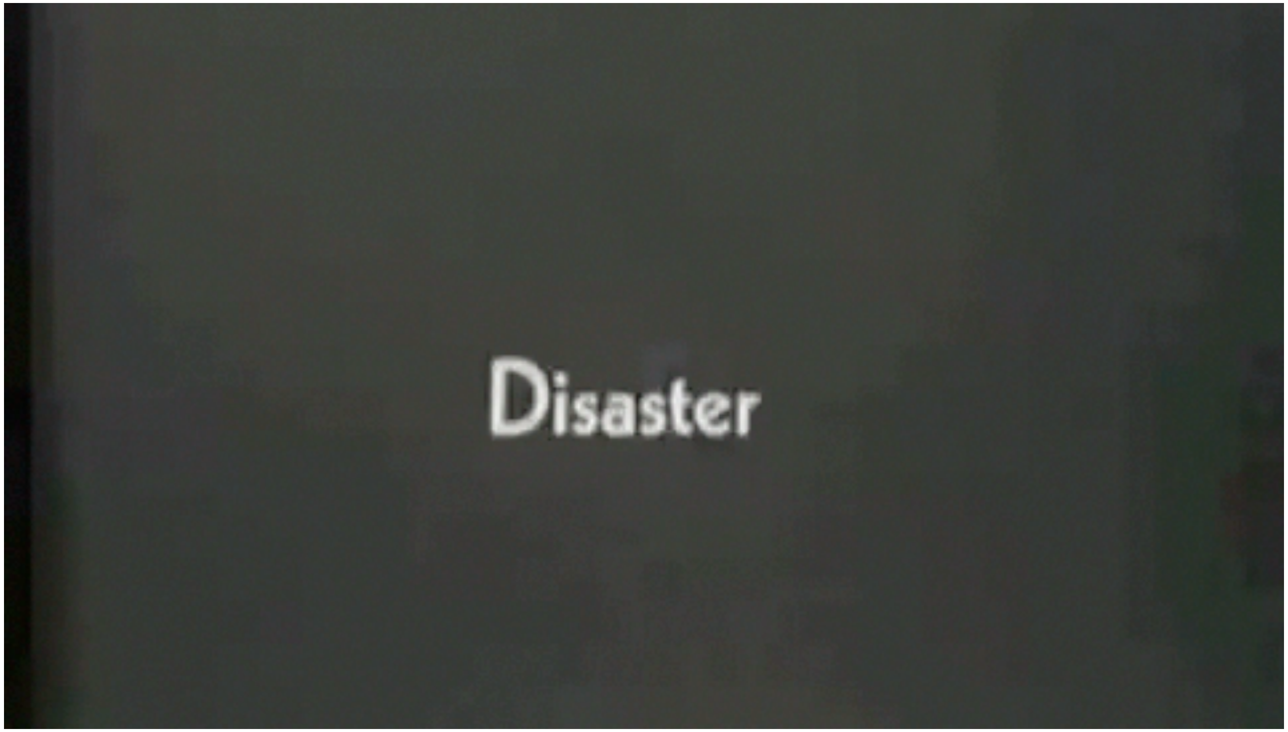
Chapter 15

Oscillations



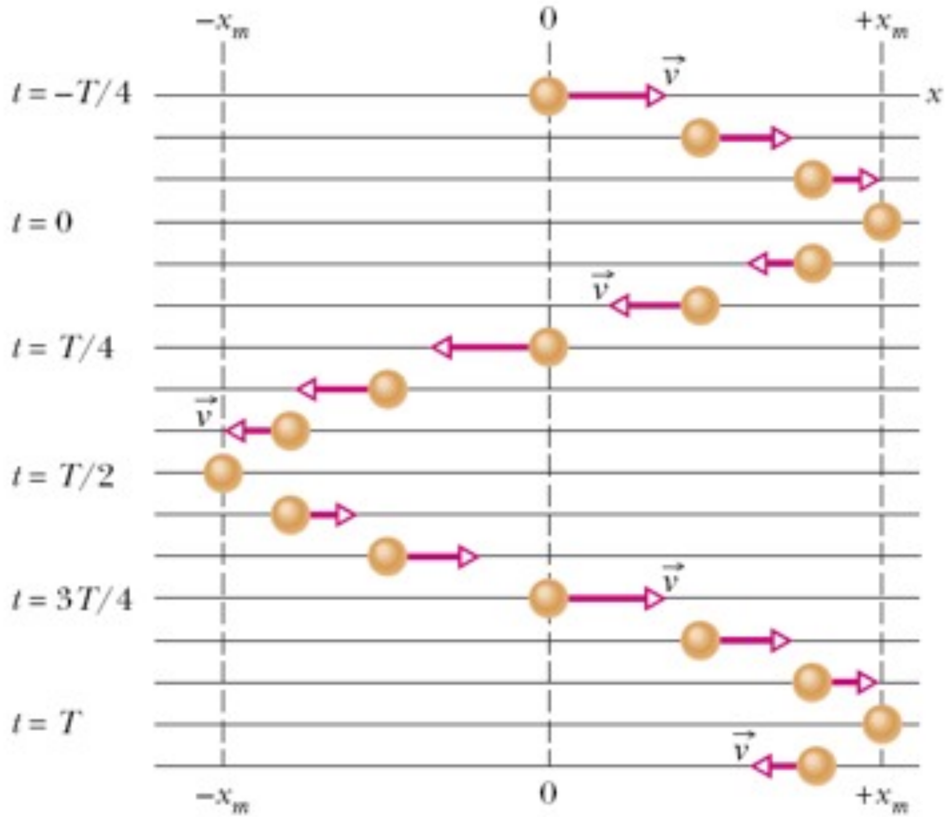
15.1 Oscillatory motion, oscillations

Motion which is periodic in time, that is, motion that repeats itself in time.

Why do you care?

- Bridges break
- Clocks, timing, circuits
- Signal analysis
- Sound perception
- Relation to waves (radio/all EM, ocean surface)
- etc etc

15.2 Simple Harmonic Motion



In the figure snapshots of a simple oscillatory system is shown. A particle repeatedly moves back and forth about the point $x = 0$.

The time taken for one complete oscillation is the period, T. In the time of one T, the system travels from $x = +x_m$, to $-x_m$, and then back to its original position x_m.

The velocity vector arrows are scaled to indicate the magnitude of the speed of the system at different times. At $x = \pm x_m$, the velocity is zero.

Frequency of oscillation is the number of oscillations that are completed in each second.

The symbol for frequency is f, and the SI unit is the hertz (abbreviated as Hz). Note that 1 Hz = 1/sec.

