call the set of variables x_j that repeat in each Π the *basis*. Each dimensionless number, Π_i, refers to a particular physical aspect of the system and we shall call it a *regime*. We shall call a collection of regimes an *ensemble*. If, in a system of n variables and a dimensional matrix[7] of rank r, the ensemble of regimes contains $n-r$ regimes, we shall call such an ensemble a *complete ensemble*. If x_k is a variable that occurs in both Π_i and Π_j, then we shall refer to x_k as a *contact variable* or as a *pivot*. These two terms will be used interchangeably.

Thus the regime Π_i offers us a dimensionally homogeneous equation connecting the variable y_i with the basis variables $x_1, \ldots, x_r$. For the rest of this section we will use the following as the *product form* relationship obtained from the regime:

$$y_i = \Pi_i \times x_1^{-\alpha_{i1}} \cdots x_r^{-\alpha_{ir}}$$

where $1 \leq i \leq n-r$.

2.3.2. *Analysis*

We need machinery for the following kinds of analyses:

(1) analysis within a regime, *intra-regime analysis*, for examining how the variables within a regime are related to one another;
(2) analysis across regimes, *inter-regime analysis*, to see how different regimes are related to one another through contact variables;
(3) analysis across ensembles, *inter-ensemble analysis*, to reason about the behavior of a device or system consisting of coupled components or subsystems.

[7] The columns of this matrix correspond to the basic dimensions and the rows correspond to the variables. The element (i, j) is exponent of the jth dimension in the dimensional representation of the ith variable.

2.3.2.1. *Intra-regime partials*
From the expression obtained from regime, Π_i, we can obtain partials of y_i with respect to x_j where x_j is a basis variable that occurs in Π_i. Expressions for intra-regime partial derivatives are of the form:

$$\partial y_i / \partial x_j = -(\alpha_{ij} y_i)/x_j \, .$$

Thus knowing the signs of the exponents, α_{ij}, we can easily determine the signs of the intra-regime partials.

2.3.2.2. *Inter-regime partials*
Inter-regime partials are used to relate performance variables y_i and y_j that occur in the regimes Π_i and Π_j respectively. An inter-regime partial is defined with respect to a *contact variable*, i.e. it only exists when the regimes Π_i and Π_j share some variable. The notation for inter-regime partial is:[8]

$$[\partial y_i / \partial y_j]^{x_p} \, ,$$

where x_p is a contact variable for regimes Π_i and Π_j. If there are several contact variables, then we can obtain an inter-regime partial for each. Intuitively, the inter-regime partial models the changes in y_i and y_j in response to a change in the contact variable x_p, all other variables remaining constant. Inter-regime partials can be related to intra-regime partials as discussed below.

Earlier we saw that each regime specifies a dimensionally homogeneous equation for its performance variable. Thus using the contact variable x_p, we can obtain an equation relating y_i and y_j and hence obtain the inter-regime partial:

$$[\partial y_i / \partial y_j]^{x_p} = (\alpha_{ip}/\alpha_{jp})(y_i/y_j) \, .$$

From the regimes Π_i and Π_j we can compute the following partials:

$$\partial y_i / \partial x_p = -y_i \alpha_{ip} / x_p \, ,$$

$$\partial y_j / \partial x_p = -y_j \alpha_{jp} / x_p \, .$$

Thus the inter-regime partial is the ratio:

$$\frac{\partial y_i / \partial x_p}{\partial y_j / \partial x_p} = (\alpha_{ip}/\alpha_{jp})(y_i/y_j) \, .$$

This result can also be obtained using more rigorous arguments.

[8] We use square brackets [ ], to distinguish the notation from the standard partial differential notation $(\partial y / \partial x)_{u, v, \ldots, w}$. The superscript denotes the variable that is *not* held constant; this is different from the usual notation of subscripting all variables that *are* held constant.

2.3.2.3. *Inter-ensemble partials*

Inter-ensemble analysis is the generalization of inter-regime analysis. The objective is to reason across ensembles. When dealing with a device with several components, we obtain an ensemble (of regimes) for each component or subsystem. Now in order to reason about the behavior of the entire device we need to reason about the coupling, which manifests itself in terms of coupling quantities. This knowledge cannot be obtained from dimensional analysis; it has to come from clues such as device function and paths. The coupling quantities are used to obtain coupling regimes. We illustrate this technique using the pressure regulator, in the next section.

In order to obtain inter-ensemble partials we need *contact regimes* (a generalization of contact variables). Consider two ensembles A and B, regimes Π_{Ai} and Π_{Bj} belonging to these ensembles, and variables y_i and y_j that are described by the regimes. Our objective will be to compute the inter-ensemble partial $\partial y_i/\partial y_j$. This will require a coupling regime to Π_C. The inter-regime notation can be extended in the obvious way to specify inter-ensemble partials:

$$[\partial y_{Ai}/\partial y_{Bj}]^{\Pi_C} .$$

2.3.3. *Implications*

We now know how to compute the sign of the partial derivative of performance variables with respect to basis variables, groups of basis variables and other performance variables. Using the equations developed here we can analyze the behavior of any component or process of the device (through intra-regime analysis), or the relationship between components or processes (through inter-regime and inter-ensemble analyses).

2.4. Reasoning with dimensionless numbers

Here we shall make several remarks about the physical role of the dimensionless numbers and how they can be used in reasoning. After these remarks, we shall use the machinery of this section to solve examples commonly used in the qualitative physics literature in Section 3.

2.4.1. *The role of the basis*

The basis plays a crucial role in the construction of regimes. For Buckingham's procedure to work, the variables in the basis, r in all, must satisfy the following dimensional criteria:

- Every dimension that occurs in the dimensional representation of the n variables characterizing the system must occur in the dimensional representation of one or more basis variables.
- The dimensional representations of the basis variables should be linearly independent.

When reasoning we consider the basis variables to be the independent variables. The objective is to compute the direction of change of a performance variable in response to a change in the basis variable(s). Another intuitive interpretation is in terms of causality. In this context the exogenous variables will usually be in the basis, as discussed in Section 3.7. However, since the basis has to satisfy the dimensional constraints mentioned earlier, it might not always be possible to place all the exogenous variables in the basis. In order to reason about change in a performance variable y_i as a result of a change in some exogenous variable z_j:

- If z_j is in the basis and occurs in Π_i, then use intra-regime partials.
- If z_j is in the basis but not in Π_i, then reason using chains of inter-regime partials.
- If z_j is not in the basis, then use the appropriate inter-regime partial linking Π_i and Π_j.

This is the basic kernel of the reasoning strategy. Using it we can reason in more complex situations, e.g. with respect to collections of variables rather than a single variable, and reason across coupled ensembles. Examples of such reasoning will be presented in Section 3.

2.4.2. *The regimes*

Although Buckingham's theorem tells us that it is possible to extract $n-r$ dimensionless numbers to represent a physical situation, it does not tell us their physical role. In particular, it does not guarantee that each number contains only one variable that is not in the basis. Understanding this single-variable aspect of dimensionless numbers follows separately and is key to why they can be considered to have physical meaning as regimes, and why they can be used to reason about the physical situation. This single-variable aspect follows from Hall's theorem in combinatorial theory which we present without proof [15, 16]. *It is this theorem that takes dimensional analysis from being a tool for modeling problems in engineering to a method for problem solving in artificial intelligence.*[9]

Hall's Theorem. *Let I be a finite set of indices, $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let S_i be a subset of a set S. A necessary and sufficient condition for the existence of distinct representatives x_i, $i = 1, 2, \ldots, n$, $x_i \in S_i$, $x_i \neq x_j$, when $i \neq j$, is condition C: For every $k = 1, \ldots, n$ and choice of k distinct indices $i_1, \ldots, i_k$, the subsets $S_{i_1}, \ldots, S_{i_k}$ contain between them at least k distinct elements.*

[9] Buckingham's theorem was published in 1914, and he has no reference to combinatorial theory in his paper; Hall's theorem, published some 20 years later, in 1935, has no reference to physical implications.

Consider a system of $r + 2$ variables. Let r variables be in the basis, and call these variables the x_j. Let the set of variables in the two regimes Π_1 and Π_2 be written in the form $S_1 = \{y_1, x_1, x_2, \ldots, x_r\}$ and $S_2 = \{y_2, x_1, x_2, \ldots, x_r\}$. Now if the r variables x_j are eliminated, we are left with two sets, which are both distinct subsets of a set $S = \{y_1, y_2\}$. Since the set of these two subsets, viz. $\{\{y_1\}, \{y_2\}\}$, consists exclusively of sets that are singletons, a system of distinct representatives of these two sets is the set $S = \{y_1, y_2\}$ itself. By induction, this scheme must extend to any set of $n - r$ variables. Hence each regime represents exactly one variable not in the basis. If a particular Π_i is held constant, it is possible to see how the variable of interest is related to the other variables in the basis.

