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ALL POSSIBLE GENERATORS OF SUPERSYMMETRIES OF THE *S*-MATRIX

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A generator of a symmetry or supersymmetry of the *S*-matrix has to have three simple properties (see sect. 2). Starting from these properties one can give a complete analysis of the possible structure of the pseudo Lie algebra of these generators. In a theory with non-vanishing masses one finds that the only extension of previously known relations is the possible appearance of "central charges" as anticommutators of Fermi charges. In the massless case (disregarding infrared problems and symmetry breaking) the Fermi charges may generate the conformal group together with a unitary internal symmetry group.

1. Introduction

We have chosen the title to indicate the close relationship of this study to ref. [1]. The results of the latter paper were generally accepted as the last and most powerful in a series of "no-go" theorems, destroying the hope for a fusion between internal symmetries and the Poincare group by a relativistic generalization of SU(6). Recently Wess and Zumino discovered field theoretical models with an unusual type of symmetry (originally called "supergauge symmetry" and now "supersymmetry") which connects Bose and Fermi fields and is generated by charges transforming like spinors under the Lorentz group [2]. These spinorial charges give rise to a closed system of commutation-anticommutation relations, which may be called

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a "pseudo Lie algebra"*. It turns out that the energy-momentum operators appear among the elements of this pseudo Lie algebra, so that in some sense a fusion between internal and geometric symmetries occurs [2,4].

The possibility of supersymmetries was not envisaged in [1]** but most of the ideas in [1] apply also to this case. We shall use them to determine all supersymmetry structures which are allowed in a theory without zero-mass particles and long-range forces.

An equally satisfactory and complete discussion of the zero-mass situation is not attempted here. However, the simplest case in which there is no infrared problem and no degeneracy of the vacuum will be treated in sect. 5. The allowed supersymmetry structure is then much more interesting than in the massive case, because it gives a complete fusion between internal and geometric symmetries which is furthermore essentially unique.

In assessing the results, one should bear in mind that the scope of this investigation is limited in three directions: First, we deal only with visible symmetries, i.e., with symmetries of the S-matrix; the fundamental equations may have a higher symmetry. Second, all impossibility statements below have to be reexamined when infrared problems or vacuum degeneracy occur. Third, we assume that each mass multiplet contains only a finite number of different types of particles. This again is eminently reasonable in the massive case, but there is one interesting alternative, namely to use the supersymmetry structure in the context of an idealization which assigns zero mass to all particles. If the total number of particle types is infinite, then this idealization is not covered here. With these limitations in scope being understood, the conclusions of our analysis may be summarized as follows: the most general pseudo Lie algebra of generators of supersymmetries and ordinary symmetries of the S-matrix in a massive theory involves the following Bose type operators: the energy-momentum operators P_μ; the generators of the homogeneous Lorentz group $M_{\mu\nu}$; and a finite number of scalar charges B_l. It will involve Fermi-type operators, all of which commute with the translations and transform like spinors of rank 1 under the homogeneous Lorentz group. Using the spinor notation of van der Waerden‡, we may divide them into a set Q^L_α ($L = 1, ..., \nu$; $\alpha = 1, 2$) and a set $\bar{Q}^L_{\dot\alpha}$, indicating the different transformation character by dotted or undotted indices. Since the Hermitian conjugate of a supersymmetry generator is again such a generator, we can choose the basis of the pseudo Lie algebra so that $\bar{Q}^L_{\dot\alpha} = (Q^L_\alpha)^\dagger$,

* The spinorial charges may be considered as generators of a continuous group whose parameters are elements of a Grassmann algebra [3].

** Spinorial charges were considered but prematurely discarded in [5].

‡ $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$; $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$; $\epsilon^{12} = \epsilon_{21} = 1$ (same for dotted indices). $(\sigma^\mu)_{\alpha\dot\beta} = (1, \sigma_i)$; $(\bar\sigma^\mu)^{\dot\alpha\beta} = (1, -\sigma_i)$. $\sigma^{\mu\nu} = \frac{1}{2}i(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$ and $\bar\sigma^{\mu\nu} = \frac{1}{2}i(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)$. A quick orientation about this formalism may be obtained from [6]. The conventions adopted there are, however, slightly different from ours.

which is equivalent to a Majorana condition in a four-spinor notation.

The algebra of these quantities can be reduced to the following form*:

$$\{Q_\alpha^L, Q_\beta^M\} = \epsilon_{\alpha\beta} \sum_l (a^l)^{LM} B_l \equiv \epsilon_{\alpha\beta} Z^{LM} , \tag{1.1}$$

where

$$[Z^{LM}, G] = 0 \tag{1.2}$$

for all G in the pseudo Lie algebra,

$$\{Q_\alpha^L, \bar{Q}_{\dot\beta}^M\} = \delta^{LM} \sigma^\mu{}_{\alpha\dot\beta} P_\mu , \tag{1.3}$$

$$[Q_\alpha^L, B_l] = \sum_M s_l^{LM} Q_\alpha^M , \tag{1.4}$$

$$[B_l, B_m] = i \sum_k c_{lm}^k B_k , \tag{1.5}$$

together with the Poincaré transformation properties of spinorial and scalar charges:

$$[Q_\alpha^L, P_\mu] = [B_l, P_\mu] = [B_l, M_{\mu\nu}] = 0 , \tag{1.6}$$

$$[Q_\alpha^L, M_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^L . \tag{1.7}$$

The c_{lm}^k in (1.5) are the structure coefficients of a compact Lie group, the s_l in (1.4) are the (Hermitian) representation matrices of the generators of this group in a ν-dimensional representation, the matrices a^l in (1.1) are restricted by a condition (to be discussed later) which connects them to the s_l (eq. (3.17)) and by eq. (1.2).

The fact that in a massive theory the only symmetry generators of Bose type which do not commute with the whole Poincaré group, are the vector P^μ and the skew tensor $M^{\mu\nu}$, has been demonstrated from a variety of angles in the past (see [1] and [5] and the literature quoted there). The counterpart of this fact for Fermi type generators is our result that all of them commute with the translations and transform like spinors of rank 1 under the Lorentz group. Furthermore, the structure relations of the pseudo Lie algebra are severely restricted. In fact, apart from (1.1), all the structural relations which are allowed have already been previously found: relation (1.3) in [2,4], relation (1.4) in [7,8]. So there is little additional freedom in the supersymmetry scheme, unless one incorporates zero-mass particles. In particular, it is not possible in a massive theory to obtain the angular momentum or non-central charges (see eq. (1.2)) from anticommutators of supersymmetry transformations.