It follows that $Tf = 1$ or

$$T = \frac{1}{f}$$

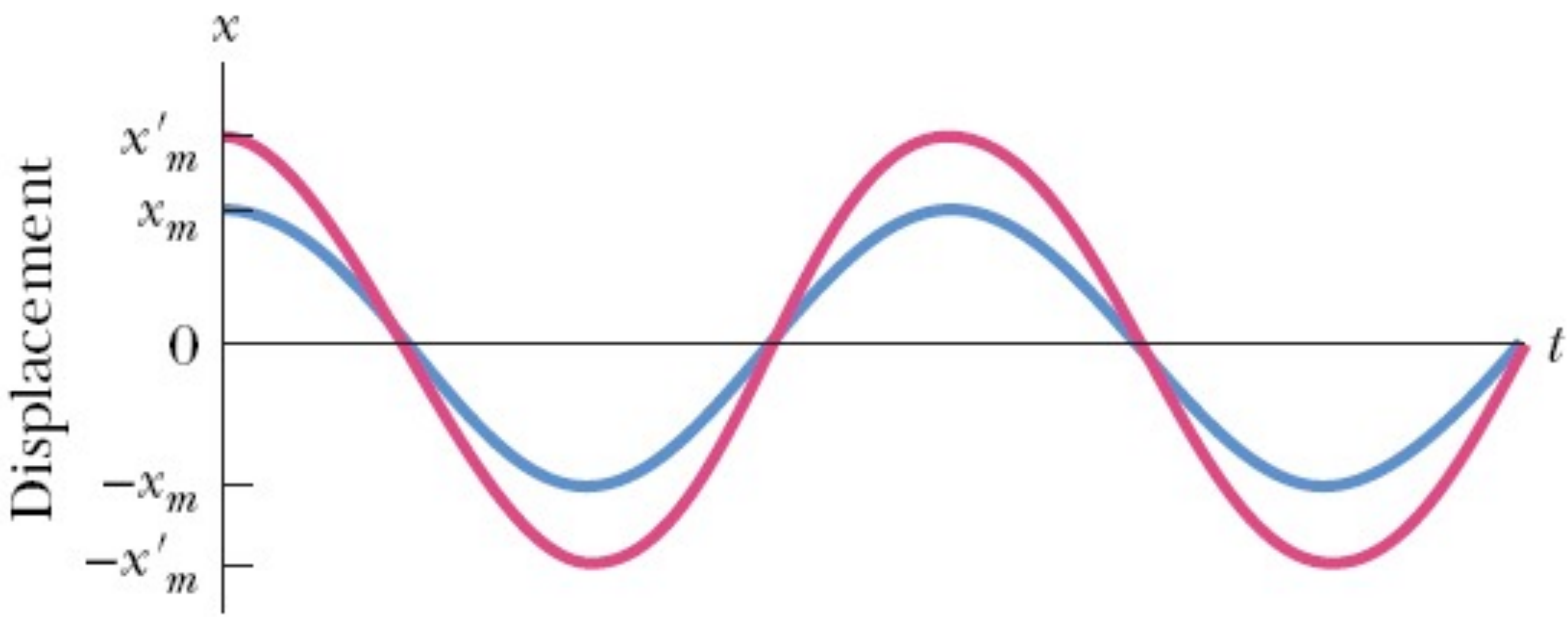
Any motion that repeats itself is *periodic* or *harmonic*.

If the motion is a sinusoidal function of time, it is called *simple harmonic motion* (SHM):

$$x(t) = x_m \cos(\omega t + \phi)$$

Here,

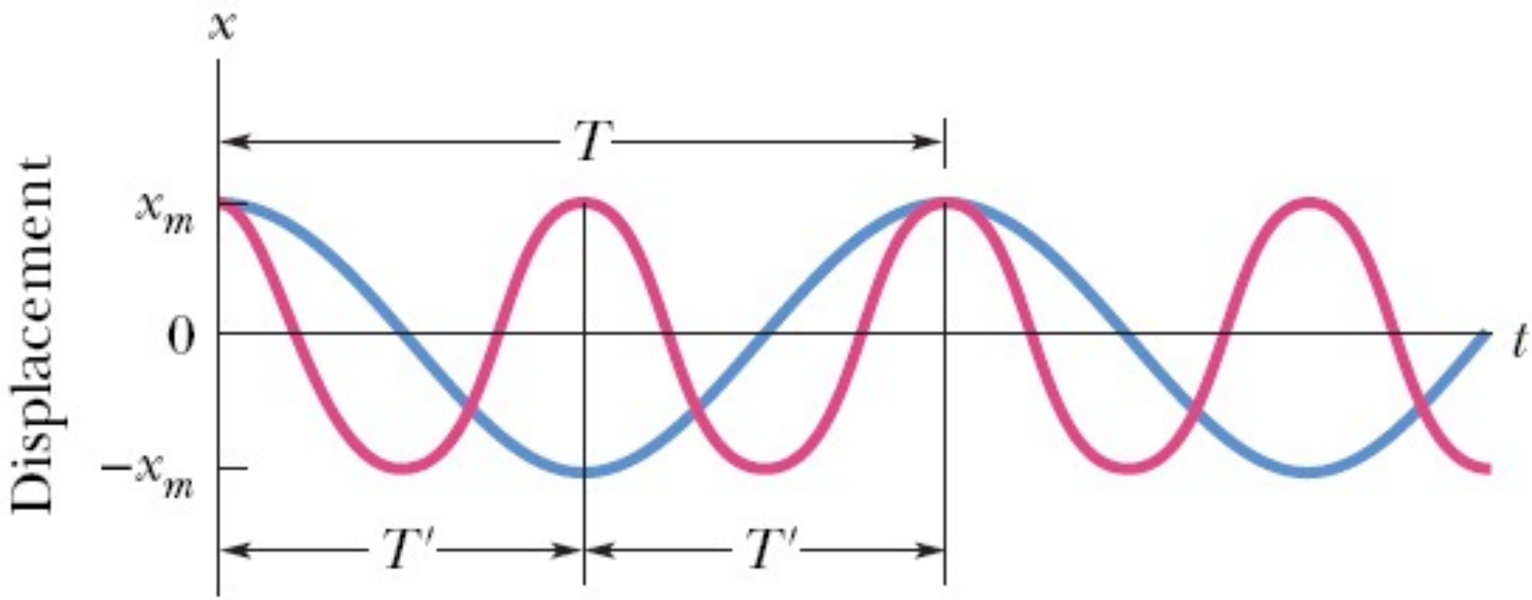
- x_m is the amplitude (maximum displacement, SI: m)
- ω is the angular frequency (SI: rad/s), and
- ϕ is the phase constant or phase angle (SI: rad)



$$x(t) = x_m \cos(\omega t)$$

$$x(t) = x'_m \cos(\omega t)$$

Displacement of two SHM systems that are different in amplitudes, but have the same angular frequency.



$$x(t) = x_m \cos(\omega t)$$

$$x(t) = x_m \cos(\omega' t)$$

Displacement of two SHM systems which are different in periods but have the same amplitude.
T is the *period* of the motion. Note that