3. Qualitative Physics

This is the long-awaited examples section; we apply the technique of dimensional analysis to a broad spectrum of problems drawn from the qualitative physics literature, such as the pressure regulator [9], springs and projectiles [42], heat exchangers [42] and circuit analysis [43]. For some of the examples we have included a graphical representation of the ensemble (Figs. 3, 5 and 7); ensemble structure is discussed in Section 3.6.

The dimensional analysis consists of the following procedure:

Step 1. List the n variables that characterize the problem, and write their dimensional representations (e.g. the dimensional representation of force is MLT^{-2}); let r be the number of distinct dimensions that are used.[10]

Step 2. By Buckingham's theorem $(n - r)$ dimensionless products (later referred to as Πs) can be calculated as follows:

(a) Select r variables to be the basis, $x_1, \ldots, x_r$; let δ_i be the dimensional representation of x_i.

(b) Each Π_i, $r + 1 \leq i \leq n$, has the form

$$y_i x_1^{\alpha_{i1}} \cdots x_r^{\alpha_{ir}}$$

and replacing each x and y by its corresponding dimensional representation δ leads to the expression

$$\delta_i \delta_1^{\alpha_{i1}} \cdots \delta_r^{\alpha_{ir}} .$$

As Π_i is dimensionless, the exponents of each dimension must add up to zero. This yields $(n - r)$ systems of r equations in r unknowns which can be solved to obtain the values of the exponents α_{ij}, and hence the expressions for Π_i.

[10] As mentioned earlier, r is correctly the rank of the dimensional matrix, and in some cases may turn out to be less than the number of dimensions.

Step 3. We can use the Πs to reason about system and component behavior by computing partial derivatives of the form $\partial y_i/\partial x_j$ and $[\partial y_i/\partial y_j]^{x_k}$. We can also reason about groups of variables.

This procedure is quite simple—but it should also be quite clear from this outline of the procedure that dimensional analysis cannot proceed without physical knowledge. The knowledge required is exactly the same as that required in any other qualitative reasoning method. What is different is where and how this knowledge is represented and organized—the choice of relevant variables, and the choice of the basis, because of their dimensionality, constrain the set of behaviors of which the system is capable. The apparent simplicity of dimensional analysis stems completely from (i) the familiarity, in physics and engineering, of the dimensional representation of a variable and (ii) the compactness of the dimensional representation of a set of physical variables. This compactness conceals a substantial amount of physical knowledge in the dimensional representation of a variable. For example, the dimensional representation for force implies knowledge of Newton's second law (which is often simplified to $F = ma$). The dimensional representation of force is MLT^{-2} precisely because $F = ma$. In fact, it is fair to say, from a parochial dimensional view, it is possible to characterize Newton's contribution as finding out that the dimensional representation of force and gravity are the same [27]!

Equally, it should be quite clear that any of the devices we describe here can in fact be modeled differently. If the devices are modeled differently, i.e. different inputs, different outputs and different physical conditions are chosen, then different behaviors will result. In fact, a different device altogether will have been modeled. Like many modeling techniques, dimensional analysis offers only minor syntactic checks as to the correctness of the inputs.

3.1. Horizontal oscillation of a block and spring

We use the problem discussed by Weld [42]. The device is presented in Fig. 2. Once the block is displaced from its rest position, it oscillates about this position. The objective is to reason about the behavior of this system in terms of the time period of oscillation. The variables describing this problem and their associated dimensions are:

time period	t	$[T]$,
mass	m	$[M]$,
spring constant	K	$[MT^{-2}]$.

We have three quantities (t, m and K) and two dimensions (M and T); thus we have a single Π. As t is the variable of interest, we write Π as:

$$\Pi = tm^{\alpha}K^{\beta} .$$

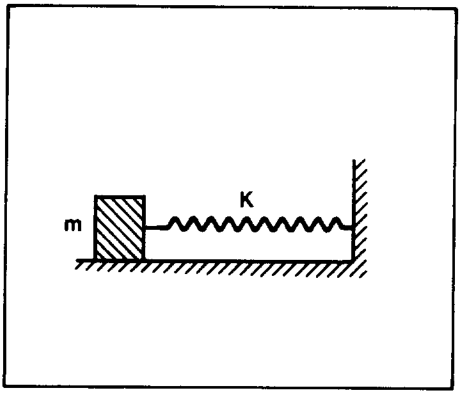

Fig. 2. Oscillation of a frictionless block.

As Π is dimensionless, the exponents of the M and the T dimensions should each add up to zero. Thus we have the equations:

$$\begin{aligned} &M\text{-homogeneity:} \quad \alpha + \beta = 0\,, \\ &T\text{-homogeneity:} \quad 1 - 2\beta = 0\,. \end{aligned}$$

Solving these equations for α and β results in

$$\Pi = tm^{-1/2}K^{1/2}\,.$$

From $\Pi = t\sqrt{K/m}$ we compute the following partial differentials:

$$\partial t/\partial m > 0\,, \qquad \partial t/\partial K < 0\,.$$

Hence we can reason that a heavier mass will oscillate with a larger time period while a stiffer spring will cause the mass to oscillate with a smaller time period. Combined effects of K and m can also be argued, e.g. if m increases and K decreases, t will increase since both changes exert a positive influence on t. Alternately combined effects of K and m can be reasoned about using the partials

$$\partial t/\partial(K/m) < 0\,, \qquad \partial t/\partial(m/K) > 0\,.$$

This ability to reason about combinations is important, for, thus, under certain conditions, we are able to eliminate one of the basic ambiguities of the qualitative calculus. If a $[+,\ 0,\ -]$ scale is used, then it is not possible to compute any relationship of the form $[x_1] - [x_2]$, given that $[x_1] = +$ and $[x_2] = +$. Here we have two ways of resolving such problems. Firstly, as discussed above, we can group the variables K and m together as K/m and reason using the partial $\partial t/\partial(K/m)$. Alternately, we can compute the total derivative of t with respect to time (we use the symbol τ for time) as:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial t}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}\tau} + \frac{\partial t}{\partial K}\frac{\mathrm{d}K}{\mathrm{d}\tau} .$$

Substituting the expressions for partials of t we obtain that $\mathrm{d}t/\mathrm{d}\tau$ is positive if:

$$\frac{\mathrm{d}m/\mathrm{d}\tau}{\mathrm{d}K/\mathrm{d}\tau} > \frac{m}{K} .$$

We could have used as an additional variable, g, the acceleration due to gravity, especially if we were taking a cue from the case of the simple pendulum in Section 1. This would introduce an additional variable, the *acceleration due to gravity*, g, with dimensions $[LT^{-2}]$. Now we have four quantities and three dimensions, so that we still have only one Π, which is of the form

$$tm^{\alpha}K^{\beta}g^{\gamma} .$$

Note that g introduces dimension $[L]$ that does not occur in the dimensional representation of any of the other variables. Hence it will not have any effect on the expression for the Π. The reader may verify the fact that inclusion of g will lead to the solution: $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and $\gamma = 0$. The apparent implication is that the variable g is not relevant to this problem.

3.2. Motion of a projectile

A projectile is shot vertically with a certain initial velocity v. It rises to a certain height and then falls back to earth. The objective of this example is to reason about the height attained and the times of rise and fall. The relevant variables and dimensions in this case are:

time of rise	t_1	$[T]$,
time of fall	t_2	$[T]$,
acceleration due to gravity	g	$[LT^{-2}]$,
maximum height	h	$[L]$,
initial velocity	v	$[LT^{-1}]$.

In this case, we have five quantities and two dimensions (L, T); thus using Buckingham's theorem we can compute three Πs. As t_1, t_2 and h (i.e. time of rise, time of fall, and maximum height attained, respectively) are the variables

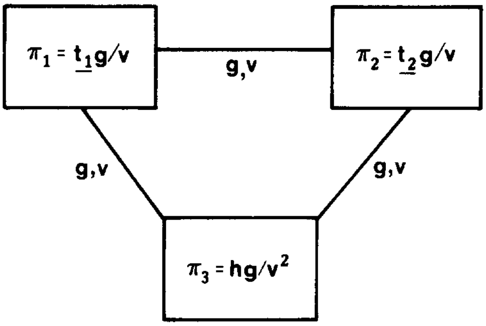

Fig. 3. The projectile ensemble.

of interest, we choose g and v as the basis (Fig. 3). The resulting Πs are:

$$\Pi_1 = \frac{t_1 g}{v}, \quad \Pi_2 = \frac{t_2 g}{v}, \quad \Pi_3 = \frac{hg}{v^2}.$$

Now we can reason about the behavior of the system using the following partials:

$$\frac{\partial t_1}{\partial v} > 0, \quad \frac{\partial t_2}{\partial v} > 0, \quad \frac{\partial h}{\partial v} > 0,$$

$$\frac{\partial t_1}{\partial g} < 0, \quad \frac{\partial t_2}{\partial g} < 0, \quad \frac{\partial h}{\partial g} < 0.$$

Thus if the initial velocity of the projectile is increased, the rise time, the fall time and the height attained all increase, i.e. $\Delta v > 0$ leads to $\Delta t_1 > 0$, $\Delta t_2 > 0$ and $\Delta h > 0$.

