AMERICAN JOURNAL of PHYSICS

Klein's paradox

Barry R. Holstein

Citation: American Journal of Physics **66**, 507 (1998); doi: 10.1119/1.18891
View online: http://dx.doi.org/10.1119/1.18891
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/66/6?ver=pdfcov
Published by the American Association of Physics Teachers

Klein's paradox

Barry R. Holstein
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003

(Received 24 September 1997; accepted 4 November 1997)

Klein's paradox is shown to be resolved by a careful consideration of Feynman's picture of antiparticles as negative energy solutions traveling backward in time. Implications associated with the spin-statistics theorem are pointed out.

I. INTRODUCTION

The topic of Klein's paradox[1] is commonly treated as a component of an introductory discussion of relativistic quantum mechanics.[2] However, a careful resolution, although given in the research literature, has yet to be presented in a pedagogical forum. Indeed, early discussions by Sauter,[3] Sommerfeld,[4] and Hund[5] are in German. More recent work by Nikishov has been translated from Russian but is not widely known.[6] The best modern discussion—that of Hansen and Ravndal—is published in the journal Physica Scripta.[7] An interesting historical perspective by Telegdi is unpublished.[8] It thus makes sense to rectify this situation by presenting here an elementary treatment of relativistic barrier scattering which demonstrates specifically how the strictures of unitarity can be manifested by accounting properly for the phenomenon of pair creation. This involves a discussion of Feynman's picture of antiparticles as negative energy solutions (of relativistic wave equations) which travel backward in time, and as a bonus such a picture reveals the intimate connection between spin and statistics. The subject of strong field pair creation which is thereby encountered also has important implications for the subject of black hole decay, which is of critical importance to modern cosmology.

We begin by examining the conventional analysis of the problem. However, before considering the relativistic barrier scattering case, we first review the corresponding nonrelativistic problem. Thus consider a particle of mass m and charge e moving under the influence of a scalar potential

$$\phi(z)=\begin{cases}\phi_0 & z\geqslant 0\\ 0 & z<0\end{cases}, \tag{1}$$

as shown in Fig. 1. For a particle incident from the left with momentum p and energy $E=p^2/2m$, the solution of the time-independent Schrödinger equation

$$\left(-\frac{1}{2m}\frac{d^2}{dz^2}+e\phi\right)\psi(z)=E\psi(z) \tag{2}$$

is

$$\psi(z)=\begin{cases}\frac{1}{\sqrt{p}}\,e^{ipz}+r(\rho)\,\frac{1}{\sqrt{p}}\,e^{-ipz} & z\leqslant 0\\[2ex] t(\rho)\,\frac{1}{\sqrt{q}}\,e^{iqz} & z>0\end{cases}, \tag{3}$$

where $q=\sqrt{2m(E-e\phi_0)}$ is the momentum to the right of the barrier, $\rho=q/p$ is the ratio of the left, right momenta, and

$$r(\rho)=\frac{1-\rho}{1+\rho},\quad t(\rho)=\frac{2\sqrt{\rho}}{1+\rho} \tag{4}$$

are the reflection and transmission amplitudes, respectively. Note that we have written our solution in terms of "WKB"-like solutions, which have the proper normalizations to account for local probability conservation. This means that the corresponding current densities are normalized to unity so that $|t(\rho)|^2$, $|r(\rho)|^2$ measure the ratio of outgoing to incoming fluxes directly, and the reflection and transmission coefficients are thus given by

$$R=|r(\rho)|^2=\left|\frac{1-\rho}{1+\rho}\right|^2,\quad T=|t(\rho)|^2=\frac{4\rho}{|1+\rho|^2}. \tag{5}$$

Then if $E>e\phi_0$, q is real and both reflection and transmission occur with unitarity satisfied via $R+T=1$. On the other hand, if $E<e\phi_0$, then q is imaginary so that $R=1$ and there is complete reflection.