$$\omega T = 2\pi$$

hence

$$\omega = \frac{2\pi}{T} = 2\pi f$$

More generally, oscillatory (or periodic) motion has

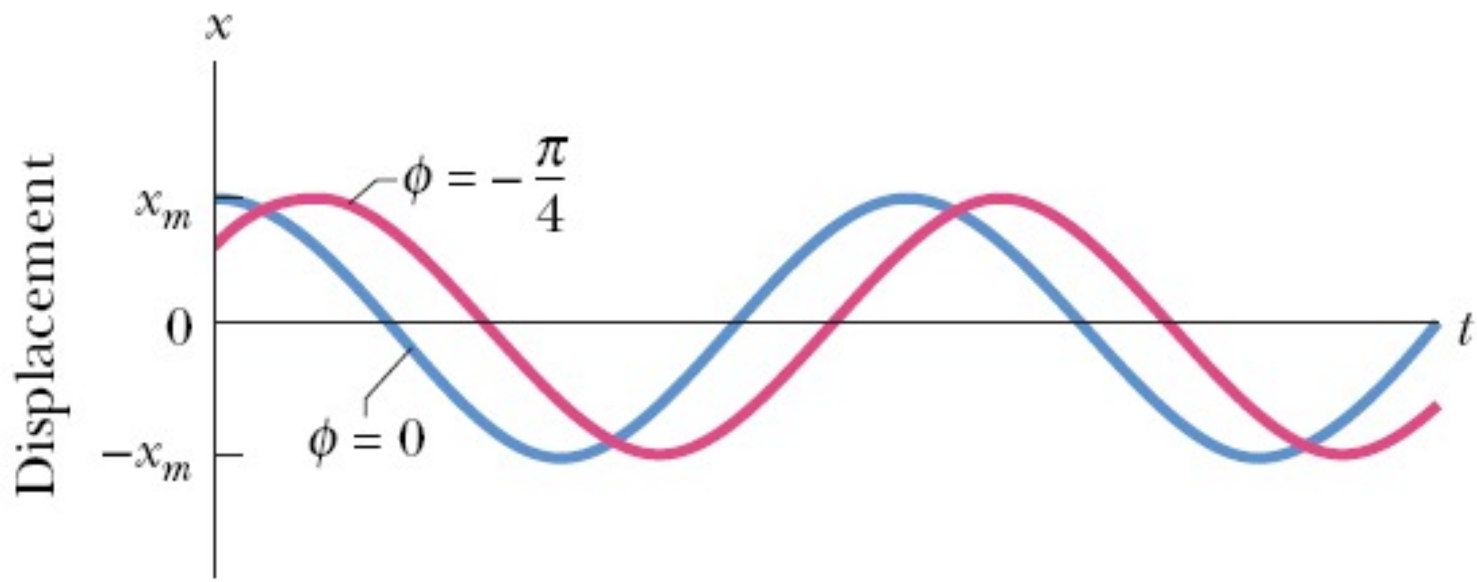
$$x(t) = x(t+T)$$

For simple harmonic motion this means

$$\begin{aligned} x_m \cos(\omega t) &= x_m \cos(\omega(t+T)) \\ &= x_m \cos(\omega t + \omega T)) \\ &= x_m \cos(\omega t + 2\pi)) \end{aligned}$$

where we have used

$$\omega T = 2\pi$$



$$x(t) = x_m \cos(\omega t) \qquad x(t) = x_m \cos(\omega t - \tfrac{\pi}{4})$$

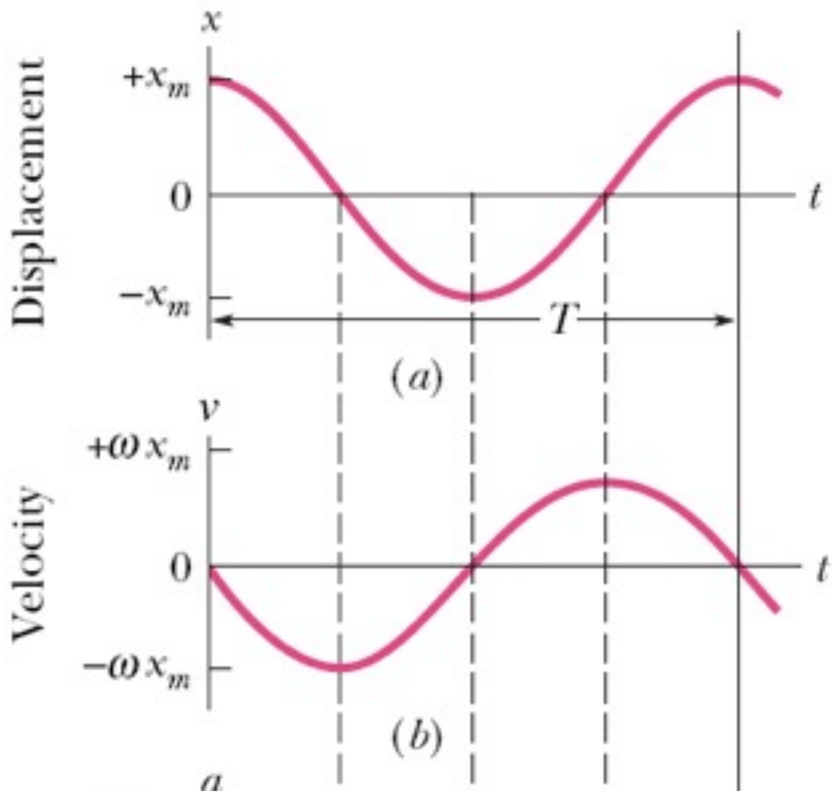
The figure plots the displacement of two SHM systems having the same period and amplitude, but different phase constants.

The velocity of SHM:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos(\omega t + \phi)\right]$$

$$\boxed{v(t) = -\omega x_m \sin(\omega t + \phi)}$$

The maximum value (amplitude) of velocity is $v_m = \omega x_m$.



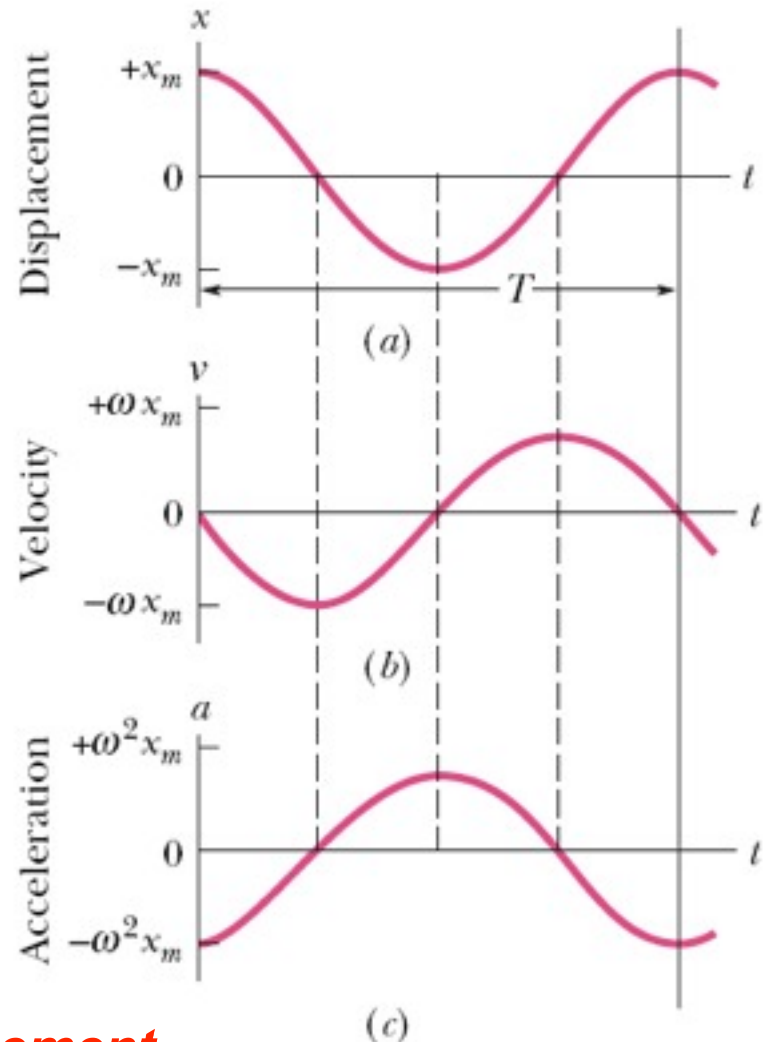
The acceleration of SHM is:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[-v_m \sin(\omega t + \phi)\right]$$