What happens if the initial characterization is incomplete? Let us assume that on initial analysis, g was not included in the list of quantities. Now we are left with the variables t_1, t_2, h and v. Since t_1, t_2, h are the variables of interest, only v is available for the basis. Since two dimensions are involved, L and T, the basis must have at least *two* variables. Thus we have an incomplete specification.

Now let us try to model the projectile problem more realistically by introducing air resistance.[11] We already have g which can be thought of as gravitational force per unit mass. Say we do not wish to introduce mass (and hence a new dimension), then we can think of air resistance as force per unit area per unit mass. Also we need to introduce the surface area of the

[11] This generalization was suggested by Sesh Murthy.

projectile.[12] So we add the following quantities:

air resistance r $[L^{-1}T^{-2}]$,
surface area S $[L^2]$.

This leads to two new dimensionless products rv^4/g^3 and Sg^2/v^4. These can be combined into a single product:

$$\Pi_4 = rS/g .$$

Using Π_1 and Π_4 we can obtain the inter-regime partial:

$$[\partial t_1/\partial(rS)]^g < 0 .$$

From this inter-regime partial we can reason that $\Delta(rS) > 0$ will lead to $\Delta t_1 < 0$. Thus a projectile with a larger surface area will have a shorter rise time. Also a projectile travelling in medium with greater air resistance per unit mass per unit area will have a shorter rise time. A more elaborate model would also be able to relate the air resistance with the speed of the projectile. Here again the essence of our technique has been based on the partial of the variable of interest with respect to a suitable group of variables. We have also shown how to add variables of interest to a dimensional analysis; this addition is modular.

3.3. A simple heat exchanger

We consider a simple heat exchanger (similar but not identical to that used by Weld [42]) that consists of a pipe immersed in a coolant chamber (see Fig. 4). Hot oil flows through the pipe and is cooled by the fluid in the coolant chamber, e.g. water. In this example we wish to reason qualitatively about the drop in temperature of the hot oil, i.e. how much it has cooled.

The system may be characterized by the following variables:

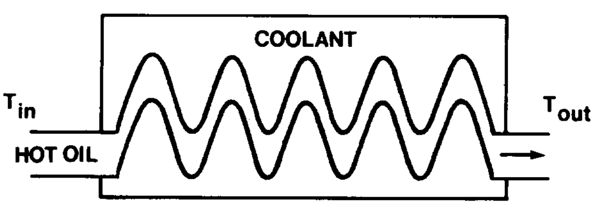

Fig. 4. A simple heat exchanger.

[12] Of course, knowing that S is an interesting variable itself requires knowledge. This kind of knowledge, about how to decide on the variables of interest, is outside the scope of this paper.

density of oil	ρ	$[ML^{-3}]$,
heat transfer area	A	$[L^2]$,
velocity of oil	v	$[LT^{-1}]$,
oil temperature at inlet	T_{in}	$[\theta]$,
oil temperature at outlet	T_{out}	$[\theta]$,
thermal conductivity of pipe material	k	$[MLT^{-3}\theta^{-1}]$.

In this case, we have six quantities and four dimensions; choosing ρ, A, k and T_{in} as the basis leads to the following two Πs (Fig. 5):

$$\Pi_1 = \frac{T_{out}}{T_{in}}, \qquad \Pi_2 = v\left(\frac{\rho A^{1/2}}{kT_{in}}\right)^{1/3}.$$

The following partials can be computed:

(1) $\partial T_{out}/\partial T_{in} > 0$, from Π_1;
(2) $\partial v/\partial T_{in} > 0$, from Π_2.

We shall now reason about the impact of various parameters on T_{out}, the temperature of the oil as it leaves the exchanger. In general, $T_{out} < T_{in}$ as oil loses some heat as it flows through the heat exchanger. However, T_{out} itself can change, while still remaining less than T_{in}. If T_{out} increases then we say that the "oil exits hotter," i.e. the oil loses less heat in the exchanger. Alternately if T_{out} decreases then we say that the "oil exits cooler."

What happens if the velocity of oil is increased, i.e. $\Delta v > 0$? From the partials we know:[13]

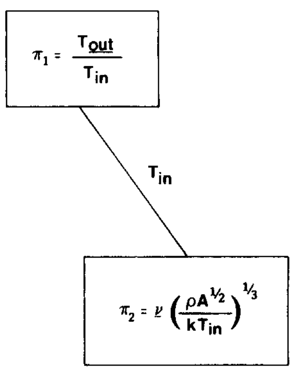

Fig. 5. The heat exchanger ensemble.

[13] For clarification, $dT_{out} = dT_{in}(\partial T_{out}/\partial T_{in})$ and $dv = (\partial v/\partial \rho)\,d\rho + (\partial v/\partial A)\,dA + (\partial v/\partial k)\,dk + (\partial v/\partial T_{in})\,dT_{in}$. Now, setting $d\rho$, dA and dk all equal to 0, we have: $dT_{out}/dv = (\partial T_{out}/\partial T_{in})/(\partial v/\partial T_{in})$.

$$\left[\frac{\partial T_{\text{out}}}{\partial v}\right]^{T_{\text{in}}} = \frac{\partial T_{\text{out}}/\partial T_{\text{in}}}{\partial v/\partial T_{\text{in}}} > 0\,.$$

Hence $\Delta v > 0$ will lead to $\Delta T_{\text{out}} > 0$ or the oil exits hotter.

What happens if the thermal conductivity of the pipe material is increased, i.e. $\Delta k > 0$? In this case, k is in the basis and it does not occur in Π_1. Thus we need to obtain the inter-regime partial $\partial T_{\text{out}}/\partial k$. We modify the basis replacing v by k; this change results in Π_2' which takes the form:

$$\Pi_2' = k(T_{\text{in}}/\rho v^3 A^{1/2})\,.$$

Now we can compute the inter-regime partial $[\partial T_{\text{out}}/\partial k]^{T_{\text{in}}}$ as

$$\frac{\partial T_{\text{out}}/\partial T_{\text{in}}}{\partial k/\partial T_{\text{in}}} < 0\,.$$

Hence $\Delta k > 0$ will lead to $\Delta T_{\text{out}} < 0$ or the oil will exit cooler (having lost more heat through improved thermal conduction).

We have presented here a very simple model of a heat exchanger. A more elaborate model would consist of two ensembles corresponding to the *hot* and *cold* sides of the heat exchanger. These ensembles would be coupled by a Π which is the ratio of rates of heat loss and gain respectively. Another aspect of the model presented above is that we used a single variable which contains only the length dimension—heat transfer area A $[L^2]$. If we introduce variables such as length of the exchanger, pipe thickness and pipe diameter—all with dimensional representation $[L]$—then some additional knowledge would be needed. This is a consequence of the fact that dimensional analysis cannot be used to distinguish between variables of identical dimensionality.

3.4. The pressure regulator

The function of the pressure regulator is to maintain a constant pressure at the output. We shall analyze this device in terms of two components—a pipe with an orifice and a spring valve assembly (see Fig. 6). Each of these components will be modeled as an ensemble using dimensional analysis. The objective of this example is to demonstrate how we reason across coupled ensembles.

3.4.1. *Pipe orifice ensemble*

The pipe with an orifice is a familiar system in fluid mechanics; the pertinent quantities are as follows:

outlet pressure	p_{out}	$[ML^{-1}T^{-2}]$,
orifice flowrate	Q	$[L^3T^{-1}]$,
inlet pressure	p_{in}	$[ML^{-1}T^{-2}]$,
orifice opening	A_{open}	$[L^2]$,
fluid density	ρ	$[ML^{-3}]$.

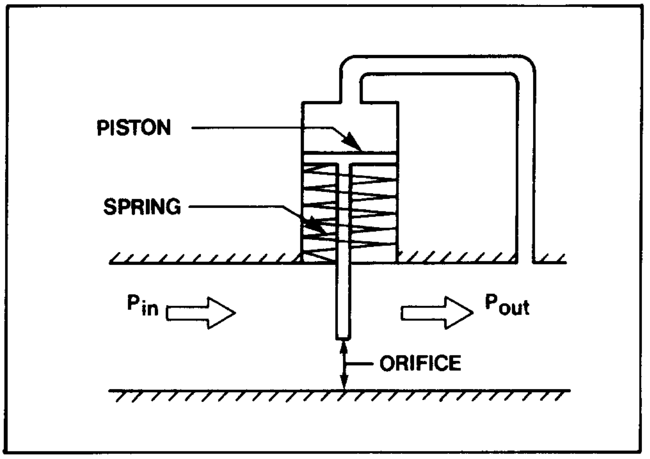

Fig. 6. The pressure regulator.