* All scalar charges can be taken as Hermitian. This is done, and we omit writing down the relations which can be obtained by taking Hermitian conjugates.

In the case of a massless theory, we may have the following alternative to the above described scheme: The Fermi part of the pseudo Lia algebra consists of two sets of spinors of rank 1, denoted by Q_α^L and $Q_\alpha^{(1)L}$ ($L = 1, ..., \nu$) and their conjugates $\bar{Q}_{\dot\alpha}^L$, $\bar{Q}_{\dot\alpha}^{(1)L}$. The anticommutators of these Fermi charges give a set of Bose symmetries which form the Lie algebra of $\mathcal{C} \otimes \mathcal{G}$. Here $\mathcal{C}$ is the conformal group and $\mathcal{G} = \mathrm{U}(\nu)$ for $\nu \neq 4$ respectively SU(4) for $\nu = 4$. The full algebra of internal Bose symmetries is the Lie algebra of $\mathcal{G}' \otimes \mathcal{G}''$ where $\mathcal{G}''$ commutes with all Fermi charges and $\mathcal{G}'$ is $\mathrm{U}(\nu)$ for $\nu \neq 4$ and either U(4) or SU(4) for $\nu = 4$. An example of this structure for $\nu = 1$ was given in [2], compare also [13], for $\nu = 2$ in [8]. The interesting fact is now, that it is impossible to have both Q^L and $Q^{(1)L}$ without getting the full conformal group together with an internal symmetry group, and that the latter must be precisely the unitary group $\mathrm{U}(\nu)$, except for $\nu = 4$ where it may be either U(4) or SU(4).

2. Assumptions and basic facts

A generator of a symmetry or supersymmetry of S is any operator G in the Hilbert space of physical states which has two properties: (i) it commutes with the S-matrix; (ii) it acts additively on the states of several incoming particles. The second of these requirements can be most conveniently expressed in the following way:

Let $a_{i,r}^{*(\mathrm{in})}(\boldsymbol{p})$ denote the creation operator of an incoming particle of type i with momentum $\boldsymbol{p}$ and spin orientation r and $a_{i,r}^{(\mathrm{in})}(\boldsymbol{p})$ the corresponding destruction operator*. Then

$$G = \sum_{\substack{i,j \\ r,s}} \int \mathrm{d}^3p\, \mathrm{d}^3p'\, a_{j,s}^{*\mathrm{in}}(\boldsymbol{p}')\, K_{js;ir}(\boldsymbol{p}', \boldsymbol{p})\, a_{i,r}^{\mathrm{in}}(\boldsymbol{p})\,, \tag{2.2}$$

where K is a c-number kernel. If in the sum over particle types only pairs (i, j) occur which refer to particles of equal statistics (both Bose or Fermi), G is of Bose type and generates an ordinary symmetry; if only pairs (i, j) with opposite statistics occur, G is of Fermi type and generates a supersymmetry. Since the S-matrix conserves statistics, both the Bose part and the Fermi part separately have the property (i), so that we can always take G to be either of pure Bose or of pure Fermi type.

The justification of the requirements (i) and (ii) can be given in a variety of ways. In the frame of a local field theory, the more fundamental requirement would be that G induces an infinitesimal transformation of the basic field quantities $\psi \to \psi + \epsilon\, \delta_G \psi$ such that**

* We use the canonical normalization

$$[a_{i,r}^{(\mathrm{in})}(\boldsymbol{p}), a_{j,s}^{*(\mathrm{in})}(\boldsymbol{q})] = \delta_{ij}\, \delta_{rs}\, \delta^3(\boldsymbol{p} - \boldsymbol{q})\,. \tag{2.1}$$

** In (2.3) we denote the anticommutator by $[...]_+$, while elsewhere curly brackets $\{...\}$ are used.

$$\delta_G(\psi(x)) = i\,[G, \psi(x)]_\pm\,, \tag{2.3}$$

is again local. On the right-hand side of (2.3) the anticommutator occurs when both G and ψ are of Fermi type, the commutator in all other cases. The usual way to construct such a G is to start from a conserved local current,

$$\partial_\mu j^\mu(x) = 0\,, \tag{2.4}$$

and define

$$G = \int_{x_0 = t} \mathrm{d}^3x\, j^0(x)\,. \tag{2.5}$$

Due to (2.4) this yields the same operator for an arbitrary choice of the time t, and hence $\delta_G(\psi(x))$ is local for arbitrary x. Note that no assumptions about the covariance of $j^\mu(x)$ under the Poincare group have to be made at this point, i.e., $j^\mu(x)$ need not be a four-vector field, and it may depend not only on the basic fields at the point x but also explicitly on the position coordinates*.

In a massive theory where the assymptotic relations as described in [9] hold, the locality of $\delta_G\psi(x)$ implies that G has the properties (i) and (ii) (compare [10–12, 5]). In fact it also implies a third property:

(iii) G connects only particle types (i, j) which have the same mass. The kernel K in (2.2) is of the form

$$K(p', p) = \sum_n K^{(n)}(p)\,\partial^n\,\delta(p - p')\,, \tag{2.6}$$

where ∂^n stands for a monomial in the derivatives $\partial/\partial p_i$ and the sum has a finite number of terms.

It is interesting to note that property (iii) follows on the one hand from the locality of $\delta_G\psi$ as indicated in [10–12, 5]**, and that it can also be derived from the properties (i) and (ii) and the assumption that the S-matrix is not trivial. This was done in [1].

We now look first at those generators which commute with the translations. Let us call the set of these $\mathcal{S}^{(0)}$; for $G \in \mathcal{S}^{(0)}$ the kernel in (2.2) is of the form

$$K_{js;ir}(p, p') = K_{js;ir}(p)\,\delta^3(p - p')\,, \tag{2.7}$$

and the matrix K does not couple particles with different mass. We shall consider the submatrix for one mass multiplet at a time. This will be meant when we speak of the matrix $K(p)$ below.