$$a(t) = -\omega v_m \cos(\omega t + \phi)$$

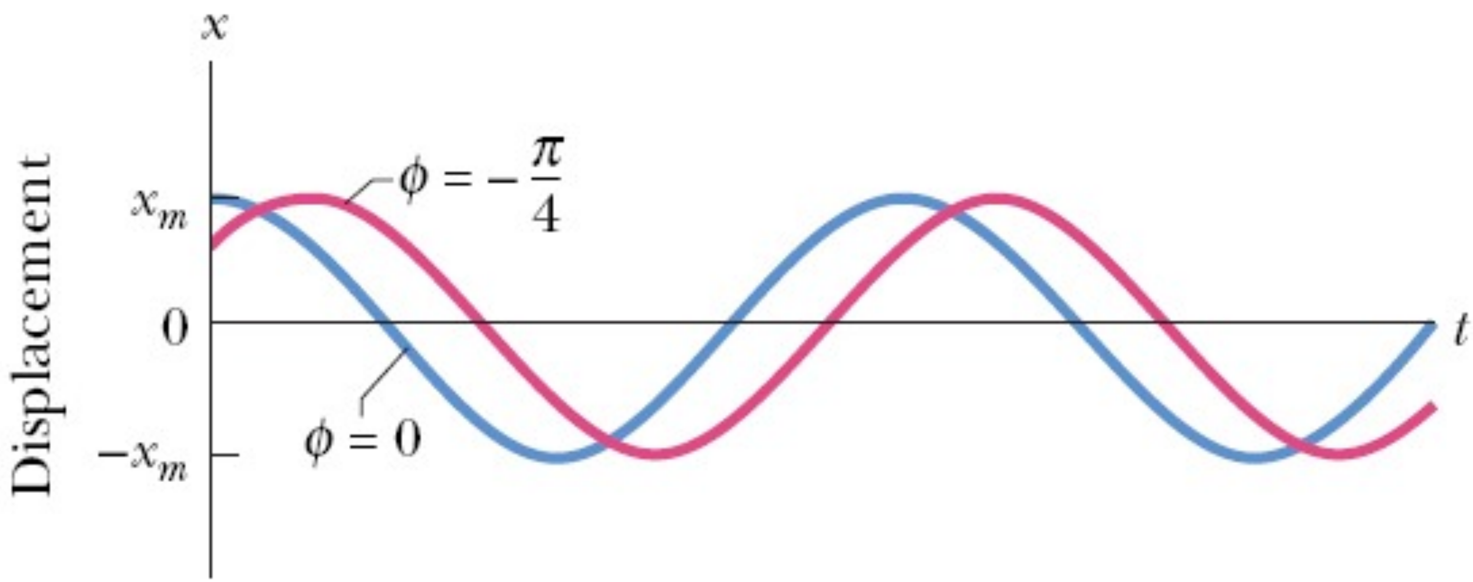
$$\boxed{a(t) = -\omega^2 x_m \cos(\omega t + \phi)}$$

The acceleration amplitude is $a_m = \omega^2 x_m$



In SHM $a(t)$ is proportional to the displacement but opposite in sign.

$$a(t) = -\omega^2 x(t)$$



$$x(t) = x_m \cos(\omega t) \qquad x(t) = x_m \cos(\omega t - \tfrac{\pi}{4})$$

The figure plots the displacement of two SHM systems having the same period and amplitude, but different phase constants.

The value of the phase constant, ϕ, and the amplitude x_m determine the value of the displacement and the velocity of the system at time $t = 0$ (and vice-versa!). Note that ω is ***not*** determined by initial conditions.

$$x(0) = x_m \cos(0) = x_m \qquad x(0) = x_m \cos(-\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}} x_m$$

$$v(0) = -\omega x_m \sin(0) = 0 \qquad v(0) = -\omega x_m \sin(-\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}} \omega x_m$$

15.3 Force Law for SHM

From Newton's 2nd law (linear motion):

$$F = ma = -m\omega^2 x$$

SHM is the motion executed by a system subject to a force that is proportional to the displacement of the system but opposite in sign.

Sounds familiar... Hooke's law is just like this!

$$F = -kx$$

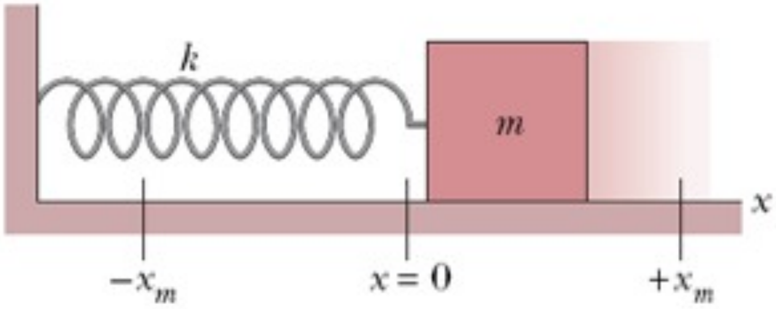
Comparing we see that a spring mass system executes SHM with angular frequency such that

$$m\omega^2 = k$$

So

$$\omega = \sqrt{\frac{k}{m}}$$

The block-spring system forms a linear SHM oscillator.



and $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Emphasize again: ω does not depend on initial displacement or velocity. It is a characteristic of the system (it is fixed for a given mass and spring, regardless on when the oscillations start, at what displacement and with what velocity).

Further connection: recall $a = F/m$ as differential equation:

Find $x(t)$ that satisfies: $\frac{d^2x}{dt^2} = -\frac{k}{m}x$

We have just shown that $x(t) = x_m \cos(\omega t + \phi)$ is a solution for **any** x_m and ϕ provided

$$\omega = \sqrt{\frac{k}{m}}$$

And to fully specify the solution (to give x_m and ϕ) we need initial conditions (initial displacement and velocity)

Example:

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

(b) What is the amplitude of the oscillation?

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

(d) What is the magnitude a_m of the maximum acceleration of the block?

(e) What is the phase constant ϕ for the motion?

(f) What is the displacement function $x(t)$ for the spring–block system?

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(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65 \text{ N/m}}{0.68 \text{ kg}}} = 9.78 \text{ rad/s}$$
$$\approx 9.8 \text{ rad/s.} \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78 \text{ rad/s}}{2\pi \text{ rad}} = 1.56 \text{ Hz} \approx 1.6 \text{ Hz.} \qquad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56 \text{ Hz}} = 0.64 \text{ s} = 640 \text{ ms.} \qquad \text{(Answer)}$$

(b) What is the amplitude of the oscillation?

KEY IDEA With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm.} \qquad \text{(Answer)}$$

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m}) = 1.1 \text{ m/s.} \quad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-4*a* and 15-4*b*, where you can see that the speed is a maximum whenever $x = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m}) = 11 \text{ m/s}^2. \quad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path. At those points, the force acting on the block has its maximum magnitude; compare Figs. 15-4*a* and 15-4*c*, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times.