Using $(p_{\text{in}}, A_{\text{open}}, \rho)$ as the basis we can obtain the following Πs (Fig. 7):

$$\Pi_{A1} = \frac{Q\rho^{1/2}}{A_{\text{open}} p_{\text{in}}^{1/2}}, \qquad \Pi_{A2} = \frac{p_{\text{out}}}{p_{\text{in}}}.$$

From this ensemble the intra-regime partials $\partial Q/\partial p_{\text{in}}$, $\partial Q/\partial A_{\text{open}}$, and $\partial p_{\text{out}}/\partial p_{\text{in}}$ are all positive. Hence the inter-regime partial $[\partial p_{\text{out}}/\partial Q]^{p_{\text{in}}}$ is also positive. If the input pressure p_{in} increases, $\Delta p_{\text{in}} > 0$, then from the intra-regime partials we can infer that the flowrate Q and outlet pressure p_{out} will increase. Similarly if the orifice opening A_{open} decreases then we can conclude that Q will increase. Lastly, since the inter-regime partial $[\partial p_{\text{out}}/\partial q]^{p_{\text{in}}}$ is positive, an increase in Q will lead to an increase in p_{out}.

3.4.2. *Spring valve ensemble*

Now we analyze the spring valve assembly as a separate component. Pressure is applied to a piston that is connected to a spring. The quantities that characterize this system are:

spring displacement	x	$[L]$,
pressure	P	$[ML^{-1}T^{-2}]$,
spring constant	K	$[MT^{-2}]$.

From these quantities we can obtain:

$$\Pi_{B1} = xP/K .$$

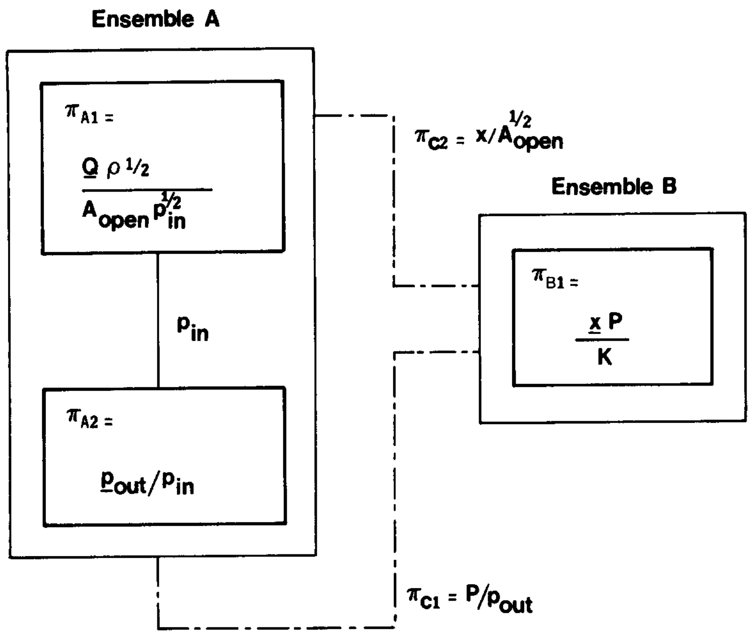

Fig. 7. Inter-ensemble analysis of the pressure regulator.

Here we have used P and K as the basis; even though three dimensions appear, the rank of the dimensional matrix is in fact two.[14] (There is a simple intuitive argument for this; the combination $[MT^{-2}]$ appears both in P and K, thus it can logically be seen as a single dimension.) The essential behavior that we are interested in follows from the intra-regime partial $\partial x/\partial P$ which is negative. As the pressure P applied on the piston increases, x decreases.

3.4.3. *Coupling the ensembles*

We now consider the coupling of the two components of the pressure regulator. The information needed for coupling the ensembles comes in two flavors—topology and geometric constraints. Coupling regimes are closely tied to the connections between components and thus are ratios of pertinent

[14]The dimensional matrix here is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & -2 \end{bmatrix}.$$

The third column is a multiple of the first column. In more intuitive physical terms it is possible to characterize the system using two dimensions, viz. $[L]$ and $[MT^{-2}]$.

quantities with identical dimensionality modulo exponent. In this example there are two coupling regimes:

$$\Pi_{C1} = P/p_{out}, \qquad \Pi_{C2} = x/A_{open}^{1/2}.$$

The regime Π_{C1} comes from the connection that transmits the outlet pressure in the pipe to the piston in the spring valve assembly. Thus $\partial P/\partial p_{out}$ is positive; so an increase in p_{out} leads to an increase in P. The second coupling regime, Π_{C2}, encodes the geometric constraint that motion of the piston affects the orifice opening; more specifically as the spring is compressed, the orifice reduces. This behavior is captured by the partial $\partial x/\partial A_{open}$ which is positive. Dimensional analysis requires that the coupling quantities are not linearly independent; additional information is needed, as we saw above, to ensure that the coupling regimes produce partials consistent with the topology and geometry.

3.4.4. *Behavior of the pressure regulator*

The function of the device is to maintain outlet pressure p_{out} at some constant value p_*. Based on the machinery developed so far, we will now reason that the pressure regulator will exhibit the following behavior:

> An increase in p_{in} leads to an increase in p_{out} (from Π_{A2}). This increase in p_{out} leads to an increase in P in the spring valve ensemble (from coupling regime Π_{C1}). The increase in P causes x to decrease (from Π_{B1}). This time using the coupling regime Π_{C2}, decrease in x leads to a decrease in A_{open}. Now in the pipe orifice ensemble, this decrease in A_{open} leads to a decrease in Q. Finally through the inter-regime partial $[\partial p_{out}/\partial Q]^{p_{in}}$, the decrease in Q leads to a decrease in p_{out}. Thus we have derived the feedback behavior, i.e. an increase in p_{out} eventually leads to a decrease in p_{out}. This might seem like a contradiction but is not; we are talking about changes in p_{out} at temporally distinct points.[15]

A similar account can be developed for the case where the initial disturbance was a decrease in p_{in}. The purpose of the account was to show how to reason about instantaneous changes that eventually culminate in equilibrium behavior. As with the other examples, we have chosen to present a simple characterization. A more detailed model would take into account aspects such as the viscosity of the fluid, and its effect on the pressure drop in the orifice. Another variant would be to study the flow under supersonic or choked conditions.

[15] Our use of *leads to* in the above account is an informal encoding of the temporal aspect of the behavior.

3.5. Circuit analysis

We now move on to the domain of electrical circuits and systems. Two different systems of dimensions have been used for electrical quantities—$MLTQ$ and $LTI\Phi$. The Q, I, and Φ dimensions stand for charge, current and electric potential respectively. For this paper we will adopt the $LTI\Phi$ system.

A simple resistor–capacitor circuit is analyzed [44]. It consists of a resistor and a capacitor connected in parallel. The voltage drop across the circuit is V_{in} and the currents in the resistor and capacitor components are I_R and I_C respectively. We will model the resistor and capacitor ensembles separately and then couple them. We will also see how different special laws can be derived by setting the Πs to specific values.

The quantities characterizing the *resistor ensemble* are:

resistor current	I_R	$[I]$,
voltage drop	V_{in}	$[\Phi]$,
resistance	R	$[I^{-1}\Phi]$.

Using R, and I_R as the basis, we obtain the following dimensionless product:

$$\Pi_1 = I_R R / V_{\text{in}} \,.$$

Π_1 captures the essence of the resistor model; in fact setting its value to 1 leads to the resistor model. Also we can obtain the intra-regime partials:

$$\partial I_R/\partial R < 0 \,, \qquad \partial I_R/\partial V_{\text{in}} > 0 \,, \qquad \partial I_R/\partial(V_{\text{in}}/R) > 0 \,.$$

The *capacitor ensemble* is characterized by the following quantities:

rate of change of voltage drop	$\dot{V}_{\text{in}}$	$[\Phi T^{-1}]$,
capacitance	C	$[TI\Phi^{-1}]$,
capacitor current	I_C	$[I]$.

Using I_C and C as the basis we obtain the following dimensionless product:[16]

$$\Pi_2 = \dot{V}_{\text{in}} C / I_C \,.$$

Here again setting Π_2 to 1 leads to the capacitor model. From this dimensionless product we can compute intra-regime partials such as $\partial \dot{V}_{\text{in}}/\partial C < 0$.

Finally we need to couple the resistor and capacitor ensembles. This is

[16] This is another example where the rank of the dimensional matrix is two even though three dimensions are used to characterize the system.

accomplished using the simple dimensionless product:

$$\Pi_3 = I_R / I_C .$$

Setting Π_3 to -1 yields the Kirchhoff Current Law for the circuit. This is additional knowledge that is not discovered automatically. Now we can use the coupled ensembles to make inferences of the following form:

> From Π_1, an increase in V_{in} leads to an increase in I_R. Since the coupling regime Π_3 is set to -1, an increase in I_R leads to a decrease in I_C. Eventually, from Π_2, decrease in I_C leads to a decrease in $\dot{V}_{in}$.