In the corresponding relativistic situation, we employ the time-independent Klein–Gordon equation

$$\left((E-e\phi(z))^2-\frac{d^2}{dz^2}+m^2\right)\psi(z)=0, \tag{6}$$

whose solution has the same form as Eq. (3) but with

$$p=\sqrt{E^2-m^2},\quad q=\sqrt{(E-e\phi_0)^2-m^2}. \tag{7}$$

Then if $E>e\phi_0$, we have both reflection and transmission as before, with unitarity satisfied via $R+T=1$. Also, if $E<e\phi_0$ but $e\phi_0-E<m$, then one has total reflection as in the nonrelativistic case. However, in the situation that $E<e\phi_0$ but $e\phi_0-E>m$ we have Klein's paradox—q is again real so that despite the presence of a large barrier, reflection *and* transmission again occur. The resolution of this problem is generally suggested to be associated with pair production—$e\phi_0>E+m>2m$ means that the potential is strong enough to generate particle–antiparticle pairs from the vacuum. The problem here is, of course, that one is trying to interpret a multiparticle phenomenon using a simple single-particle wave function.[9] However, while arguing that a proper treatment can be found using quantum field theory, none of the existing discussions (including my own!)[2] undertake a truly quantitative verification of this assertion, and it is this omission which this note will address.

In the next section we examine the problem of spinless relativistic barrier scattering using the Feynman picture and demonstrate the incorrectness of the above "explanation." Instead, we show that the resolution of the unitarity problem is associated with the possibility of creating multiple particle pairs. The discussion is aided by a simple field theoretic construction. In the following section, we examine the analogous case of spin 1/2 particles and show that a similar analysis can be made consistent with unitarity provided one includes the stricture of Fermi statistics. Finally, we summarize our results in Sec. V.

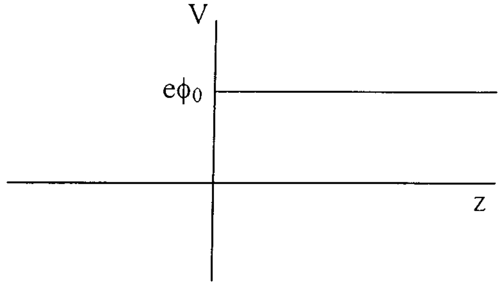

Fig. 1. The barrier potential associated with Klein's paradox.

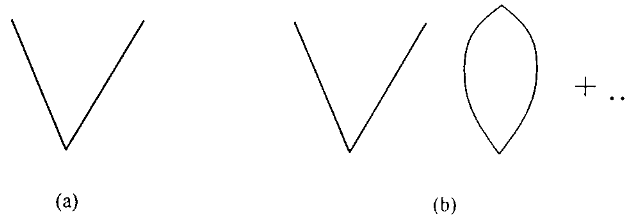

Fig. 2. (a) Pair creation from the vacuum; (b) pair creation from the vacuum can be accompanied by one or more particle–antiparticle creation and annihilation ''bubbles.''

II. KLEIN'S PARADOX: $S=0$

We begin our discussion of Klein's paradox by noting that Feynman's picture of antiparticles as particle states moving backward in time implies that the associated momentum is given by

$$\tilde{\mathbf{p}}=\frac{d\mathbf{x}}{d(-t)}=-\frac{d\mathbf{x}}{dt}=-\mathbf{p}, \tag{8}$$

which implies that (for $q>0$) our interpretation of the wave function Eq. (3) is incorrect—the component e^{iqz} for $z>0$ is actually associated with an antiparticle moving to the *left*![10] This is consistent with the well-known feature that for the Klein–Gordon equation, the quantity $j_\mu=\phi^* i\partial_\mu\phi - i\partial_\mu\phi^*\phi$ is to be associated with the electromagnetic- (*not* probability-) current density. Thus T must be positive since the negative sign associated with moving to the left is compensated by the negative sign connected with the antiparticle content. The process depicted therein then represents total reflection of the incoming particle from the high barrier accompanied by particle–antiparticle *annihilation* with probability T. By time reversal this is also the probability for single pair creation! However, if the probability for the creation of one pair is T, then there exists the probability T^2 for the creation of two pairs, T^3 for three pairs, etc. Thus the *total* relative probability for pair creation is

$$P_{\text{pair}}^{\text{rel}}=T+T^2+T^3+\cdots=\frac{T}{1-T}=\frac{4\rho}{(1-\rho)^2}. \tag{9}$$

Of course, there is also some probability K_0 that the vacuum state remains unchanged in the presence of the strong electric field—i.e., that *no* pairs are created. The absolute probability for particle creation is then

$$P_{\text{pair}}^{\text{abs}}=K_0 P_{\text{pair}}^{\text{rel}} \tag{10}$$

and by unitarity we require that

$$K_0+P_{\text{pair}}^{\text{abs}}=K_0(1+P_{\text{pair}}^{\text{rel}})=1,$$

i.e,

$$K_0=1-T=\left(\frac{1-\rho}{1+\rho}\right)^2. \tag{11}$$

Note that $K_0<1$, as required.