(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos\phi. \tag{15-14}$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad.} \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-3. Substituting known quantities into that equation gives us

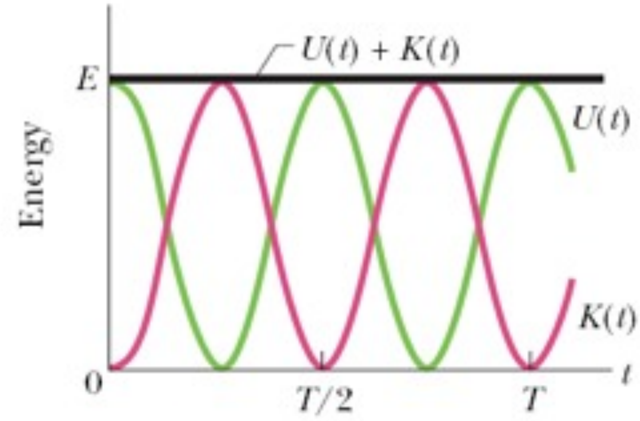
$$\begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ &= (0.11 \text{ m}) \cos[(9.8 \text{ rad/s})t + 0] \\ &= 0.11 \cos(9.8t), \qquad \text{(Answer)} \end{aligned}$$

where x is in meters and t is in seconds.

15.4: Energy in Simple Harmonic Motion

The potential energy of a spring-mass linear oscillator is associated entirely with the spring.

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^{\ 2}\cos^2(\omega t + \phi)$$

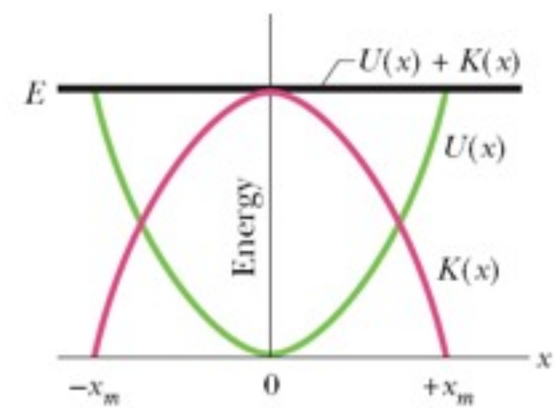


The kinetic energy of the system is associated entirely with the speed of the block.

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 x_m^{\ 2}\sin^2(\omega t + \phi) = \frac{1}{2}kx_m^{\ 2}\sin^2(\omega t + \phi)$$

The total mechanical energy of the system:

$$E = U + K = \frac{1}{2}kx_m^{\ 2}$$



Example, energy in SHM:

Many tall building have mass dampers, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

Suppose that the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring-block system?

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(a) What is the total mechanical energy E of the spring-block system?

KEY IDEA The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$\begin{aligned} k &= m\omega^2 = m(2\pi f)^2 \\ &= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2 \\ &= 1.073 \times 10^9 \text{ N/m.} \end{aligned}$$

We can now evaluate E as

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \\ &= 0 + \tfrac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2 \\ &= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J.} \qquad \text{(Answer)} \end{aligned}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$2.147 \times 10^7\ \text{J} = \tfrac{1}{2}(2.72 \times 10^5\ \text{kg})v^2 + 0,$$

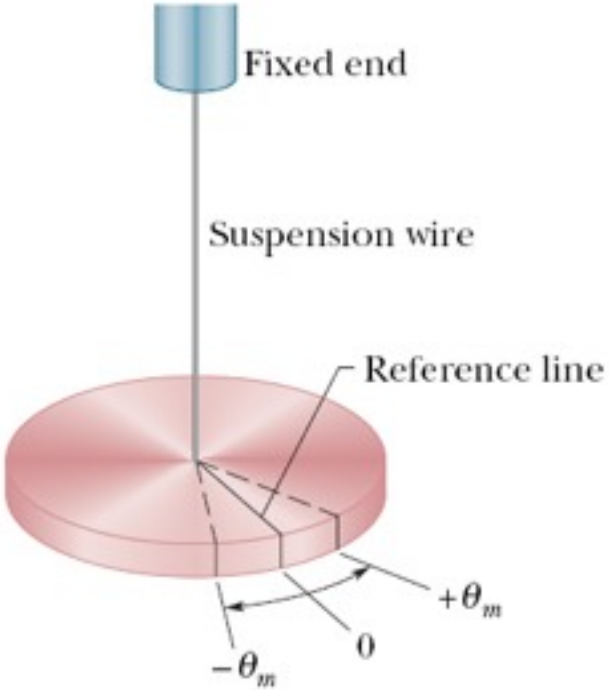
or

$$v = 12.6\ \text{m/s}. \qquad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.

15.5: An Angular Simple Harmonic Oscillator

The figure shows an example of angular SHM. In a torsion pendulum involves the twisting of a suspension wire as the disk oscillates in a horizontal plane.

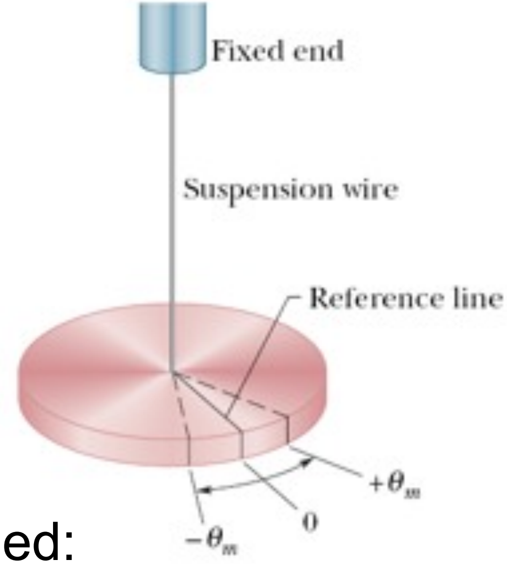


The torque τ associated with an angular displacement of θ is given by:

$$\tau = -\kappa\theta$$

κ is the torsion constant, and depends on the length, diameter, and material of the suspension wire.

15.5: An Angular Simple Harmonic Oscillator



To find the angular frequency we use Newton's 2nd Law for rotation of a rigid body about a fixed axis:

$$\tau = I\alpha$$

together with

$$\tau = -\kappa\theta$$

Compare the resulting equation to what we have learned:

$$I\alpha = -\kappa\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad \text{vs.} \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Since mass-block system has solution $x(t) = x_m \cos(\omega t + \phi)$ with $\omega = \sqrt{\frac{k}{m}}$

we have

$$\boxed{\theta(t) = \theta_m \cos(\omega t + \phi)}$$

with

$$\boxed{\omega = \sqrt{\frac{\kappa}{I}}}$$

Example, angular SHM:

Figure (a) shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object X, is then hung from the same wire, as in Fig. (b), and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?