An interesting aspect of the resistor–capacitor circuit is the time taken to discharge the capacitor, i.e. the time for the capacitor current to fall to some specified *near-zero* level. Let this time be denoted by the variable τ. Using the circuit parameters R and C as the basis, we can obtain the dimensionless product τ/RC. From this it follows that $\partial\tau/\partial(RC)$ is positive. This enables us to reason about the effects of changes in R and C on the variable τ.

This problem illustrates how setting a particular Π to an interesting value in the domain can yield interesting results, such as the Kirchhoff Current Law.

3.6. Structure of ensembles

In this section we review the structure of the ensembles that were derived for the different examples. The discussion is focused on ensembles produced for the projectile, heat exchanger and pressure regulator. We will also use the ensemble structure to show what kinds of partials need be computed.

Ensemble structure is represented as an undirected graph (see Figs. 3, 5 and 7) where each node is a Π; for ease of understanding we have included the expression for the Π in the node and the quantity of interest (also referred to as the performance variable) has been underlined. The edges encode the existence of contact variables and the labels are the contact variables themselves.

The projectile ensemble (Fig. 3) is fully connected, i.e. every regime has an edge to every other regime, and each edge is labeled by the entire basis, i.e. the variables g and v. All partials of interest can be computed directly as intra-regime partials since all the basis variables occur in each of the regimes.

In the modified projectile, where air resistance and surface have been included, we introduce an additional regime Π_4 which is rS/g; this modification is not shown in Fig. 3. This new regime will be connected to all the existing regimes by edges labeled g. Consider that we need to reason about the effect of air restance r on time of rise t_1. The variables t_1 and r are performance variables in the regimes Π_1 and Π_4 respectively. To reason across these regimes, we need an edge connecting them; and this is accomplished by the inter-regime partial using g as the contact variable.

The heat exchanger example presents a simple ensemble (Fig. 5) consisting of two regimes and a single contact variable T_{in}. An extension of such an ensemble can be a linear chain of regimes. For instance Π_2 can be connected to another regime, e.g. modeling geometric aspects of the heat exchanger, with heat transfer area A as the contact variable. In such a case reasoning across regimes that are not immediate neighbors will involve using more than one inter-regime partial.

In general, the regimes in an ensemble must be connected either directly or indirectly. Why is an ensemble with a disconnected regime not meaningful? Consider an attempt to model a heat exchanger where all variables except two do not contain the temperature dimension $[\theta]$, and the remaining two variables are some temperatures T_1 and T_2. Hence T_1/T_2 will fall out as a disconnected regime. This signals the fact that the problem is underspecified, i.e. we have included temperatures but missed quantities that capture the heat transfer phenomenon (e.g. thermal conductivity).

The pressure regulator consists of two connected ensembles (Fig. 7) each with two regimes. The inter-ensemble structure is captured by dashed edges that encode coupling regimes and are thus labeled. In this example multi-ensemble modeling has been used to encode feedback. This requires at least two coupling regimes since effects need to be transmitted in both directions. As we saw earlier, such information is provided by the topology of component interconnection. Thus while coupling ensembles A and B we will need coupling regimes with the following flavor:

- couple a performance variable in A to a basis variable in B;
- couple a performance variable in B to a basis variable in A.

The coupling regimes for the pressure regulator accomplish this kind of interconnection. Sometimes it might also be useful to couple basis variables across ensembles. However, multi-ensemble modeling is not exclusively for encoding feedback or interconnection. It can be used for hierarchical refinement of the initial model. In this case some basis variable x_i in ensemble A is coupled to a performance variable z_j in a lower-level ensemble B. This lower-level ensemble B has a different basis that models the device at a finer granularity. In terms of causality, x_i is an exogenous variable with respect to the ensemble A. Ensemble B represents the refined model and considers the changes that affect this variable.

3.7. Heuristics for basis selection

A crucial step in computing the Πs is the selection of r basis variables. In principle there are $\binom{n}{r}$ choices; however, many of these do not yield an ensemble, i.e. there are fewer equations than there are unknowns.

We have found the following *basis selection heuristics* to be useful. These heuristics assume that for all candidates for membership in the basis, the property that basis dimensions cover the dimensional space of the problem holds.

- A variable of interest, whose behavior is to be reasoned about, should not be included in the basis.
- Exogenous variables, *whenever possible*, should be included in the basis.
- Other things being equal, dimensional richness (e.g. MLT^{-2} is richer than L) is the criterion for including a variable in the basis.
- Given several variables with the same dimensional representation, only one should be included in the basis.

Implementing a system for dimensional reasoning is largely a matter of selecting the input variables and output variables that characterize a particular device or process. The heuristics that we have listed above are guides to implementing device models rather than heuristics that can be used by a system to select basis variables.

4. Regimes as Representation

4.1. Regimes as physical process

A complete ensemble of regimes represents a physical process, and it is possible to produce such an ensemble for any process.[17] If the basis in an ensemble is simply the set of exogenous variables, then the individual regimes are ways of characterizing the relationship of each of the performance variables, the y_i, in terms of the exogenous variables. Thus the regimes can be thought of as representing decomposable subsystems. In this sense, the regimes are *tertiary variables* that use the earlier levels we have defined, where the dimensions are primary variables and the variables in the system are secondary variables.

Two aspects of Πs as representations are important; first, they allow a certain modularity in the analysis. Second, they embody combinatorial knowledge among the variables rather than relational knowledge such as equations.

Because the Πs are a system of distinct representatives of the subsets of the set of variables (see Section 2.4.2), adding variables will add Πs in a modular fashion. This can be understood by reasoning as follows: if a new variable is introduced to the problem, it can either introduce a new dimension or not.

[17] Compare with other uses of the term *process*. Forbus: "A *physical process* is something that acts through time to change the parameters of objects in a situation." [14, pp. 104–105]; Iwasaki and Simon ". . . *mechanism* . . . to refer to distinct conceptual parts in terms of whose functions the working of the whole system is explained. Mechanisms are such things as laws describing physical processes or local components that can be described as operating according to such laws." [19, p. 8] Our conception of a physical process is similarly informal.

Case 1. If it introduces a new dimension, then the rank of the dimensional matrix, r, increases by 1, n increases by 1, $n - r$ is not increased, therefore no new Π is possible. (Actually, because only one variable has the new dimension, it can be directly eliminated, as mass m was eliminated in the simple pendulum case, leaving the ensemble unchanged.)

Case 2. If no new dimension is introduced, n is increased, r is unchanged, and $n - r$ is increased by 1. Thus, a new Π can be computed. This new variable is not needed in the basis, and therefore a new Π can be constructed for it.

The important thing is that new Πs can be added as assumptions are relaxed, with no requirements for a cross-product of the partials of the new variable with respect to the existing system of differential equations, whether they are qualitative, ordinary or partial.

The second important aspect of the Πs as representation is that they are derived from combinatorial knowledge. Relationships between variables are not available to the algorithm that computes the Πs. Instead, only combinatorial knowledge is available in the form of the set of relevant variables. The information about relationships is implicit, being contained in the constraints placed by the principle of dimensional homogeneity, the product theorem and Buckingham's theorem, on the dimensional representation of each variable.

4.2. In-principle reducibility

Unlike many qualitative expressions of physical laws, dimensional analysis does not require that numerical information be substituted with nonnumerical, qualitative information. Instead, an ensemble of regimes with the appropriately chosen variables contains all the physical information that a set of laws and geometrical constraints contain. As in many other representations, this reducibility is an *in-principle reducibility*, implying that it is often inefficient to carry out the reduction, but that it may be done if necessary. Inappropriately chosen variable sets can of course lose information. Thus, for example in many equilibrium or conservation situations ($\Phi + Y = 0$ *or* $\Phi = Y$) it may be necessary to have at least one variable that represents $\Delta\Phi$ or ΔY, so that conservation laws may be appropriately represented in the regimes.

4.3. Conservation of dimensions

Deriving the ensemble for a physical system does not require any direct or explicit knowledge of the laws of physics. Once this physical knowledge is available dimensional analysis can be conceived of as a combinatorial exercise rather than an equation-based semantic reasoning mechanism. However, the constraints on the scheme are derived from the principle of dimensional homogeneity. If this principle is derivable from principles of set theory and the theory of types, then it is correct to term the exercise mathematical (or combinatorial.)[18] However, if the principle of dimensional homogeneity or dimensional conservation is also a short-hand method for expressing some of the conservation laws of physics, then the exercise must be termed physical.[19] We have, however, not been able to find any discussion, in the physics literature we have examined, about the basis of the principle of dimensional homogeneity.