It is interesting to note in this regard that Hund has pointed out that the average number of pairs produced can be found from the relation

$$\bar{n}=|t(-q)|^2=\frac{4\rho}{(1-\rho)^2}. \tag{12}$$

This follows from the feature that when $q\to -q$ the solution Eq. (3) is recognized as being associated with an outgoing antiparticle state, so that $t(\rho)$ becomes the amplitude for pair *production*. Hund's relation Eq. (12) is found to be identical to the result directly calculated from Eq. (9):

$$\bar{n}=K_0(T+2T^2+3T^3+\cdots)=K_0\,\frac{T}{(1-T)^2}=\frac{4\rho}{(1-\rho)^2}. \tag{13}$$

Diagramatically what is going on is that accompanying *any* amplitude such as that for pair production, as shown in Fig. 2(a), is the amplitude for the vacuum to remain a vacuum—i.e., for any number of associated bubble diagrams representing creation and subsequent annihilation of particle–antiparticle pairs from the vacuum—Fig. 2(b).

One can, of course, pursue a parallel discussion of scattering from the barrier. However, in order to do so it is useful to first define a formalism involving creation and annihilation operators and to develop a simple S-matrix theory. We begin by defining a set of properly normalized purely incoming or outgoing single-particle states:

$$\begin{aligned}
&\phi_p^{\text{in}}(z)=\frac{1}{\sqrt{p}}\,e^{ipz}, \quad \phi_p^{\text{out}}(z)=\frac{1}{\sqrt{p}}\,e^{-ipz},\\
&\tilde{\phi}_q^{\text{in}}(z)=\frac{1}{\sqrt{|q|}}\,e^{iqz}, \quad \tilde{\phi}_q^{\text{out}}(z)=\frac{1}{\sqrt{|q|}}\,e^{-iqz}.
\end{aligned} \tag{14}$$

Note here that $\phi^{\text{in,out}}(z)$, $\tilde{\phi}^{\text{in,out}}(z)$ represent incoming, outgoing states for particles, antiparticles, respectively. In terms of such states then we can construct solutions of the Klein–Gordon equation which contain only an asymptotically incoming or outgoing particle for $z>0$ as

$$\begin{aligned}
\psi_1(z)&=\begin{cases}\dfrac{1+\rho}{2\sqrt{\rho}}\,\phi_p^{\text{in}}(z)+\dfrac{1-\rho}{2\sqrt{\rho}}\,\phi_p^{\text{out}}(z) & z<0\\ \tilde{\phi}_q^{\text{in}}(z) & z>0\end{cases},\\
\psi_2(z)&=\begin{cases}\dfrac{1-\rho}{2\sqrt{\rho}}\,\phi_p^{\text{in}}(z)+\dfrac{1+\rho}{2\sqrt{\rho}}\,\phi_p^{\text{out}}(z) & z<0\\ \tilde{\phi}_q^{\text{out}}(z) & z>0\end{cases}.
\end{aligned} \tag{15}$$

Alternatively, one can also construct solutions containing only an asymptotically incoming or outgoing particle state for $z<0$:

$$\psi_3(z)=\begin{cases}\phi_p^{\text{in}}(z) & z<0\\ \dfrac{1+\rho}{2\sqrt{\rho}}\,\tilde{\phi}_q^{\text{in}}(z)-\dfrac{1-\rho}{2\sqrt{\rho}}\,\tilde{\phi}_q^{\text{out}}(z) & z>0\end{cases},$$
$$\psi_4(z)=\begin{cases}\phi_p^{\text{out}}(z) & z<0\\ -\dfrac{1-\rho}{2\sqrt{\rho}}\,\tilde{\phi}_q^{\text{in}}(z)+\dfrac{1+\rho}{2\sqrt{\rho}}\,\tilde{\phi}_q^{\text{out}}(z) & z>0\end{cases}. \tag{16}$$