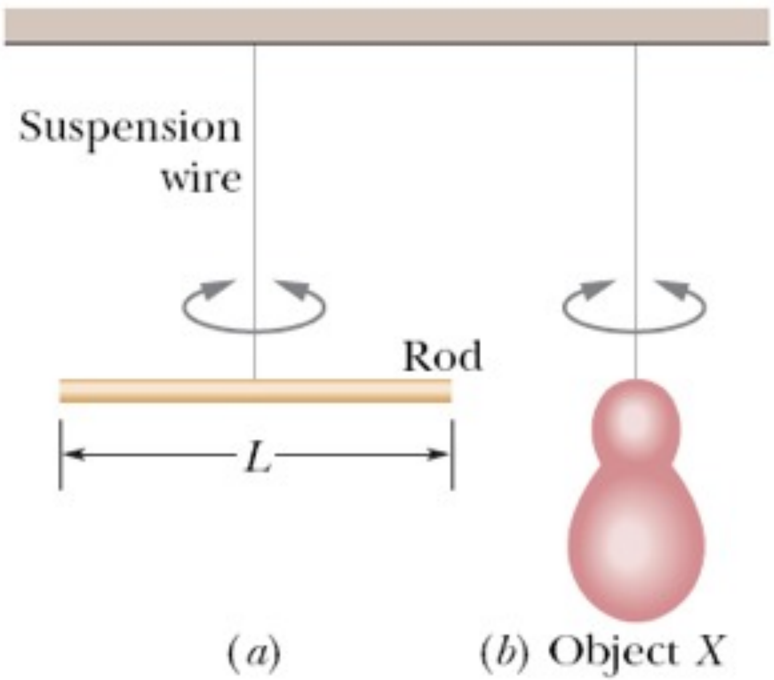
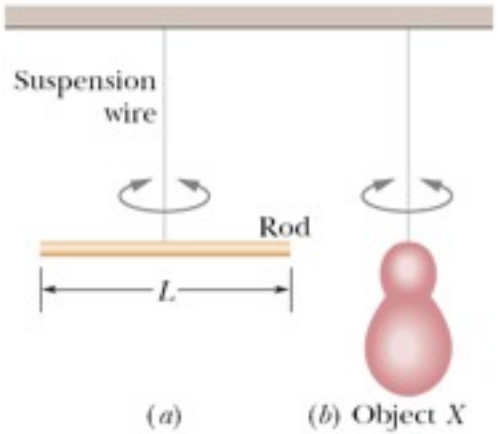


Figure (a) shows a thin rod whose length L is 12.4 cm and whose mass m is 135 g, suspended at its midpoint from a long wire. Its period T_a of angular SHM is measured to be 2.53 s. An irregularly shaped object, which we call object X, is then hung from the same wire, as in Fig. (b), and its period T_b is found to be 4.76 s. What is the rotational inertia of object X about its suspension axis?



Answer: The rotational inertia of either the rod or object X is related to the measured period. The rotational inertia of a thin rod about a perpendicular axis through its midpoint is $\frac{1}{12} mL^2$. Thus, we have, for the rod in Fig. (a),

$$I_a = \tfrac{1}{12}mL^2 = (\tfrac{1}{12})(0.135\text{ kg})(0.124\text{ m})^2$$
$$= 1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2.$$

Now let us write the periods, once for the rod and once for object X:

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b. The result is

$$I_b = I_a\frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2)\,\frac{(4.76\text{ s})^2}{(2.53\text{ s})^2}$$
$$= 6.12 \times 10^{-4}\text{ kg}\cdot\text{m}^2. \qquad \text{(Answer)}$$

15.6: Pendulums

In a *simple pendulum*, a particle of mass *m* is suspended from one end of an unstretchable massless string of length *L* that is fixed at the other end.

The restoring torque acting on the mass when its angular displacement is θ, is:

$$\tau = -LF_g \sin\theta$$

Using Newton's 2nd Law, $\tau = I\alpha$ gives

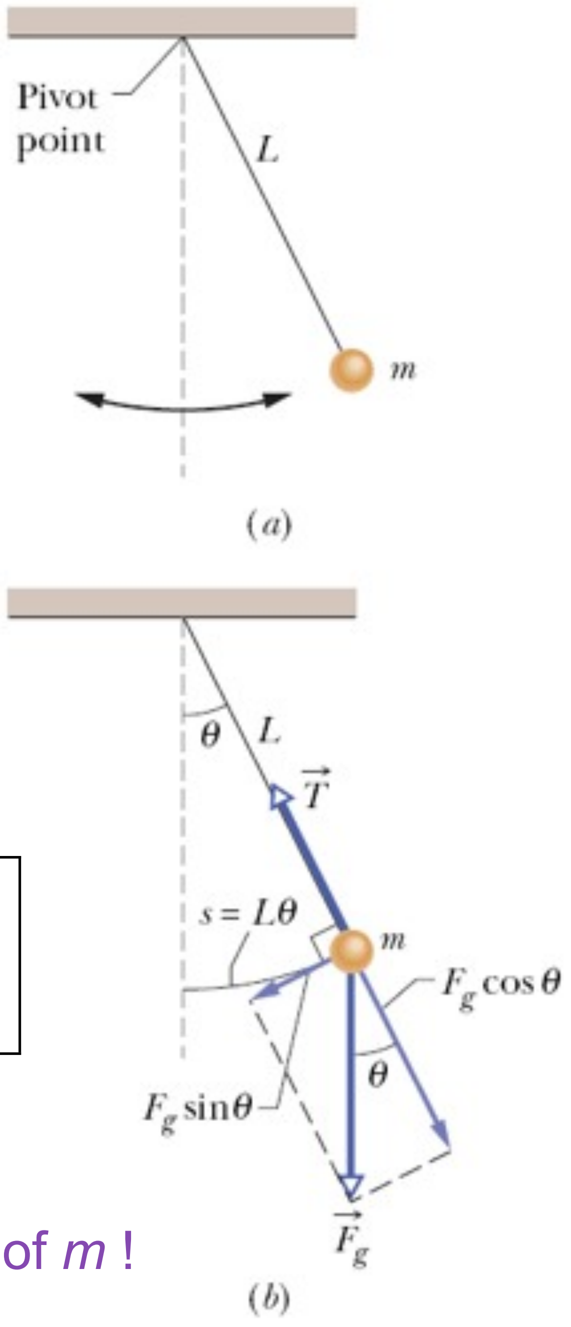
$$\alpha = -\frac{Lmg}{I}\sin\theta$$

Hard to solve in general. But for small θ, use $\sin\theta \approx \theta$

$$\theta(t) = \theta_m \cos(\omega t + \phi)$$ with $$\omega = \sqrt{\frac{Lmg}{I}}$$

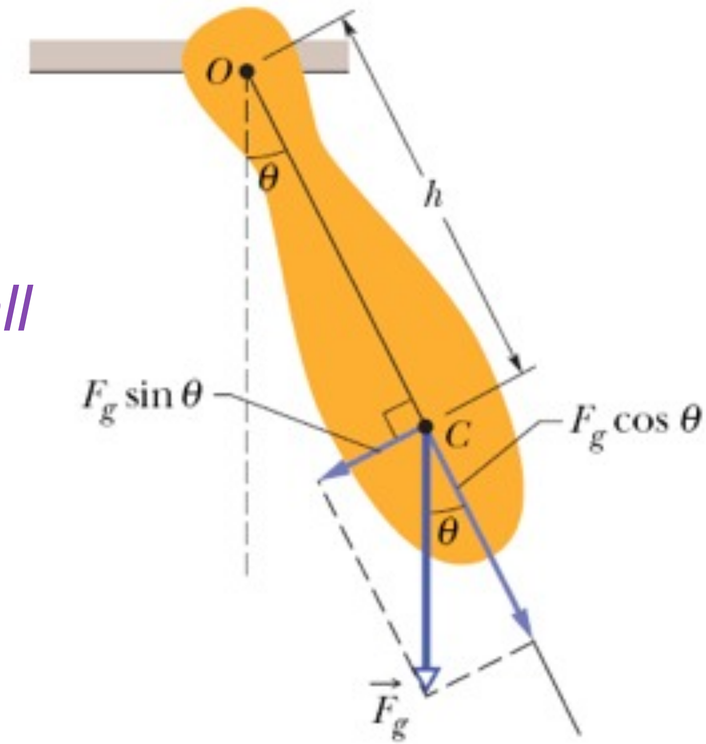
Moreover, since $I = mL^2$

$$\omega = \sqrt{\frac{g}{L}}$$ independent of *m* !