4.4. Power versus generality

In a paper for an operations research audience in 1969, Newell discussed the simplex tableau in terms of a power–generality tradeoff [29]. The task of inventing new representations and ways of handling them mechanically, Newell then implied, were tasks in artificial intelligence. We think the regimes are a way of representing physical processes and the partials are mechanical ways of handling them.

Dimensional methods require a special representation, viz. the dimensional representation of physical variables. This special representation is widely understood and used in the physical sciences and in engineering, but is still quite specialized. Intransmutability of dimensions is hard to come across in fields that are not physical. This specialization, however, allows the use of weak methods that are fairly robust, such as the weak method called ordinal reasoning.[20] This power–generality tradeoff is an important part of the appeal of the dimensional calculus and makes it a point or region on the continuum between weak and strong methods.

5. Related Concepts

In this section we shall discuss how dimensional analysis is related to other work on qualitative physics. Our approach here has been to choose systems that are either representative of the field of qualitative physics or are very closely related to our approach, rather than to be exhaustive. In particular, we have left out at least two important aspects of qualitative physics: diagnosis, where we have not discussed the excellent work of Brian Williams and Johan de Kleer [11] and order-of-magnitude reasoning of the sort proposed by Raiman [30]. These are important parts of qualitative physics, but we have not yet started to tackle them with dimensional analysis.

With regard to the theories in qualitative physics we do discuss, our

[18] This is an unsupported conjecture by us.

[19] We are intrigued by the possibility of a relation between dimensional homogeneity and conservation laws. For example, it is possible to construct a system of units where G (Newton's constant), h (Planck's constant), and c (the velocity of light) can all be set equal to one. Such a system of "geometrized units" is used in general relativity [27].

[20] The partials are obviously elements in the representation, and the method used to construct them is however not a weak method, but instead the strong method known as differential calculus.

description of the relationship will necessarily be brief. Our intention is to describe the central characteristic of each of these points of view and then describe its theoretical relationship with dimensional analysis.

5.1. Critical hypersurfaces

Kokar's work on the use of dimensions, dimensional analysis and critical hypersurfaces [20–23] is closely related to our work (a hypersurface is an expression for a Π; a *critical* hypersurface is a particular value of a Π). The main similarity is the fact that dimensions and dimensionless numbers play a role in both schemes. In spite of this common approach, there are several differences, which we enumerate below. We are concerned primarily with the symbolic content of ordinary physics; Kokar's concern is with using numerical values to establish values in a quantity space. We elaborate, in order to clarify the fundamental differences between our approach and his.

Criticality
Kokar concentrates on *critical* hypersurfaces which correspond to a particular numerical value of a Π. Dimensionless numbers are used similarly in traditional engineering, for example, a plot of Reynolds' number versus drag coefficients [41, p. 217]. Our use of dimensional analysis, however, is independent of the actual numerical values of a particular dimensionless number.

Use of partial derivatives
We use partial derivatives to reason about physical behavior. Kokar suggests that a single partial derivative, if chosen well, will establish a critical hypersurface. Beyond establishing a value for a hypersurface, partial derivatives play no role in his scheme.

The role of the basis
Kokar states that the actual choice of basis (the "dimensional base") is not important, because any choice will allow the establishment of a critical hypersurface. We think that the choice of basis will actually determine the utility of a particular ensemble; physical knowledge, represented dimensionally in a model, can be used to select exogenous variables for the basis.

Devices
Although his analysis is about physical laws and processes, devices themselves are not considered.

Choosing relevant variables
Kokar is concerned with the "Complete Relevance Problem." We offer some clues to it, but it is outside the scope of dimensional analysis.

Use of the Πs
Finally, Kokar uses critical hypersurfaces in order to use a quantity space. We

reason directly with partial derivatives and do not use a quantity space generated in some other representational scheme.

5.2. Naive physics and qualitative process theory

Both naive physics [17, 18] and qualitative process theory (QPT) [14] have been subject to substantial theoretical development as well as practical exploration (for example, see [8]), and both have devoted some attention to problems of reasoning about fluids. For example, Hayes has argued implicitly that fluid flows and the fluids that are actually flowing can be allowed to be conceptually distinct entities that may (sometimes) be reasoned about separately from one another. This is embodied in the pair of ontologies that have been proposed, the *piece-of-stuff* ontology and the *contained-liquid* ontology.

Fluid mechanics in fact also uses two approaches, the so-called *control mass* approach and the so-called *control volume* approach. It is generally accepted that the two approaches are equivalent in a variety of ways, though conversion is sometimes tedious. The control mass approach and the control volume approach bear a striking resemblance to the piece-of-stuff ontology and the contained-liquid ontology.[21] This is not surprising; representations are reflective of the phenomena that they represent and Hayes' ontologies, QPT and fluid mechanics are all attempting to represent understanding of the behavior of the fluids, albeit at different levels, and with different aspirations to generality. We think that dimensional calculus can be used in conjunction with qualitative process theory, because the representational approaches in both schemes are complementary.

5.3. Confluences

Regimes and confluences are similar. They are both direct representations of physical situations. Further, intra-regime analysis is similar to the component heuristic, and inter-regime analysis is similar to the propagation heuristic. In general, regimes are derivable from dimensional analysis without explicit knowledge of physical laws, while confluences can be considered, at some level of abstraction, to be qualitative, specialized restatements of physical laws.[22] In general, these restatements cannot be derived, but are carefully crafted. Because of this care of crafting, confluences can be made to yield useful and relevant results for analyzing and understanding devices, whereas regimes do not always necessarily yield useful results [9].

[21] In fact, some books on engineering thermodynamics literally define control mass as "a piece of matter." For example see [33, p. 105].

[22] It is worth noting that a confluence, however, unlike a physical law, need not be dimensionally homogeneous [9, Footnote 1, p. 8].

5.4. Comparative analysis

(1) Both DQ analysis and the method of exaggeration need a limited amount of equation knowledge, whereas dimensional analysis does not [42].
(2) DQ analysis is not always do-able, while regimes can always be derived. On the other hand DQ analysis, where do-able, is guaranteed to be useful. An ensemble of regimes may have to be manipulated somewhat before its results are useful or relevant.
(3) Comparative analyses use qualitative differentials, whereas dimensional analysis uses conventional partial differentials, and relies on ordinal reasoning about these derivatives for its qualitative aspects.

5.5. Qualitative simulation

QSIM considers states of systems to be crucial, with attention being paid to possible trajectories of states and state transitions [24]. Multiple trajectories are possible because the information that is used to compute the states uses qualitative landmark values rather than numerical values. Temporal aspects of situations become significant. Dimensional analysis, however, does not deal directly with time; instead, time has to be dealt with by considering total derivatives of the regimes with respect to time. Although we do not believe that this calls for much new apparatus in dimensional analysis, we have not explored the possibilities of representing time-based behavior with our regimes.

5.6. Causal orderings

Causal orderings are, we believe, important aspects of device behavior. Dimensional analysis does not capture this directly, but does offer two important pointers. First, the choice of the basis and a subsequent ensemble of regimes allow a device to be represented as a collection of subsystems. Second, the regimes admit of a causal ordering among themselves, so that it becomes appropriate to speak of a causal ordering among subsystems or regimes rather than among single variables. This causal ordering cannot however be generated from dimensional analysis alone, but must be done using knowledge of the device [19].

6. Conclusion

In our lifetimes, the scientific world has been dazzled by the invention of chromatography, the code of the double helix, and, in the computer age, the marvellous properties of bootstrap methods in statistics. In each case, information has been found in what was felt to be a surprising place. Dimensions of variables appeared to us as unlikely and insufficient repositories of the enormous amounts of information needed for physical reasoning; the dimensional content of ordinary physics has surprised us. In this paper, we have developed a calculus for dealing with physical variables symbolically, using the dimensional representation of the physical variables. We have done this by extending the technique of dimensional analysis. Using this calculus, we have presented analyses of some well-known problems in the qualitative physics literature. The technique is useful primarily because it can be applied when no direct knowledge of the physical laws of a device is available.

6.1. Qualitative physics in the real world

The examples presented in the paper, though representative of qualitative physics research, were essentially simple and pedagogical. Even though we did not use equation knowledge from the underlying physics, such knowledge is simple, available, and used by other researchers. The greatest challenge for qualitative physics and artificial intelligence, however, lies in large systems where the underlying physics knowledge is either ununderstood or uncomputed. Humans, we believe, have great difficulty performing qualitative reasoning about such a system primarily for reasons of *scaling* and *coupling*.

Scaling refers to the enormous number of variables (or quantities), possibly several hundred, that characterize the system. A more graphic account of this complexity can be witnessed in the profusion of dials, displays, and warning lights in the control room of a power plant or the cockpit of a jet fighter. Using dimensional calculus, we believe, can render tractable problems of larger systems, even if the characteristic equations of the system are not known.