In this way we have defined *two* complete sets of states—one representing those which contain asymptotically only an outgoing particle or antiparticle and those which asymptotically contain only an incoming particle or antiparticle. However, these two sets of states are *not* independent and are related by a unitary transformation—the so-called S or scattering matrix:

$$|\psi(\text{in})\rangle=S|\psi(\text{out})\rangle. \tag{17}$$

In particular, there exist two vacuum states which are related by

$$|0(\text{in})\rangle=S|0(\text{out})\rangle. \tag{18}$$

Now since both sets are complete we can express field operators in terms of either[11]

$$\psi(z)=\sum_k \hat{b}_k^{\text{in}}\phi_k^{\text{in}}(z)+\hat{d}_k^{\text{in}\dagger}\tilde{\phi}_k^{\text{in}}(z) \tag{19}$$

or

$$\psi(z)=\sum_k \hat{b}_k^{\text{out}}\phi_k^{\text{out}}(z)+\hat{d}_k^{\text{out}\dagger}\tilde{\phi}_k^{\text{out}}(z), \tag{20}$$

where $\hat{b}_k^{\text{in}\dagger}$, $\hat{d}_k^{\text{in}\dagger}$ is a single-particle operator which creates an incoming particle, antiparticle with momentum k:

$$\hat{b}_k^{\text{in}\dagger}|0(\text{in})\rangle=|k(\text{in})\rangle,\quad \hat{d}_k^{\text{in}\dagger}|0(\text{in})\rangle=|\tilde{k}(\text{in})\rangle, \tag{21}$$

while $\hat{b}_k^{\text{out}\dagger}$, $\hat{d}_k^{\text{out}\dagger}$ are the corresponding operators for outgoing states:

$$\hat{b}_k^{\text{out}\dagger}|0(\text{out})\rangle=|k(\text{out})\rangle,\quad \hat{d}_k^{\text{out}\dagger}|0(\text{out})\rangle=|\tilde{k}(\text{out})\rangle. \tag{22}$$

The in and out operators can be expressed in terms of each other by means of a Bogoliubov transformation. The form of this transformation can be found by determining the relation between in and out states—from Eqs. (15) and (16):

$$\begin{aligned}\psi_1(z)&=\frac{1+\rho}{2\sqrt{\rho}}\,\psi_3(z)+\frac{1-\rho}{2\sqrt{\rho}}\,\psi_4(z)\\ \psi_2(z)&=\frac{1-\rho}{2\sqrt{\rho}}\,\psi_3(z)+\frac{1+\rho}{2\sqrt{\rho}}\,\psi_4(z)\end{aligned} \tag{23}$$

or, equivalently,

$$\begin{aligned}\psi_3(z)&=\frac{1+\rho}{2\sqrt{\rho}}\,\psi_1(z)-\frac{1-\rho}{2\sqrt{\rho}}\,\psi_2(z),\\ \psi_4(z)&=-\frac{1-\rho}{2\sqrt{\rho}}\,\psi_1(z)+\frac{1+\rho}{2\sqrt{\rho}}\,\psi_2(z).\end{aligned} \tag{24}$$

Then we can write

$$\begin{aligned}\psi_4(z)&=\frac{2\sqrt{\rho}}{1-\rho}\,\psi_1(z)-\frac{1+\rho}{1-\rho}\,\psi_3(z),\\ \psi_3(z)&=\frac{2\sqrt{\rho}}{1-\rho}\,\psi_2(z)-\frac{1+\rho}{1-\rho}\,\psi_4(z),\\ \psi_2(z)&=-\frac{2\sqrt{\rho}}{1-\rho}\,\psi_3(z)+\frac{1+\rho}{1-\rho}\,\psi_1(z),\\ \psi_1(z)&=-\frac{2\sqrt{\rho}}{1-\rho}\,\psi_4(z)+\frac{1+\rho}{1-\rho}\,\psi_2(z),\end{aligned} \tag{25}$$

and the corresponding Bogoliubov relations can be read off as

$$\begin{aligned}\hat{b}_k^{\text{out}\dagger}&=-\hat{b}_k^{\text{in}\dagger}\,\frac{1+\rho}{1-\rho}+\hat{d}_k^{\text{in}}\,\frac{2\sqrt{\rho}}{1-\rho},\\ \hat{d}_k^{\text{out}\dagger}&=-\hat{b}_k^{\text{in}}\,\frac{2\sqrt{\rho}}{1-\rho}+\hat{d}_k^{\text{in}\dagger}\,\frac{1+\rho}{1-\rho},\\ \hat{b}_k^{\text{in}\dagger}&=-\hat{b}_k^{\text{out}\dagger}\,\frac{1+\rho}{1-\rho}+\hat{d}_k^{\text{out}}\,\frac{2\sqrt{\rho}}{1-\rho},\\ \hat{d}_k^{\text{in}\dagger}&=-\hat{b}_k^{\text{out}}\,\frac{2\sqrt{\rho}}{1-\rho}+\hat{d}_k^{\text{out}\dagger}\,\frac{1+\rho}{1-\rho}.\end{aligned} \tag{26}$$