* In that case $\partial_\mu j$ has to be distinguished from $i\,[P_\mu, j]$. This case is important in order to obtain also those symmetry generators which do not commute with P^μ.

** The quoted discussions are not completely adequate for our purpose here, but can be adapted.

For the generators of Bose type we can take over the analysis in [1] completely. If one assumes particle finiteness and non-triviality of S, one finds:

(A) a complete basis of Bose type operators in $\mathcal{S}^{(0)}$ is given by the energy-momentum operators P^μ and a finite number of scalar charges B_l, having the commutation relations (1.5), (1.6). The Lie algebra $\mathcal{L}$ of the B_l may be decomposed into a semi-simple part $\mathcal{L}_1$ with positive definite Cartan metric and an Abelian part $\mathcal{L}_2$ whose elements commute also with those of $\mathcal{L}_1$.

For the generators of Fermi type we can take over lemma 5 from [1] in the form:

(B) if $K(\boldsymbol{p})$ vanishes for two momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$ on a mass hyperboloid, then $K(\boldsymbol{p})$ vanishes for all momenta on this hyperboloid*.

3. Classification of translation invariant Fermi type generators

The conclusion (B) above, together with particle finiteness, tells us that there are only a finite number of linearly independent Fermi type generators in $\mathcal{S}^{(0)}$ since the matrix $K(p)$ has finite dimensionality, and according to (B), G is fixed by specifying two such matrices. Furthermore, the set $\mathcal{S}^{(0)}$ is stable under homogeneous Lorentz transformations.

Therefore this set is a representation space of a finite dimensional representation of the Lorentz group and can be decomposed into irreducibles. An irreducible representation of the connected component of the Lorentz group is labeled by a pair of indices (j, j'), each integer or half-integer; the corresponding subset of generators consists of linear combinations of spinors $Q_{\alpha_1 \dots \alpha_{2j}; \dot\beta_1 \dots \dot\beta_{2j'}}$, symmetric in the $2j$ undotted and the $2j'$ dotted indices. Due to the spin-statistics theorem, $2(j+j')$ must be odd for Fermi-type generators. Now consider the anticommutator of such a Q with its Hermitian conjugate $Q^\dagger$. This will again be an element of $\mathcal{S}^{(0)}$, because the properties (i), (ii) and (iii), as well as translational invariance are conserved**, and it will be of Bose type. If we take all indices equal, say $\alpha_1 = \alpha_2 = \dots = \dot\beta_1 = \dots = \dot\beta_{2j'} = 1$, then the anticommutator will be a component $B_{11..1;\dot 1 \dots \dot 1}$ of a spinor with $2j + 2j'$ undotted and equally many dotted indices, which is furthermore symmetric in each type of index. Hence it belongs to the irreducible representation $(j+j', j+j')$. But we know that there are no Bose type generators which belong to a representation (j, j) with $j > \frac{1}{2}$. Thus the mentioned anticommutator must vanish unless the original Q was a spinor $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. However, $QQ^\dagger + Q^\dagger Q = 0$ implies $Q = 0$. Hence no spinors of rank higher than 1 are allowed and for a spinor Q_α of rank 1 [representation $(\frac{1}{2}, 0)$] we must get:

* Actually $\boldsymbol{p}, \boldsymbol{p}'$ should be so chosen that the elastic scattering does not vanish for a pair of particles with these momenta.

** Note that the property (ii) is conserved for the anticommutator (not for the commutator) of two Fermi-type generators. This is the reason why the last part of the analysis in [1] is not applicable to the Fermi-type generators.

$$\{Q_\alpha, (Q_\beta)^\dagger\} = c\, P_{\alpha\dot\beta} = c\, \sigma^\mu{}_{\alpha\dot\beta} P_\mu \,,$$

since $(\frac{1}{2}, \frac{1}{2})$ is the spinorial description of a four-vector and the only four-vectors in $\mathcal{S}^{(0)}$ are multiples of P^μ. If $Q_\alpha \neq 0$ then $c \neq 0$. If we have several spinorial charges, say Q^L_α, we must have [remember $\bar{Q}^L_{\dot\alpha} = (Q^L_\alpha)^\dagger$]:

$$\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = c^{LM} \sigma^\mu{}_{\alpha\dot\beta} P_\mu \,, \tag{3.1}$$

where c^{LM} is a positive definite Hermitian numerical matrix, which can be brought into the form

$$c^{LM} = \delta^{LM} \,, \tag{3.2}$$

by suitable choice of the Q^L (diagonalization of c^{LM} and normalization). Consider now $\{Q^L_\alpha, Q^M_\beta\}$. It also belongs to $\mathcal{S}^{(0)}$. The antisymmetric part in the indices α, β is a scalar, which could be a linear combination of the internal symmetry generators B_l. The symmetrical part belongs to the representation (1,0) and corresponds to a self-dual skew-symmetric tensor. Since there is no such thing in $\mathcal{S}^{(0)}$, the symmetric part must vanish and we can only have

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} \sum_l (a^l)^{LM} B_l \,, \tag{3.3}$$

with

$$(a^l)^{LM} = -(a^l)^{ML} \,. \tag{3.4}$$

Finally, the commutators $[B_l, Q^L_\alpha]$ belong to the $(\frac{1}{2}, 0)$ part of $\mathcal{S}^{(0)}$ and therefore:

$$[Q^L_\alpha, B_l] = \sum_M s_l^{LM} Q^M_\alpha \,. \tag{3.5}$$

Relations (3.1)–(3.5), together with the structure relations of the ordinary internal symmetries

$$[B_l, B_m] = i \sum_k c^k_{lm} B_k \,, \tag{3.6}$$

and the fact that P_μ commutes with all B_l and Q_α, give the structure of the part of our pseudoLie algebra which lies in $\mathcal{S}^{(0)}$.