Qualitative reasoning about large systems often requires one to reason about the subsystems and the manner in which they are coupled. Knowledge about coupling, such as paths and behaviors, can isolate the quantities that participate in the coupling. Applying the dimensional calculus to these quantities can then be used to obtain the regimes that couple the subsystems. The pressure regulator, presented earlier, demonstrated coupling at the device level. Some potential applications of qualitative reasoning using dimensional calculus are power plants, process engineering and control systems and electrical machines, as well as a variety of other coupled systems, such as technological accidents and catastrophes.

6.2. A remark on ordinal reasoning

The technique has also revealed to us that it is possible to do ordinal reasoning about physical systems, and that much of physical intuition can be captured in terms of a set of appropriate partials; to get the partials, it is not always necessary to have direct knowledge of the laws of physics. Instead specifying a system's input and output variables is sufficient to produce a qualitative understanding of the device. This also suggests that reasoning in physics can be accomplished by the same methods that are used to reason in the social sciences, such as economics, where ordinal reasoning with partials plays an important role. We believe that it is probably quite difficult to distinguish between so-called "hard science" and so-called "soft science," on the basis of intrinsic distinctions in methods of reasoning or the logic used to reach conclusions and form theories.

6.3. Some difficult questions

Causal ordering problems
We have not explored the intimate connection between causal ordering and dimensional analysis. Appropriate use of dimensional analysis requires explicit representations of causality. We conjecture that exploration of the connections between Hall's theorem, Buckingham's theorem and the theorems of causal ordering, will yield useful results.

Heuristics for discovering numerical values for Πs
The example of the circuit suggests that it is possible to discover interesting physical regimes by setting a Π to some appropriate value, such as -1, 0, 1, ∞, γ, etc. We believe that heuristics for setting numerical values will be important as dimensional analysis is applied to problems of systems whose physics are usually left uncomputed.

Dimensional intransmutability in nonphysical domains
We conjecture that dimensional analysis will be applicable in domains where much of the knowledge is in the form of relationships between variables whose dimensions are intransmutable; the greater the number of such dimensions, the easier it is to discover dimensionally rich situations. We have not come across such domains, where the intransmutability is as rigid as it is in the physical world and the dimensional characterization as obvious.

Discovering the applicability of dimensional analysis to qualitative physics has been an exciting experience; we hope that we have communicated some of the excitement to our readers.

Appendix A. Product and Π-Theorem from Bridgman

In this section we present the product and Π theorems and their proofs according to Bridgman; the material has been drawn from Bridgman's book. Other proofs to these theorems can be found in the literature. This material has been included for completeness since Bridgman's book may not be easily accessible to our readers. We retain Bridgman's notation, but make every effort to motivate and clarify the material. All quotes in this section are from Bridgman.

Bridgman, and others, group physical quantities into two categories—*primary* and *secondary*. The primary quantities are regarded as "fundamental and of an irreducible simplicity." A secondary quantity is derived from measurements of associated primary quantities, e.g.

$$f(\alpha, \beta, \ldots)$$

is a secondary quantity derived from the values of the primary quantities α, $\beta, \ldots$. For example velocity is a secondary quantity derived from primary quantities, distance and time, i.e. velocity is obtained by measuring appropriate distance and time values. The product theorem shows that the function f is in product form.

Another important aspect of Bridgman's approach is the *rule of operation* by which numerical values are assigned to quantities. This covers the notions of scale and ratios that are summarized next:

Scale. A primary quantity α is measured in some units (e.g. feet in case of length). If the units are scaled so that the new unit is $1/x$ of the old unit then the new value of the measurement is $\alpha' = \alpha * x$.

Ratio. Consider two different physical situations, e.g. two different rods. Let α_1 and α_2 be the measurements (e.g. lengths) on a given scale. Now if we change the scale and measure the rods, the values are α_1' and α_2'. The rule of operation requires that:

$$\alpha_1/\alpha_2 = \alpha_1'/\alpha_2' .$$

This rule applies to primary as well as secondary quantities. Bridgman refers to this rule as the "absolute significance of relative magnitude." In other words the ratio of viscosity of two liquids will be independent of the units in which viscosity is measured.

A.1. Product theorem

Assuming absolute significance of relative magnitudes of physical quantities, the function f relating a secondary quantity to the appropriate primary quantities α, $\beta, \ldots$ has the form:

$$f = C\alpha^a\beta^b \cdots$$

where C, a, $b, \ldots$ are constants.

Proof. Let us consider two concrete secondary quantities, e.g. kinetic energies of two different particles. Let $f(\alpha_1, \beta_1, \ldots)$ and $f(\alpha_2, \beta_2, \ldots)$ be quantities derived from the measurements of the primary quantities $\alpha_1, \beta_1, \gamma_1, \ldots$ and $\alpha_2, \beta_2, \gamma_2, \ldots$ respectively. Note that α_1 and α_2 are measurements of α

corresponding to the concrete situations. Similarly β_1 and β_2 are measurements of β. Now change the unit measuring α_1 and α_2 to be $1/x$ as large; the corresponding measurements will now be $x * \alpha_1$ and $x * \alpha_2$. Similarly change the units of β_1 and β_2 using a factor $1/y$ instead of $1/x$; and so on for the remaining primary quantities. The requirement of absolute significance of relative magnitudes leads to:

$$\frac{f(\alpha_1, \beta_1, \ldots)}{f(\alpha_2, \beta_2, \ldots)} = \frac{f(x * \alpha_1, y * \beta_1, \ldots)}{f(x * \alpha_2, y * \beta_2, \ldots)} \tag{A.1}$$

for all values of $\alpha_1, \beta_1, \ldots, \alpha_2, \beta_2, \ldots$, and $x, y, \ldots$.

Rewrite this in the form:

$$f(x * \alpha_1, y * \beta_1, \ldots) = f(x * \alpha_2, y * \beta_2, \ldots) \times \frac{f(\alpha_1, \beta_1, \ldots)}{f(\alpha_2, \beta_2, \ldots)}\ . \tag{A.2}$$

Differentiate (A.2) partially with respect to x. Then we get

$$\alpha_1 \frac{\partial f(x * \alpha_1, y * \beta_1, \ldots)}{\partial(x * \alpha_1)} = \alpha_2 \frac{\partial f(x * \alpha_2, y * \beta_2, \ldots)}{\partial(x * \alpha_2)} \times \frac{f(\alpha_1, \beta_1, \ldots)}{f(\alpha_2, \beta_2, \ldots)}\ . \tag{A.3}$$

Setting $x, y, \ldots$ all equal to 1. This leads to:

$$\alpha_1 \frac{\partial f(\alpha_1, \beta_1, \ldots)/\partial \alpha_1}{f(\alpha_1, \beta_1, \ldots)} = \alpha_2 \frac{\partial f(\alpha_2, \beta_2, \ldots)/\partial \alpha_2}{f(\alpha_2, \beta_2, \ldots)}\ . \tag{A.4}$$

Holding $\alpha_2, \beta_2, \gamma_2, \ldots$ to be constant and varying $\alpha_1, \beta_1, \gamma_1, \ldots$, we get an equation of the form:

$$\frac{\alpha}{f} \frac{\partial f}{\partial \alpha} = constant \tag{A.5}$$

or

$$\frac{1}{f} \frac{\partial f}{\partial \alpha} = \frac{constant}{\alpha}\ . \tag{A.6}$$

A solution to this equation is:

$$f = C_1 \alpha^{constant}\ , \tag{A.7}$$

where C_1 is a function of $\beta, \ldots$.

This process can be repeated to yield the desired form for function f:

$$f = C\alpha^a \beta^b \cdots . \qquad \square \tag{A.8}$$

Based on the product theorem we conclude that dimensional representations are *multiplicative* or in product form. Thus, if the primary quantities have the dimensions $[L]$, $[T]$, . . . , then the dimensional representation of the secondary quantity, as given by the function f is:

$$[L]^a[T]^b \cdots .$$

A.2. Buckingham's Π-theorem

If $\phi(\alpha, \beta, \ldots) = 0$ is a complete equation, then the solution can be written in the form $F(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) = 0$, where n is the number of arguments of ϕ, and r is the basic number of dimensions needed to express the variables $\alpha, \beta, \ldots$; for all i, Π_i is a dimensionless number [3, 4, 26].

Proof. (A note of caution for the reader—the variables $\alpha, \beta, \ldots$ are *not primary* quantities as in the product theorem. In fact they are the quantities that pertain to the situation modeled by the function ϕ).

We shall now establish the notation to be used in the proof. The quantities $\alpha, \beta, \ldots$ contain r dimensions in their dimensional representation. For ease of understanding we will assume that these dimensions are $[L]$, $[T]$, . . . (r in all). For the quantity α the exponents in the dimensional representation are $\alpha_1, \alpha_2, \ldots$, i.e. the dimensional representation of α is $[L]^{\alpha_1}[T]^{\alpha_2} \cdots$. A similar notation will be used for β and all the other quantities.