Thus if we define

$$\langle 0(\text{out})|0(\text{in})\rangle\equiv e^{iW} \tag{27}$$

as the amplitude that an empty in state evolves into an empty out state, the amplitude for creation of a single pair is

$$A_{1\ \text{pair}}=\langle 0(\text{out})|\hat{d}^{\text{out}}\hat{b}^{\text{out}}|0(\text{in})\rangle. \tag{28}$$

This can be evaluated by noting that since

$$0=\hat{b}^{\text{in}}|0(\text{in})\rangle=\left(-\hat{b}^{\text{out}}\,\frac{1+\rho}{1-\rho}+\hat{d}^{\text{out}\dagger}\,\frac{2\sqrt{\rho}}{1-\rho}\right)|0(\text{in})\rangle, \tag{29}$$

we require

$$\hat{b}^{\text{out}}|0(\text{in})\rangle=\frac{2\sqrt{\rho}}{1+\rho}\,\hat{d}^{\text{out}\dagger}|0(\text{in})\rangle. \tag{30}$$

Then,

$$\begin{aligned}A_{1\ \text{pair}}&=\frac{2\sqrt{\rho}}{1+\rho}\,\langle 0(\text{out})|\hat{d}^{\text{out}}\hat{d}^{\text{out}\dagger}|0(\text{in})\rangle\\ &=\langle 0(\text{out})|[\hat{d}^{\text{out}},\hat{d}^{\text{out}\dagger}]+\hat{d}^{\text{out}\dagger}\hat{d}^{\text{out}}|0(\text{in})\rangle\\ &=\frac{2\sqrt{\rho}}{1+\rho}\,e^{iW}.\end{aligned} \tag{31}$$

Likewise, we can calculate the n-pair amplitude as[12]

$$A_{n\ \text{pair}}=\frac{1}{n!}\,\langle 0(\text{out})|(\hat{d}^{\text{out}})^n(\hat{b}^{\text{out}})^n|0(\text{in})\rangle=\left(\frac{2\sqrt{\rho}}{1+\rho}\right)^n e^{iW}. \tag{32}$$

By unitarity we require

$$1=\exp(-2\ \text{Im}\ W)\sum_{n=0}^{\infty}\left(\frac{2\sqrt{\rho}}{1+\rho}\right)^{2n}=\frac{\exp(-2\ \text{Im}\ W)}{1-\dfrac{4\rho}{(1+\rho)^2}} \tag{33}$$

or

$$\exp(-2\,\mathrm{Im}\,W)=K_0=\left(\frac{1-\rho}{1+\rho}\right)^2, \tag{34}$$

as found in Eq. (11) by less formal means.

Using this formalism we can now also treat the problem of barrier scattering. Thus consider a boson incident on this very high barrier potential. Upon reaching the barrier it will be totally reflected but may also stimulate pair production. The amplitude for the particle to be reflected with no pairs being generated is

$$\begin{aligned}A_{0\ \text{pair}}^{\text{reflec}}&=\langle 0(\text{out})|\hat{b}^{\text{out}}\hat{b}^{\text{in}\dagger}|0(\text{in})\rangle\\&=-\left(\frac{1+\rho}{1-\rho}\right)\langle 0(\text{out})|\hat{b}^{\text{out}}\hat{b}^{\text{out}\dagger}|0(\text{in})\rangle\\&\quad+\frac{2\sqrt{\rho}}{1-\rho}\langle 0(\text{out})|\hat{b}^{\text{out}}\hat{d}^{\text{out}}|0(\text{in})\rangle\\&=e^{iW}\left[-\left(\frac{1+\rho}{1-\rho}\right)+\frac{4\rho}{1-\rho^2}\right]=-\left(\frac{1+\rho}{1-\rho}\right)e^{iW},\end{aligned} \tag{35}$$