The matrices a^l, s_l are still restricted by the Jacobi identities*. The one involving

* If B denotes a Bose and F a Fermi operator, the relevant identities are:

$$[B, \{F_1, F_2\}] + \{F_1, [F_2, B]\} - \{F_2, [B, F_1]\} = 0 \,, \tag{3.7}$$

$$[F_1, \{F_2, F_3\}] + [F_2, \{F_3, F_1\}] + [F_3, \{F_1, F_2\}] = 0 \,. \tag{3.8}$$

B_l, Q_α^L, $(Q_\beta^M)^\dagger$ tells us that

$$s_l^{LM} = \bar{s}_l^{ML} , \tag{3.9}$$

i.e., that s_l is Hermitian. The Jacobi identity involving B_l, B_m, Q_α^L tells us that the matrices s_l form a representation of the Lie algebra of the B_l. The remaining three identities are

$$(Q, Q, Q): \qquad \sum_l (a^l)^{KL} s_l^{NM} = \sum_l (a^l)^{LN} s_l^{KM} , \tag{3.10}$$

$$(Q, Q, Q^\dagger): \qquad (a^l)^{KL} \bar{s}_l^{MN} = 0 , \tag{3.11}$$

$$(Q, Q, B): \qquad (s_m a^l)^{KL} - (s_m a^l)^{LK} = i \sum_n c_{nm}^l (a^n)^{KL} . \tag{3.12}$$

Eq. (3.10) is also a consequence of (3.11) with (3.9). For the analysis of the remaining relations , we remember, that $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ where $\mathcal{L}_1$ is semisimple, $\mathcal{L}_2$ Abelian. Consider the elements

$$Z^{KL} = \sum_l (a^l)^{KL} B_l . \tag{3.13}$$

Eqs. (3.6) and (3.12) tell us that the linear span of the Z^{KL} is an invariant subalgebra, say $\mathcal{L}_3 \subset \mathcal{L}$. Using in addition (3.11), we see that

$$[Z^{KL}, Z^{MN}] = 0 . \tag{3.14}$$

Thus the intersection of $\mathcal{L}_1$ and $\mathcal{L}_3$ would have to be an Abelian, invariant subalgebra of $\mathcal{L}_1$ which does not exist because $\mathcal{L}_1$ is semisimple. Thus $\mathcal{L}_3 \subset \mathcal{L}_2$, i.e.,

$$[Z^{KL}, B_m] = 0 . \tag{3.15}$$

Finally, eq. (3.11) tells us that Z^{KL} lies in the kernel of the representation s or, in other words, by (3.5), that

$$[Z^{KL}, Q_\alpha^M] = 0 . \tag{3.16}$$

Taking (3.15) and (3.16) together, we see that $\mathcal{L}_3$ must be a part of the center $\mathcal{Z}$ of the whole pseudo Lie algebra. This result has been stated above as eq. (1.2). We may write

$$\mathcal{L} = \mathcal{L}' \oplus \mathcal{Z},$$

where $\mathcal{L}'$ does not contain any central elements. We choose a basis Z_ρ in $\mathcal{Z}$, and we can replace the right-hand side of (1.1) or (3.3) by $\Sigma_\rho\, \epsilon_{\alpha\beta} (a^\rho)^{LM} Z_\rho$.

The matrices a^ρ are still restricted by (3.12): putting there $l = \rho$ and m arbitrary, the right-hand side of (3.12) is zero, because the structure constant then vanishes. Using (3.4), we can now rewrite (3.12) in the form

$$s_m a^\rho = a^\rho t_m \,, \tag{3.17}$$

where

$$t_m = -\bar{s}_m \tag{3.18}$$

is the complex conjugate representation of the Lie algebra. This means, that every matrix a^ρ must be an intertwiner of the representations s and t of $\mathcal{L}'$. This limits the number of central charges which can appear in (1.1) and (3.3). To give an example: Suppose that $\mathcal{L}'$ is the Lie algebra of SU(2) and s its basic two-dimensional representation, so that we have two spinors Q^L_α ($L = 1, 2$). Then the complex conjugate representation is unitarily equivalent to s and there is only one linearly independent intertwiner between s and t, namely a complex multiple of the matrix ϵ^{LM}. Thus there can be at most two real central charges Z_1, Z_2 in the pseudo Lie algebra which are not completely trivial, and (1.1) becomes for this example

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta}\, \epsilon^{LM}(c_1 Z_1 + i c_2 Z_2) \,.$$

4. Symmetry generators which do not commute with P_μ

Let us call $\mathcal{S}^{(N)}$ the set of those symmetry generators for which the kernel (2.6) contains only derivatives up to (and including) the order N. An operator which belongs to $\mathcal{S}^{(N)}$ but not to $\mathcal{S}^{(N-1)}$ will be called a symmetry of degree N*.

The analysis proceeds now from the following observations:

(a) All symmetries commute with P^2 (sect. 2, property (iii))**.

(b) If $G \in \mathcal{S}^{(N)}$ then $[P_\mu, G] \in \mathcal{S}^{(N-1)}$.

This is, because the commutation with P_μ conserves the three properties (i), (ii), (iii) and the order of the derivatives in (2.6) (or, alternatively, the degree of the polynomial of x in the current density) is lowered. In fact, the degree of a symmetry is lowered in all cases precisely by one.

* Such operators arise typically if in (2.5)

$$j_0(x) = \Sigma \Pi (x_\nu)^{n_\nu}\, \zeta_{n_0, n_1 n_2 n_3}(x) \,,$$

where ζ_{n_ν} are Poincaré covariant fields and $N = \sup \Sigma\, n_\nu$.

** This holds for the restriction of the symmetry operators to the single particle subspace [i.e. for the kernel in (2.2)], which we need to consider only and which we also denote by G. Furthermore, in the zero-mass case, the discussion refers to a multiplet of particles all of which are massless.

(c) $\mathcal{S}^{(N)}$ contains a finite number of linearly independent elements because G is fixed up to addition of elements from $\mathcal{S}^{(0)}$ by $[P_\mu, G]$.

$\mathcal{S}^{(N)}$ is also closed under Lorentz transformations. We can then classify the elements of $\mathcal{S}^{(N)}$ according to their Lorentz transformation character (j, j').