Corresponding to each of the r dimensions are units; the units are $m_1, m_2, \ldots$. We will be modifying the units by scale factors; the scale factors for the units $m_1, m_2, \ldots$ are $x_1, x_2, \ldots$.

Decreasing the units $m_1, m_2, \ldots$ by the scale factors $x_1, x_2, \ldots$ will, according to the product theorem lead to the *primed* measurements as shown below:

$$\alpha' = \alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots , \qquad \beta' = \beta x_1^{\beta_1} x_2^{\beta_2} \cdots , \qquad \ldots .$$

Since, $\phi(\alpha, \beta, \ldots) = 0$ is a complete equation, it must hold independent of the units chosen to measure $\alpha, \beta, \ldots$; hence it follows that

$$\phi(\alpha', \beta', \ldots) = 0 \tag{A.9}$$

or

$$\phi(\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots , \beta x_1^{\beta_1} x_2^{\beta_2} \cdots , \ldots) = 0 . \tag{A.10}$$

Differentiating (A.9) with respect to x_1 and setting all the x_i to 1 we get:

$$\alpha_1 \alpha \frac{\partial \phi}{\partial \alpha} + \beta_1 \beta \frac{\partial \phi}{\partial \beta} + \cdots = 0 . \tag{A.11}$$

Now introduce new independent variables:

$$\alpha'' = \alpha^{1/\alpha_1}\,, \qquad \beta'' = \beta^{1/\beta_1}\,, \quad \ldots .$$

In terms of these new variables, (A.11) can be rewritten as

$$\alpha'' \frac{\partial \phi}{\partial \alpha''} + \beta'' \frac{\partial \phi}{\partial \beta''} + \cdots = 0\,. \tag{A.12}$$

Let ζ'' be the last of the n variables α'', β'', ... Now introduce a set of variables z_i such that:

$$z_1 = \alpha''/\zeta''\,, \qquad z_2 = \beta''/\zeta''\,, \quad \ldots, \quad z_n = 1\,.$$

Now we substitute the z_i into the basic function ϕ. We then have:

$$\phi(\alpha'', \beta'', \ldots, \zeta'') \equiv \phi(z_1\zeta'', z_2\zeta'', \ldots, \zeta'')\,. \tag{A.13}$$

It can be shown that function ϕ is independent of ζ''. Differentiating ϕ partially with respect to ζ'' and using (A.12), we can obtain $\partial\phi/\partial\zeta'' = 0$. So ϕ is a function of $n-1$ variables z and we can write an equivalent function:

$$\Psi(z_1, z_2, \ldots, z_{n-1}) = 0\,. \tag{A.14}$$

Note that the arguments, z_i, are dimensionless in the first dimension. Now starting with (A.14) the process can be repeated for each of the remaining $r-1$ dimensions. Each time a dimension is eliminated, the number of arguments in the function is reduced by 1. Thus when all the r dimensions have been eliminated, we will be left with a function of the form:

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) = 0\,. \qquad \square$$

ACKNOWLEDGEMENT

We are grateful to Mahzarin Banaji, Doug Cooper, Dick Hayes, Se June Hong, Sesh Murthy, Asha Radhakrishnan and Wlodek Zadrozny as well as to Ben Kuipers and an anonymous referee for comments on previous drafts.

REFERENCES

1. D.G. Bobrow, Retrospectives: A note from the editor, *Artificial Intelligence* **23** (1984) 247.
2. D.G. Bobrow, Qualitative reasoning about physical systems: An introduction, *Artificial Intelligence* **24** (1984) 1–5 (Special Volume).
3. P.W. Bridgman, *Dimensional Analysis* (Yale University Press, New Haven, CT, 1922).
4. E. Buckingham, On physically similar systems; Illustrations of the use of dimensional equations, *Phys. Rev.* **IV** (4) (1914) 345–376.

5. E. Buckingham, The principle of similitude, *Nature* **96** (2406) (1915) 396–397.
6. A.B. Cambel, *Plasma Physics and Magnetofluidmechanics* (McGraw-Hill, New York, 1963).
7. S. Chandrasekhar, *Hydrodynamics and Hydromagnetic Stability* (Clarendon Press, Oxford, 1963).
8. J.W. Collins and K.D. Forbus, Reasoning about fluids via molecular collections, in: *Proceedings AAAI-87*, Seattle, WA (1987) 590–594.
9. J. de Kleer and J.S. Brown, A qualitative physics based on confluences, *Artificial Intelligence* **24** (1984) 7–83.
10. J. de Kleer and J.S. Brown, Theories of causal ordering, *Artificial Intelligence* **29** (1986) 33–61.
11. J. de Kleer and B.C. Williams, Diagnosing multiple faults, *Artificial Intelligence* **32** (1987) 97–130.
12. B. Ellis, *Basic Concepts of Measurement* (Cambridge University Press, Cambridge, England, 1969).
13. R.P. Feynmann, R.B. Leighton and M. Sands, *The Feynmann Lectures on Physics* (Addison-Wesley, Reading, MA, 1963).
14 K.D. Forbus, Qualitative process theory, *Artificial Intelligence* **24** (1984) 85–168.
15. M. Hall, *Combinatorial Theory* (Blaisdell, Waltham, MA, 1967).
16. P. Hall, On representative of subsets, *J. London Math. Soc.* **10** (1935) 26–30.
17. P.J. Hayes, The second naive physics manifesto, in: J. Hobbs, *Formal Theories of the Commonsense World* (Ablex, Norwood, NJ, 1985).
18. P.J. Hayes, Naive physics I: Ontology for liquids, in: J. Hobbs, ed., *Formal Theories of the Commonsense World* (Ablex, Norwood, NJ, 1985).
19. Y. Iwasaki and H.A. Simon, Causality in device behavior, *Artificial Intelligence* **29** (1986) 3–32.
20. M.M. Kokar, Discovering functional formulas through changing representation base, in: *Proceedings AAAI-86*, Philadelphia, PA (1986) 455–459.
21. M.M. Kokar, Determining arguments of invariant functional descriptions, *Mach. Learning* **1** (1986) 403–422.
22. M.M. Kokar, Critical hypersurfaces and the quantity space, in: *Proceedings AAAI-87*, Seattle, WA (1987) 616–620.
23. M.M. Kokar, Generating qualitative representations of continuous physical processes, in: Z.W. Ras and M. Zemankova, eds., *Methodologies for Intelligent Systems* (North-Holland, New York, 1987).
24. B.J. Kuipers, Qualitative simulation, *Artificial Intelligence* **29** (1986) 289–338.
25. I. Lakatos, *Proofs and Refutations* (Cambridge University Press, Cambridge, England, 1976).
26. H.L. Langhaar, *Dimensional Analysis and Theory of Models* (Wiley, New York, 1951).
27. C. Misner, K.S. Thorne and J. Wheeler, *Gravitation* (Freeman, San Francisco, CA, 1973).
28. S.S. Murthy, Qualitative reasoning at multiple resolutions, in: *Proceedings AAAI-88*, St. Paul, MN (1988).
29. A. Newell, Heuristic programming: Ill-structured problems, in: J. Aronofsky, ed., *Progress in Operations Research* III (Wiley, New York, 1969) 360–414.
30. O. Raiman, Order of magnitude reasoning, in: *Proceedings AAAI-86*, Philadelphia, PA (1986).
31. Rayleigh, The principle of similitude, *Nature* **95** (2368) (1915) 66–68.
32. Rayleigh, The principle of similitude, *Nature* **95** (2389) (1915) 644.
33. W.C. Reynolds and H.C. Perkins, *Engineering Thermodynamics* (McGraw-Hill, New York, 1970).
34. D. Riabouchinsky, The principle of similitude, *Nature* **95** (2387) (1915) 591.
35. H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979).
36. M. Schwarzschild, *The Structure and Evolution of the Stars* (Dover, New York, 1958).

37. L.I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic Press, New York, 1959); Translation from the Russian Eighth Edition by M. Holt.
38. I.R. Shames, *Mechanics of Fluids* (McGraw-Hill, New York, 1962).
39. H.A. Simon, Some trivial but useful mathematics, CIP Working Paper Series No. 459, Department of Psychology, Carnegie-Mellon University, Pittsburgh, PA (1988).
40. H.A. Simon, *Models of Man* (Wiley, New York, 1957).
41. V.L. Streeter and E.B. Wylie, *Fluid Mechanics* (McGraw-Hill, New York, 1975).
42. D.S. Weld, Theories of comparative analysis, Ph.D. Thesis, MIT Artificial Intelligence Laboratory, Cambridge, MA (1988).
43. B.C. Williams, Qualitative analysis of MOS circuits, *Artificial Intelligence* **24** (1984) 281–346.
44. B.C. Williams, The use of continuity in a qualitative physics, in: *Proceedings AAAI-84*, Austin, TX (1984) 350–354.

Received October 1988; revised version received September 1989