15.6: Pendulums

A *physical pendulum* can have a complicated distribution of mass. If the center of mass, *C*, is at a distance of *h* from the pivot point (figure), then for *small angular amplitudes*, the motion is simple harmonic.



The analysis is as before:

$$\alpha = -\frac{Lmg}{I}\sin\theta$$

Here, *I* is the rotational inertia of the pendulum about *O*.

Hard to solve in general. But for small θ, use $\sin\theta \approx \theta$

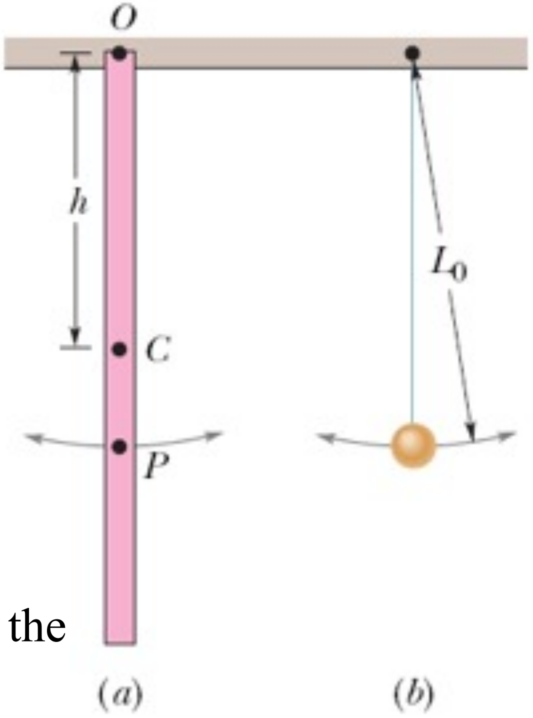
$$\theta(t) = \theta_m \cos(\omega t + \phi) \quad \text{with} \quad \boxed{\omega = \sqrt{\frac{mgh}{I}}}$$

In the **small-angle approximation** we can assume that $\theta << 1$ and use the approximation $\sin\theta \approx \theta$. Let us investigate up to what angle θ is the approximation reasonably accurate?

θ (degrees)	θ (radians)	$\sin\theta$
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: If we keep $\theta < 10°$ we make less than 1 % error.

Example, pendulum:



In Fig. *a,* a meter stick swings about a pivot point at one end, at distance h from the stick's center of mass.
(a)What is the period of oscillation T?

KEY IDEA: The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

Calculations: The period for a physical pendulum depends on the rotational inertia, I, of the stick about the pivot point. We can treat the stick as a uniform rod of length L and mass m. Then $I = 1/3\, mL^2$, where the distance h is L.
Therefore,

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg(\frac{1}{2}L)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

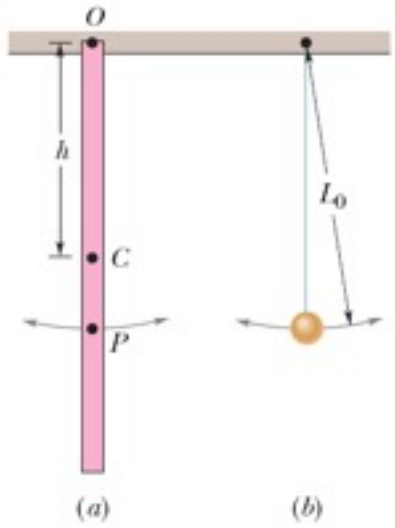
$$= 2\pi\sqrt{\frac{(2)(1.00\text{ m})}{(3)(9.8\text{ m/s}^2)}} = 1.64\text{ s.} \qquad \text{(Answer)}$$

Note the result is independent of the pendulum's mass *m*.

Example, pendulum, continued:

(b) What is the distance L_0 between the pivot point O of the stick and the center of oscillation of the stick? (That is, the length of a simple pendulum with the same period of oscillation).

Calculations: We want the length L_0 of the simple pendulum (drawn in Fig. *b)* that has the same period as the physical pendulum (the stick) of Fig. *a*.



$$T = 2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}.$$

$$L_0 = \tfrac{2}{3}L = \left(\tfrac{2}{3}\right)(100\text{ cm}) = 66.7\text{ cm}. \quad \text{(Answer)}$$

15.7: Simple Harmonic Motion and Uniform Circular Motion

Consider a reference particle *P'* moving in uniform circular motion with constant angular velocity ω.

The projection of the particle on the *x*-axis is a point *P*, describing motion given by:

$$x(t) = x_m \cos(\omega t + \phi)$$

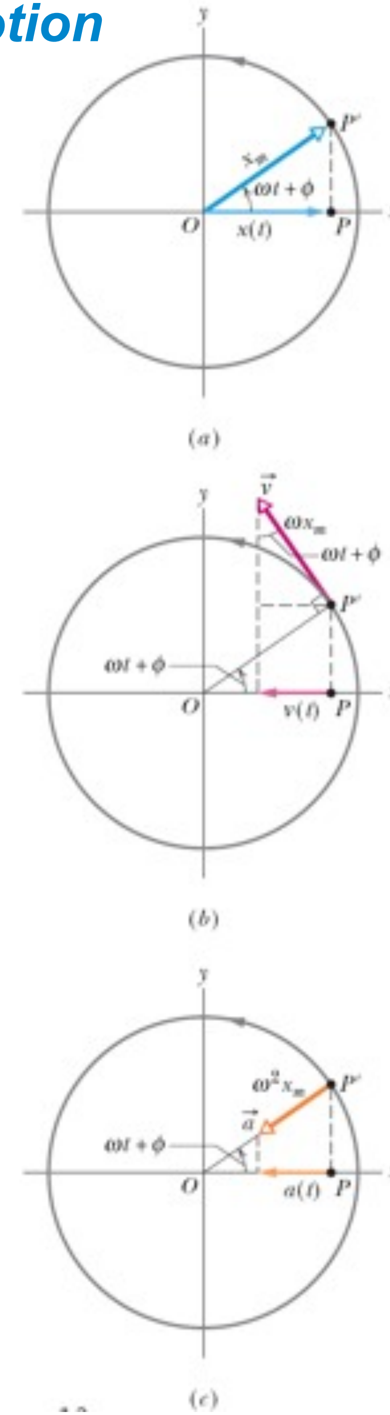
This is the displacement equation of SHM.
The *x*-component of velocity and acceleration follow:

$$v_x(t) = -\omega x_m \sin(\omega t + \phi)$$

$$a_x(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

Note that x_m is just the radius *R* of the circle, and the angular velocity is just the angular frequency of SHM.