4.1. Bose symmetries of degree 1

Here $[P_\mu, G] \in \mathcal{S}^{(0)}$. Since all Bose symmetries in $\mathcal{S}^{(0)}$ belong either to $(\frac{1}{2}, \frac{1}{2})$ or $(0, 0)$, G can belong only to $(\frac{1}{2}, \frac{1}{2})$, $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$. The last four can be combined into a general second rank tensor $T_{\mu\nu}$. The first is a vector V_μ. Then we must have

$$[P_\mu, V_\nu] = g_{\mu\nu} \sum_l c_l B_l \,,$$

$$[P_\rho, T_{\mu\nu}] = icP_\rho g_{\mu\nu} + ib^+(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu + i\epsilon_{\rho\sigma\mu\nu}P^\sigma)$$
$$+ ib^-(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu - i\epsilon_{\rho\sigma\mu\nu}P^\sigma) + ia(g_{\rho\mu}P_\nu + g_{\rho\nu}P_\mu - \tfrac{1}{2}g_{\mu\nu}P_\rho) \,, \quad (4.1)$$

where the four terms on the right-hand side correspond respectively to the parts $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ of $T_{\mu\nu}$ and c_l, c, b^+, b^-, a are numerical constants. The commutativity of G with P^2 gives

$$0 = P_\mu \sum_l c_l B_l \,, \qquad 0 = ia(2P_\mu P_\nu - \tfrac{1}{2}g_{\mu\nu}P^2) \,, \qquad 0 = icg_{\mu\nu}P^2 \,.$$

Thus

$$c_l = 0 \,, \qquad a = 0 \,, \quad \text{and for} \quad m^2 \neq 0 \quad \text{also} \quad c = 0 \,.$$

The second and third terms in (4.1) agree up to the numerical factors with the commutators between P_ρ and the $(1, 0)$ respectively $(0, 1)$-parts of the angular momenta $M_{\mu\nu}$. Since $\mathcal{S}^{(0)}$ does not contain any skew tensor, the irreducible parts of $T_{\mu\nu} - T_{\nu\mu}$ are multiples of those of $M_{\mu\nu}$.

Thus we have the result

(A) In the massive case all Bose symmetry generators of degree 1 are linear combinations of $M_{\mu\nu}$.

(B) For zero-mass one may have in addition one scalar element $D \in \mathcal{S}^{(1)}$ with commutation relations

$$[P_\mu, D] = iP_\mu \,. \qquad (4.2)$$

By this D is fixed up to an additive scalar from $\mathcal{S}^{(0)}$.

4.2. Bose symmetries of degree 2

Since the Bose part of $\mathcal{S}^{(1)}$ contains at most the covariants $(0, 0)$, $(1, 0)$, $(0, 1)$, the Bose part of $\mathcal{S}^{(2)}$ can contain at most $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$.

The case $(\frac{3}{2}, \frac{1}{2})$ may be ruled out as follows (see appendix for definitions of $P_{\alpha\dot{\beta}}, M_{\alpha_1\alpha_2}$ and their commutation relations):

$$[P_{\alpha\dot{\beta}}, A_{\gamma_1\gamma_2\gamma_3;\dot{\gamma}}] = f \sum \epsilon_{\alpha\gamma_1} \epsilon_{\dot{\beta}\dot{\gamma}} M_{\gamma_2\gamma_3} ,$$

(where the sum runs over the permutations of $\gamma_1, \gamma_2, \gamma_3$), is the only possible covariant ansatz. This yields

$$[P_{\alpha_1\dot{\beta}_1}, [P_{\alpha_2\dot{\beta}_2}, A_{\gamma_1\gamma_2\gamma_3;\dot{\gamma}}]] = -2if \sum \epsilon_{\alpha_2\gamma_1} \epsilon_{\alpha_1\gamma_2} \epsilon_{\dot{\beta}_2\dot{\gamma}} P_{\gamma_3\dot{\beta}_1} .$$

Now this quantity should be symmetric under the simultaneous interchange $\alpha_1 \leftrightarrow \alpha_2$, $\dot{\beta}_1 \leftrightarrow \dot{\beta}_2$ (Jacobi identity). Specializing to the choice $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\alpha_1 = \alpha_2 = 2$, $\dot{\gamma}_1 = \dot{\beta}_1 = 1$ and $\dot{\beta}_2 = \dot{2}$, this symmetry requirements becomes

$$-12 i f P_{1\dot{1}} = 0 , \quad \text{i.e.,} \quad f = 0 .$$

A corresponding argument rules out the case $(\frac{1}{2}, \frac{3}{2})$.

For a $(\frac{1}{2}, \frac{1}{2})$ covariant K_μ we have the ansatz

$$[P_\mu, K_\nu] = a g_{\mu\nu} D + b M_{\mu\nu} + c \epsilon_{\mu\nu\kappa\lambda} M^{\kappa\lambda} .$$

We look at the Jacobi identity involving P_μ, P_ν, K_ρ and find with the help of (4.2) that $c = 0$, $b = -a$. The existence of $D \neq 0$ is therefore a necessary condition for the non-vanishing of K_ν.

Result: (C) In the massive case there is no Bose symmetry of degree 2.

(D) For zero-mass, if there is a $D \in \mathcal{S}^{(1)}$ then there may be a $K_\nu \in \mathcal{S}^{(2)}$, whose commutator with P_μ may be normalized to

$$[P_\mu, K_\nu] = 2i(g_{\mu\nu} D - M_{\mu\nu}) . \tag{4.3}$$

By (4.3), K_ν is only fixed up to an additive multiple of P_μ. We shall make use of this freedom below.

4.3. Bose symmetries of degree $N > 2$

By the same technique one finds

(E) No Bose symmetry of degree 3 exists and hence no Bose symmetry of any higher degree.

We only indicate the steps of the argument. A G of degree 3 can only be a component of a general tensor $A_{\mu\nu}$, whose commutator with P_ρ will be of the form (4.1) with K replacing P. Commuting again with P_σ, the result sould be symmetric in ρ, σ (Jacobi identity). Evaluating the expressions, using (4.3), one finds that this requires the vanishing of all coefficients, i.e., $G = 0$.