while the amplitude to be reflected along with the generation of n pairs is

$$\begin{aligned}A_{n\ \text{pair}}^{\text{reflec}}&=\frac{1}{\sqrt{n!(n+1)!}}\langle 0(\text{out})|(\hat{d}^{\text{out}})^n(\hat{b}^{\text{out}})^{n+1}\hat{b}^{\text{in}\dagger}|0(\text{in})\rangle\\&=-\left(\frac{1+\rho}{1-\rho}\right)\frac{n+1}{\sqrt{n!(n+1)!}}\langle 0(\text{out})|(\hat{d}^{\text{out}})^n(\hat{b}^{\text{out}})^n|0(\text{in})\rangle\\&\quad+\frac{2\sqrt{\rho}}{1-\rho}\langle 0(\text{out})|(\hat{d}^{\text{out}})^{n+1}(\hat{b}^{\text{out}})^{n+1}|0(\text{in})\rangle\\&=-\left(\frac{1-\rho}{1+\rho}\right)\sqrt{n+1}\left(\frac{2\sqrt{\rho}}{1+\rho}\right)^n e^{iW}.\end{aligned} \tag{36}$$

We verify then that the total probability for scattering accompanied by the production of no, one, two, etc., pairs is

$$\begin{aligned}\text{Prob}^{\text{tot}}&=\sum_{n=0}^{\infty}|A_{n\ \text{pair}}^{\text{reflec}}|^2\\&=\exp(-2\,\mathrm{Im}\,W)\left(\frac{1-\rho}{1+\rho}\right)^2\sum_{n=0}^{\infty}(n+1)\left(\frac{2\sqrt{\rho}}{1+\rho}\right)^{2n}\\&=\left(\frac{1-\rho}{1+\rho}\right)^4\left(\frac{1}{1-\frac{4\rho}{(1+\rho)^2}}\right)^2=1,\end{aligned} \tag{37}$$

which is unitary, as expected.

We have demonstrated, then, how for spin-zero boson scattering from a high barrier potential, Klein's paradox is completely resolved in terms of a correct interpretation of pair production. Indeed, the incident particle is completely reflected, as expected, but is accompanied in the outgoing state by one, two, etc., particle–antiparticle pairs generated by interaction with the potential. A corresponding calculation can be made for the case of a Dirac particle, with intriguing differences associated with statistics, as we now show.

III. KLEIN'S PARADOX: $S=1/2$

A Klein's paradox also arises for the case of Dirac particles scattering from a high barrier, and the resolution can be developed as above, with a few interesting modifications. Thus consider the time-independent Dirac equation in the presence of the potential Eq. (1):

$$(E-e\phi(z))\psi(z)=\gamma_0\left(i\gamma_z\frac{d}{dz}+m\right)\psi(z) \tag{38}$$

and write the solution as

$$\psi(z)=\begin{cases}\sqrt{\dfrac{E+m}{p}}\,e^{ipz}\begin{pmatrix}\chi_+\\ \dfrac{p}{E+m}\chi_+\end{pmatrix}+r(\rho)\sqrt{\dfrac{E+m}{p}}\,e^{-ipz}\begin{pmatrix}\chi_+\\ -\dfrac{p}{E+m}\chi_+\end{pmatrix} & z<0\\[2ex] t(\rho)\,\dfrac{E-e\phi_0+m}{\sqrt{|q|}}\,e^{iqz}\begin{pmatrix}\chi_+\\ \dfrac{q}{E-e\phi_0+m}\chi_+\end{pmatrix} & z>0\end{cases}, \tag{39}$$

where χ_+ is a spinor quantized along the z direction—i.e., $\sigma_z\chi_+=\chi_+$. Continuity of the wave function at $z=0$ yields

$$t(\rho)=\frac{2\sqrt{\dfrac{q}{p}}\sqrt{\dfrac{E+m}{E-e\phi_0+m}}}{1+\dfrac{q}{p}\dfrac{E+m}{E-e\phi_0+m}},$$

$$r(\rho)=\frac{1-\dfrac{q}{p}\dfrac{E+m}{E-e\phi_0+m}}{1+\dfrac{q}{p}\dfrac{E+m}{E-e\phi_0+m}}. \tag{40}$$