4.4. Fermi charges of degree 1

Possible covariants are $(\frac{1}{2}, 0)$, $(1, \frac{1}{2})$ and their conjugates. $(1, \frac{1}{2})$ is excluded by the following argument. If $Q^{(1)}_{\gamma_1\gamma_2;\dot\gamma}$ is of degree 1 then

$$[P_{\alpha\dot\beta}, Q^{(1)}_{\gamma_1\gamma_2;\dot\gamma}] = a\,\epsilon_{\dot\beta\dot\gamma}(\epsilon_{\alpha\gamma_1} Q_{\gamma_2} + \epsilon_{\alpha\gamma_2} Q_{\gamma_1}) \ ,$$

where Q on the right-hand side is some (non-vanishing) Fermi charge in $\mathcal{S}^{(0)}$. The condition $[P^2, Q^{(1)}_{\gamma_1\gamma_2;\dot\gamma}] = 0$ gives

$$0 = a(P_{\gamma_1\dot\gamma} Q_{\gamma_2} + P_{\gamma_2\dot\gamma} Q_{\gamma_1}) \ ,$$

and if we anticommute that with $\bar{Q}_{\dot\alpha}$, we get by (3.1)

$$0 = a(P_{\gamma_1\dot\gamma} P_{\gamma_2\dot\alpha} + P_{\gamma_2\dot\gamma} P_{\gamma_1\dot\alpha}) \ .$$

The bracket belongs to the representation (1,1) and is the spinor equivalent of $P_\mu P_\nu - \frac{1}{4} g_{\mu\nu} P^2$ which cannot vanish identically on a mass shell. Hence $a = 0$.

This leaves us to consider Fermi charges of degree 1 belonging to $(\frac{1}{2}, 0)$. Denoting such an element by $Q^{(1)}_\alpha$ we have

$$[P_{\alpha\dot\beta}, Q^{(1)}_\gamma] = i\,\epsilon_{\alpha\gamma} \bar{Q}_{\dot\beta} \ ,$$

with $\bar{Q} \in \mathcal{S}^{(0)}$, $\bar{Q} \neq 0$. From this and (3.1) we compute

$$\epsilon^{\gamma\zeta}\,\epsilon^{\dot\delta\dot\eta}\,[P_{\alpha\dot\beta}, [P_{\gamma\dot\delta}, \{Q^{(1)}_\zeta, \bar{Q}^{(1)}_{\dot\eta}\}]] = c P_{\alpha\dot\beta} \ ,$$

with $c \neq 0$. This means that $\{Q^{(1)}, \bar{Q}^{(1)}\}$ is a Bose symmetry of degree 2 (see observation (b)). Thus if $Q^{(1)}$ exists, then also K_μ must exist. The converse is true also since, as we shall see later, $[K_\mu, Q_\alpha]$ cannot vanish and belongs to $\mathcal{S}^{(1)}$.

Thus: (F) In the massive case there are no Fermi charges of degree 1 or higher.

(G) In the zero-mass case Fermi charges $Q^{(1)}_\alpha$, $\bar{Q}^{(1)}_{\dot\alpha}$ appear if and only if $K_\mu \in \mathcal{S}^{(2)}$ exists.

4.5. Fermi charges of higher degree

By the same technique used in eliminating Bose symmetries of higher degree, we find:

(H) No Fermi charges of degree $N > 1$ exist.

5. Complete algebraic structure

The discussion of the massive case is finished since there the only symmetry generators which are not in $\mathcal{S}^{(0)}$ are the $M_{\mu\nu}$, whose commutation relations with all other quantities are known. The most general structure is then given by eqs. (1.1) through (1.7).

In the massless case, we may distinguish two situations. If there is no K_μ then the situation remains essentially unchanged. The only element which can possibly be added is the dilatation D and we have to supplement (1.1) through (1.7) by the assignment of dimensions to the previous quantities. The algebra with K_μ on the other hand is significantly richer in elements and more restrictive in structure. We shall discuss this in the rest of this section.

The first step is to show that K_μ, P_μ, $M_{\mu\nu}$, D give the structure relations of the conformal group $\mathcal{C}$. Beyond the specified Lorentz transformation properties and the previously obtained relations (4.2) and (4.3), which fix the definition of D, we need for that purpose still the two relations

$$[K_\mu, K_\nu] = 0 \,, \tag{5.1}$$

$$[K_\mu, D] = -iK_\mu \,. \tag{5.2}$$

To obtain them, one may note that on the right-hand side of (5.1) we can only have a linear combination of $M_{\mu\nu}$ and its dual, since there are no other skew tensors in the Lie algebra. The Jacobi identity between P_ρ, K_μ, K_ν shows then that we can use the remaining freedom in the definition of K_μ (addition of a multiple of P_μ) to achieve (5.1), (5.2) and that then K_μ is uniquely fixed.

Next, one sees that all Bose charges B_l commute with the whole conformal group, specifically that

$$[K_\mu, B_l] = 0 \,, \tag{5.3}$$

$$[D, B_l] = 0 \,. \tag{5.4}$$

A general ansatz for the right-hand side of (5.3) would be $c_l K_\mu + c_l' P_\mu$. The Jacobi identity involving P_μ, K_ν, B_l then demands $c_l = 0$ and also gives (5.4). The one involving K_μ, K_ν, B_l gives $c_l' = 0$.

To find the action of K_μ, P_ν, D on the Fermi charges, let us start from the zero-degree charges Q_α^L, $\bar{Q}_{\dot\alpha}^L$ ($L = 1, ..., \nu$) and define charges $Q_\alpha^{(1)L}$, $\bar{Q}_{\dot\alpha}^{(1)L}$ by

$$[K_{\alpha\dot\beta}, Q_\gamma^L] = 2i\epsilon_{\alpha\gamma}\bar{Q}_{\dot\beta}^{(1)L} \,. \tag{5.5}$$

The Hermitian conjugate of this is

$$[K_{\beta\dot\alpha}, \bar{Q}_{\dot\gamma}^L] = 2i\epsilon_{\dot\alpha\dot\gamma}Q_\beta^{(1)L} \tag{5.6}$$

From the Jacobi identity between P_μ, D, Q_α^L one learns that $[Q_\alpha^L, D]$ commutes with P_μ and is therefore of degree zero. So

$$[Q_\alpha^L, D] = \sum_M d^{LM} Q_\alpha^M \,, \tag{5.7}$$

$$[\bar{Q}^L_{\dot\alpha}, D] = -\sum_M \bar{d}^{LM} \bar{Q}^M_{\dot\alpha} . \tag{5.8}$$

The Jacobi identity between $Q, \bar{Q}, D$ in conjuction with (3.1) (3.2), (4.2) gives then

$$d^{LM} - \bar{d}^{ML} = i\delta^{LM} ,$$

or

$$d^{LM} = \tfrac{1}{2} i \delta^{LM} + d'^{LM} ,$$

where d' is Hermitian. The choice of basis in Q-space is so far only limited by the convention (3.2) which still allows a unitary transformation, so that we can diagonalize d' and have

$$d^{LM} = \delta^{LM}(\tfrac{1}{2} i + d'_L) . \tag{5.9}$$

We can now use the Jacobi identity between P, K, Q^L and the information about the the commutators in (5.5), (5.7), (5.9) and (4.3). The computation is most conveniently done in spinorial notation, given in the appendix. The results are:

$$d'_L = 0 , \qquad \text{i.e.} \qquad [Q^L_\alpha, D] = \tfrac{1}{2} i Q^L_\alpha , \tag{5.10}$$

$$[P_{\alpha\dot\beta}, \bar{Q}^{(1)L}_{\dot\gamma}] = 2i \epsilon_{\dot\beta\dot\gamma} Q^L_\alpha . \tag{5.11}$$

From (5.10) and (5.11) we see immediately

$$[\bar{Q}^{(1)L}_{\dot\alpha}, D] = -\tfrac{1}{2} i \bar{Q}^{(1)L}_{\dot\alpha} , \qquad [Q^{(1)L}_\alpha, D] = -\tfrac{1}{2} i Q^{(1)L}_\alpha . \tag{5.12}$$

Also we see that the scheme is symmetric under the interchange of quantities of opposite dimension. The counterpart of (3.5), i.e., the transformation of the $Q^{(1)L}_\alpha$ under the internal Bose symmetries B_l follows from (5.5) and the Jacobi identity between K, Q^L, B_l. It is (see (3.18) for the definition of t_l):

$$[\bar{Q}^{(1)L}_{\dot\alpha}, B_l] = \sum_M s_l^{LM} \bar{Q}^{(1)M}_{\dot\alpha} , \tag{5.13a}$$

$$[Q^{(1)L}_\alpha, B_l] = \sum_M t_l^{LM} Q^{(1)M}_\alpha . \tag{5.13b}$$

The precise form of the dimensional reflection in the pseudo Lie algebra is

$$P_{\alpha\dot\beta} \leftrightarrow K_{\beta\dot\alpha} , \qquad M_{\alpha_1\alpha_2} \leftrightarrow \bar{M}_{\dot\alpha_1\dot\alpha_2} , \qquad D \leftrightarrow -D ,$$

$$Q^L_\alpha \leftrightarrow \bar{Q}^{(1)L}_{\dot\alpha} , \qquad B_l \leftrightarrow B_l . \tag{5.14}$$

By dimension counting, use of the automorphism (5.14) and the knowledge of all covariants of each dimension, we can write down the remaining structure relations:

$$\{Q^L_\alpha, Q^M_\beta\} = 0\,, \qquad \{Q^{(1)L}_\alpha, Q^{(1)M}_\beta\} = 0\,, \tag{5.15}$$

$$\{Q^L_\alpha, \bar{Q}^{(1)M}_{\dot\beta}\} = 0\,, \tag{5.16}$$

$$\{Q^{(1)L}_\alpha, \bar{Q}^{(1)M}_{\dot\beta}\} = \delta^{LM} K_{\alpha\dot\beta}\,, \tag{5.17}$$

$$\{Q^L_\alpha, Q^{(1)M}_\beta\} = a^{LM} \epsilon_{\alpha\beta} D + b^{LM} M_{\alpha\beta} + i\epsilon_{\alpha\beta} B^{LM}\,, \tag{5.18}$$

where the B^{LM} are some linear combinations of the internal Bose symmetries B_l and a^{LM}, b^{LM} are numerical matrices.

Note that in contrast to the massive case (and also the massless case with only dilatational and not conformal invariance), the right-hand side of (1.1) must vanish. This is a consequence of the fact that conformal invariance fixes uniquely the dimensions of B_l and Q.

All commutation relations are now in their final form except (5.18). There remain a few Jacobi identities which have not been used yet. They fix not only the matrices a, b in (5.18) but also determine the group of internal Bose symmetries and its representation s_l (apart from the trivial possibility of adding internal symmetries which commute with all the Fermi charges). The $(P_{\delta\dot\gamma}, Q^L_\alpha, Q^{(1)M}_\beta)$ identity yields

$$2\delta^{LM} \epsilon_{\delta\beta} P_{\alpha\dot\gamma} = a^{LM} \epsilon_{\alpha\beta} P_{\delta\dot\gamma} - b^{LM}(\epsilon_{\delta\alpha} P_{\beta\dot\gamma} + \epsilon_{\delta\beta} P_{\alpha\dot\gamma})\,,$$

from where one concludes (decomposing into symmetric and antisymmetric parts in α, β)

$$a^{LM} = -b^{LM} = \delta^{LM}\,. \tag{5.19}$$

The $(K, Q^L, \bar{Q}^M)$ identity gives

$$B^{LM} = (B^{ML})^\dagger\,. \tag{5.20}$$

The two last independent relations are

$$(\bar{Q}, Q, Q^{(1)}):\quad [\bar{Q}^L_{\dot\alpha}, B^{MN}] = \sum_K (2\delta^{LM}\delta^{NK} - \tfrac{1}{2}\delta^{MN}\delta^{LK})\bar{Q}^K_{\dot\alpha}\,, \tag{5.21}$$

$$(Q, Q^{(1)}, B):\quad [B^{LM}, B_l] = \sum_N (t_l^{MN} B^{LN} + s_l^{LN} B^{NM})\,. \tag{5.22}$$

Eq. (5.21) tells us that for $\nu \neq 4$ all B^{MN} are linearly independent, for $\nu = 4$ there is precisely one linear relation between them. For, suppose $\Sigma\, a_{MN} B^{MN} = 0$.