The corresponding probability current densities are found via

$$j_z(z)=\overline{\psi}(z)\gamma_z\psi(z) \tag{41}$$

and yield for the transmission and reflection coefficients

$$\bar{T}=|t(\bar{\rho})|^2=\frac{4\bar{\rho}}{(1+\bar{\rho})^2},\quad \bar{R}=|r(\bar{\rho})|^2=\left(\frac{1-\bar{\rho}}{1+\bar{\rho}}\right)^2, \tag{42}$$

where we have defined

$$\bar{\rho}=\frac{q}{p}\frac{E+m}{E-e\phi_0+m} \tag{43}$$

and we verify as before that $\bar{R}+\bar{T}=1$. We observe then that the expressions for the transmission and reflection coefficients for Dirac particles are identical in form to those for their Klein–Gordon analogues, except for the replacement $\rho\to\bar{\rho}$. One important difference, however, is that in the Dirac case we have $\bar{\rho}<0$ for the case of the high barrier—$e\phi_0>E+m$. Thus the transmission coefficient is actually *negative*! However, this is to be expected since we are dealing in the Dirac case with a probability density *not* a current density as in the Klein–Gordon case. The negative value for the transmission coefficient is then associated with the fact that the antiparticle content of the solution given in Eq. (39) is moving to the left—i.e., this represents particle–antiparticle annihilation, as stressed in Sec. II.

Because of the formal similarity of the Dirac and Klein–Gordon solutions we can construct asymptotic ingoing, outgoing particle, antiparticle states and an associated Bogoliubov transformation between the relevant operators as given in Eq. (26), except that in the Dirac case we replace $\rho\to\bar{\rho}$. One other critical difference is that the Dirac operators must satisfy *anticommutation* relations rather than the commutation relations which we used in Sec. II. This means, for example, that if we consider pair creation from the vacuum, only a *single pair* can be created, since more than one would violate the Pauli exclusion principle. If $\bar{K}_0$ represents the vacuum to vacuum probability for the fermion case, then since $-\bar{T}$ is the relative pair creation probability (remember that $\bar{T}$ is negative) the unitarity condition requires

$$\bar{K}_0(1-\bar{T})=1,$$

i.e.,

$$\bar{K}_0=\frac{1}{1-\bar{T}}=\left(\frac{1+\bar{\rho}}{1-\bar{\rho}}\right)^2. \tag{44}$$

This result may be also expressed in operator language, since defining

$$\langle 0(\text{out})|0(\text{in})\rangle=e^{i\bar{W}} \tag{45}$$

the pair creation amplitude is found to be

$$\begin{aligned}\bar{A}_{n\ \text{pair}}&=\delta_{n1}\langle 0(\text{out})|\hat{d}^{\text{out}}\hat{b}^{\text{out}}|0(\text{in})\rangle\\&=\delta_{n1}\frac{2\sqrt{|\bar{\rho}|}}{1+\bar{p}}\langle 0(\text{out})|\hat{d}^{\text{out}}\hat{d}^{\text{out}\dagger}|0(\text{in})\rangle\\&=\delta_{n1}\frac{2\sqrt{|\bar{\rho}|}}{1+\bar{\rho}}e^{i\bar{W}}.\end{aligned} \tag{46}$$

The unitarity condition is then

$$\begin{aligned}1&=\exp(-2\ \text{Im}\ \bar{W})\sum_{n=0}^{1}\left(\frac{2\sqrt{|\bar{\rho}|}}{1+\bar{\rho}}\right)^{2n}\\&=\exp(-2\ \text{Im}\ \bar{W})\left(1-\frac{4\bar{\rho}}{(1+\bar{\rho})^2}\right)\end{aligned} \tag{47}$$

or

$$\exp(-2\ \text{Im}\ \bar{W})=\bar{K}_0=\left(\frac{1+\bar{\rho}}{1-\bar{\rho}}\right)^2, \tag{48}$$

as found before. We can also verify Hund's result, which in this case reads

$$\bar{n}=|t(-\bar{\rho})|^2=\frac{-4\bar{\rho}}{(1-\bar{\rho})^2}. \tag{49}$$

Since for fermions only a single pair is possible we have

$$\bar{n}=\bar{K}_0|\bar{A}_{1\ \text{pair}}|^2=-\frac{\bar{T}}{1-\bar{T}}=-\frac{4\bar{\rho}}{(1-\bar{\rho})^2}, \tag{50}$$

in agreement with Hund's relation.

Finally, it is interesting to study scattering from the high barrier. In the Dirac case, since an outgoing particle is already present there is *no possibility* for pair creation, and the scattering amplitude is given by

$$\bar{A}_{n\ \text{pair}}^{\text{reflec}}=\delta_{n0}\langle 0(\text{out})|\hat{b}^{\text{out}}\hat{b}^{\text{in}\dagger}|0(\text{in})\rangle=-\delta_{n0}\left(\frac{1-\bar{\rho}}{1+\bar{\rho}}\right)e^{i\bar{W}}. \tag{51}$$

The full scattering probability is then

$$\text{Prob}^{\text{tot}}=|\bar{A}_{0\ \text{pair}}^{\text{reflec}}|^2=\left(\frac{1-\bar{\rho}}{1+\bar{\rho}}\right)^2\exp(-2\ \text{Im}\ \bar{W})=1, \tag{52}$$

as required by unitarity.

We see then how the spin and statistics are closely intertwined in order to satisfy the important stricture of unitarity. Had we chosen to use commutators for the Dirac case or anticommutators for the Klein–Gordon particles we would have found the obviously unphysical result that $K_0,\bar{K}_0>1$. That is, that the probability that the vacuum state remains a vacuum would be greater than unity. This consistency condition and its connection with the spin-statistics theorem was noted by Feynman.[13]

IV. CONCLUSION

The subject of Klein's paradox, as associated with scattering of a particle from a high potential barrier in relativistic quantum mechanics, has been discussed from the perspective of Feynman's view of antiparticles as particles traveling backward in time. In this picture a careful discussion of the related creation process, which includes the possibility of multiple pair creation, has been shown to be internally consistent provided that one assumes Bose statistics for particles satisfying the Klein–Gordon equation and Fermi statistics for corresponding Dirac particles. The critical feature in the phenomenon of pair creation is that the presence of the barrier changes the vacuum in that region so that what is a pure incoming particle solution in the potential free regime connects onto a multiparticle configuration in the corresponding outgoing state. In this way the apparent paradox can be completely and successfully resolved.

ACKNOWLEDGMENT

Research supported in part by the National Science Foundation.

[1]O. Klein, ''Die Reflexion von Elektronen an einem Potentialsprung nach der Relativistischen Dynamik von Dirac,'' Z. Phys. **53**, 157–65 (1929).

[2]See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw–Hill, New York, 1964); F. Gross, *Relativistic Quantum Mechanics and Field Theory* (Wiley, New York, 1993); and B. R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1992).

[3]F. Sauter, ''Über das Verhalten eines Elektrons im Homogenen Elektrischen Feld Nach der Relativistischen Theorie Diracs,'' Z. Phys. **69**, 742–764 (1931); and ''Zum Kleinschen Paradoxon,'' **73**, 547–552 (1931).

[4]A. Sommerfeld, *Atombau and Spektrallinien* (Friedr. Vieweg, Braunschweig, 1960), Vol. II.

[5]F. Hund, ''Materieezeugung um Anschaulichen und im Gequantelten Wellenbild der Materie,'' Z. Phys. **117**, 1–17 (1940).

[6]A. I. Nikishov, ''Pair Production by a Constant Electric Field,'' Sov. Phys. JETP **30**, 660–662 (1969); ''Barrier Scattering in Field Theory Removal of Klein Paradox,'' Nucl. Phys. B **21**, 346–358 (1970).

[7]A. Hansen, and F. Ravndal, ''Klein's Paradox and its Resolution,'' Phys. Scr. **23**, 1030–1042 (1981).

[8]V. Telegdi, ''Klein's Paradox Revisited,'' Cal Tech preprint, 1995 (unpublished).

[9]See also the discussions by R. G. Winter, ''Klein Paradox for the Klein–Gordon Equation,'' Am. J. Phys. **27**, 355–358 (1959); M. G. Fuda and E. Furlani, ''Zitterbewegung and the Klein Paradox for Spin-zero Particles,'' *ibid.* **50**, 545–549 (1982).

[10]An alternative way to see this point is to note that in Feynman's picture the wave function for $z>0$, which has negative kinetic energy $E-e\phi_0$ and positive wave number q corresponds to an *antiparticle* having the conjugate wave function—i.e. $(e^{iqz-i(E-V_0)t})^*=e^{-iqz-i(e\phi_0-E)t}$—and is recognized as a left moving, positive energy state.

[11]Note that a creation operator for antiparticles is required because of charge conservation—if the charge at location x decreases by one unit due to the application of the field $\psi(x)$, this can be associated either with destruction of a particle at that location or *creation* of an (oppositely charged) antiparticle.

[12]Recall that a unit-normalized n-particle state is given by

$$|n\rangle=\frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle. \tag{53}$$

[13]R. P. Feynman and S. Weinberg, *Elementary Particles and the Laws of Physics* (Cambridge U.P., New York, 1987).