Then, by (5.21)

$$2a_{LK} = \tfrac{1}{2}\delta^{LK}\,\mathrm{tr}\,(a)\,,$$

which for $\nu \neq 4$ is impossible and for $\nu = 4$ fixes a_{LK} up to a normalization factor. By (5.20), the real Lie algebra spanned by the B^{LM} is therefore isomorphic to the set of all Hermitian $\nu \times \nu$ matrices (for $\nu \neq 4$), respectively to all traceless such matrices (for $\nu = 4$). This part of the internal symmetry group is therefore U(ν) (respectively SU(4)). Consider now the Lie algebra $\mathcal{L}$ of all the B_l and denote the kernel of the representation s_l by $\mathcal{K}$*, the subalgebra spanned by the B^{LM} by $\mathcal{L}_1$. The quotient $\mathcal{L}/\mathcal{K}$ is faithfully represented by Hermitian matrices s in the ν-dimensional space and must therefore be contained in the Lie algebra of U(ν). Therefore for $\nu \neq 4$: $\mathcal{L}/\mathcal{K} = \mathcal{L}_1$. But $\mathcal{L}_1$ is an invariant subalgebra by (5.22) and $\mathcal{K}$ is an invariant subalgebra because it is a kernel. Therefore $\mathcal{L}$ is the direct sum

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{K}\,.$$

For $\nu = 4$ we have similarly

$$\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{K}\,,$$

where $\mathcal{L}_1$ may be either the Lie algebra of U(4) or that of SU(4).

We have seen that only the zero-mass case gives the possibility of a complete fusion between geometric and internal symmetries: the Fermi charges may then generate the full conformal group together with a unitary symmetry group, the only arbitrariness being the number ν of Fermi charges. The phenomenological application of the scheme is unfortunately plagued in that case even more than in the massive case by symmetry breaking (spontaneous or otherwise). But the consideration of lepton physics from the point of view of supersymmetry appears to be most indicated.

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Appendix

Some of the calculations are greatly simplified if one uses the spinorial notation for the angular momenta and the four-vectors which appear. We give the relevant formulas which have been used in the text.

Define the symmetric spinors $M_{\alpha_1\alpha_2}$, $\bar{M}_{\dot\beta_1\dot\beta_2}$ implicitly by

* $\Sigma\, a_l B_l \in \mathcal{K}$ if $\Sigma\, a_l s_l = 0$.

$$\sigma^{\mu}{}_{\alpha_1\dot{\beta}_1}\,\sigma^{\nu}{}_{\alpha_1\dot{\beta}_2}\,M_{\mu\nu} = M_{\alpha_1\alpha_2}\,\epsilon_{\dot{\beta}_1\dot{\beta}_2} + \bar{M}_{\dot{\beta}_1\dot{\beta}_2}\,\epsilon_{\alpha_1\alpha_2}\,, \tag{A.1}$$

then

$$(M_{\alpha_1\alpha_2})^{\dagger} = \bar{M}_{\dot{\alpha}_1\dot{\alpha}_2}\,, \tag{A.2}$$

$$M_{\alpha_1\alpha_2} = -\tfrac{1}{2}\,i\,(\sigma^{\mu\nu}\epsilon)_{\alpha_1\alpha_2}\,M_{\mu\nu}\,, \tag{A.3}$$

$$\bar{M}_{\dot{\beta}_1\dot{\beta}_2} = -\tfrac{1}{2}\,i\,(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}_1\dot{\beta}_2}\,M_{\mu\nu}\,, \tag{A.4}$$

$$M_{\mu\nu} = \tfrac{1}{4}\,i\,((\epsilon\,\sigma_{\mu\nu})^{\alpha_1\alpha_2}\,M_{\alpha_1\alpha_2} + (\bar{\sigma}_{\mu\nu}\epsilon)^{\dot{\beta}_1\dot{\beta}_2}\,\bar{M}_{\dot{\beta}_1\dot{\beta}_2})\,. \tag{A.5}$$

This splits $M_{\mu\nu}$ into its irreducible parts with respect to the connected Lorentz group, and $M_{\alpha_1\alpha_2}$ acts only on undotted, $\bar{M}_{\dot{\beta}_1\dot{\beta}_2}$ only on dotted spinor indices:

$$[Q_{\alpha}, M_{\alpha_1\alpha_2}] = -i\,(\epsilon_{\alpha\alpha_1}\,Q_{\alpha_2} + \epsilon_{\alpha\alpha_2}\,Q_{\alpha_1})\,, \tag{A.6}$$

$$[\bar{Q}_{\dot{\beta}}, \bar{M}_{\dot{\beta}_1\dot{\beta}_2}] = -i\,(\epsilon_{\dot{\beta}\dot{\beta}_1}\,\bar{Q}_{\dot{\beta}_2} + \epsilon_{\dot{\beta}\dot{\beta}_2}\,\bar{Q}_{\dot{\beta}_1})\,, \tag{A.7}$$

$$[\bar{Q}_{\dot{\beta}}, M_{\alpha_1\alpha_2}] = [Q_{\alpha}, \bar{M}_{\dot{\beta}_1\dot{\beta}_2}] = 0\,. \tag{A.8}$$

From this the transformation of spinors of higher rank can be immediately read off, e.g.:

$$[P_{\alpha\dot{\beta}}, M_{\alpha_1\alpha_2}] = -i(\epsilon_{\alpha\alpha_1}\,P_{\alpha_2\dot{\beta}} + \epsilon_{\alpha\alpha_2}\,P_{\alpha_1\dot{\beta}})\,. \tag{A.9}$$

The commutator (4.3) reads in this notation

$$[P_{\alpha_1\dot{\beta}_1}, K_{\alpha_2\dot{\beta}_2}] = 4i\,\epsilon_{\alpha_1\alpha_2}\,\epsilon_{\dot{\beta}_1\dot{\beta}_2}\,D - 2i(\epsilon_{\dot{\beta}_1\dot{\beta}_2}\,M_{\alpha_1\alpha_2} + \epsilon_{\alpha_1\alpha_2}\,\bar{M}_{\dot{\beta}_1\dot{\beta}_2})\,,$$

if we define the following connection between a four-vector V_{μ} and the corresponding spinor $V_{\alpha\dot{\beta}}$:

$$V_{\alpha\dot{\beta}} = (\sigma^{\mu})_{\alpha\dot{\beta}}\,V_{\mu}\,, \tag{A.10}$$

$$V_{\mu} = \tfrac{1}{2}(\bar{\sigma}_{\mu})^{\dot{\beta}\alpha}\,V_{\alpha\dot{\beta}}\,. \tag{A.11}$$

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