Similarly for *y*-component of velocity and acceleration:



15.7: Simple Harmonic Motion and Uniform Circular Motion

Similarly for *y*-component of velocity and acceleration:

$$y(t) = y_m \sin(\omega t + \phi)$$
$$v_y(t) = \omega y_m \cos(\omega t + \phi)$$
$$a_y(t) = -\omega^2 y_m \sin(\omega t + \phi)$$

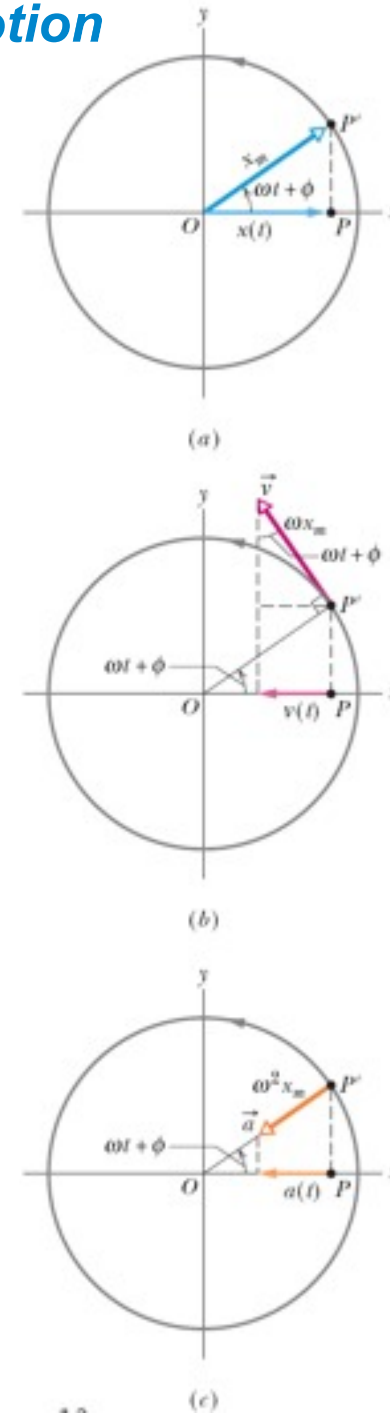
Using $x_m = y_m = R$ we then have

$$y(t)^2 + x(t)^2 = R^2(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)) = R^2$$

and

$$v = \sqrt{v_x^2(t) + v_y^2(t)} = \omega R$$
$$a = \sqrt{a_x^2(t) + a_y^2(t)} = \omega^2 R$$

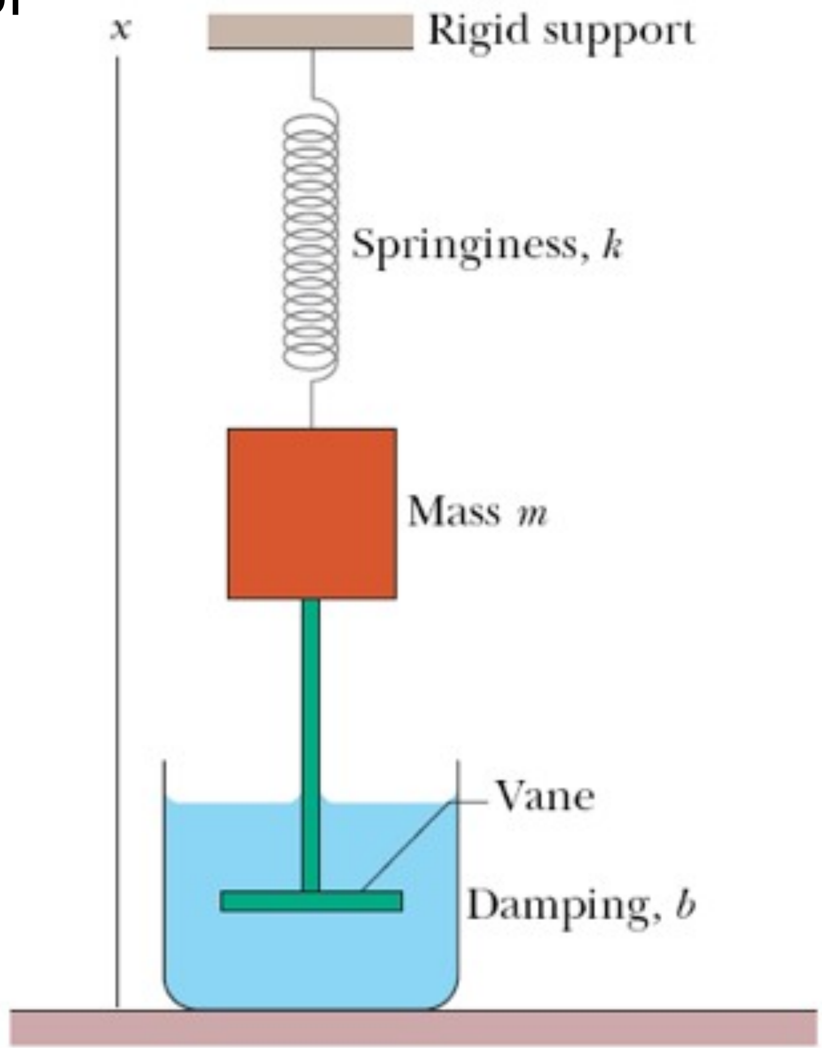
as expected for circular motion



15.8: Damped Simple Harmonic Motion

In a damped oscillation, the motion of the oscillator is reduced by an external damping force.

Example: A block of mass *m* oscillates vertically on a spring on a spring, with spring constant, *k*. From the block a rod extends to a vane which is submerged in a liquid. The liquid provides the external damping force, F_d.



15.8: Damped Simple Harmonic Motion

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -bv$$

From Newton's 2nd law, the following DE results:

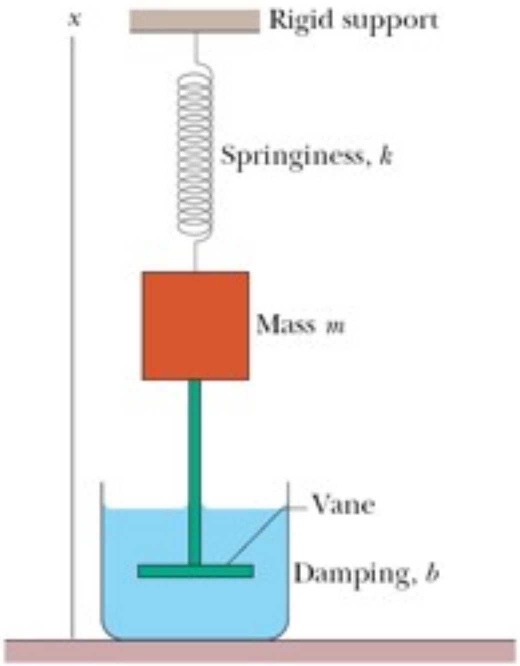
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The solution is:

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$

Here ω' is the angular frequency, and is given by:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



Let's check

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

$$x(t)=x_m e^{\frac{-bt}{2m}}\cos(\omega' t+\phi)$$

$$\frac{dx(t)}{dt}=x_m\left(\frac{-b}{2m}\right)e^{\frac{-bt}{2m}}\cos(\omega' t+\phi)-\omega' x_m e^{\frac{-bt}{2m}}\sin(\omega' t+\phi)$$

$$\frac{d^2x(t)}{dt^2}=x_m\left(\frac{-b}{2m}\right)^2 e^{\frac{-bt}{2m}}\cos(\omega' t+\phi)-2\omega' x_m\left(\frac{-b}{2m}\right)e^{\frac{-bt}{2m}}\sin(\omega' t+\phi)-\omega'^2 x_m e^{\frac{-bt}{2m}}\cos(\omega' t+\phi)$$

Adding (taking out the common factor of $x_m e^{-bt/2m}$):

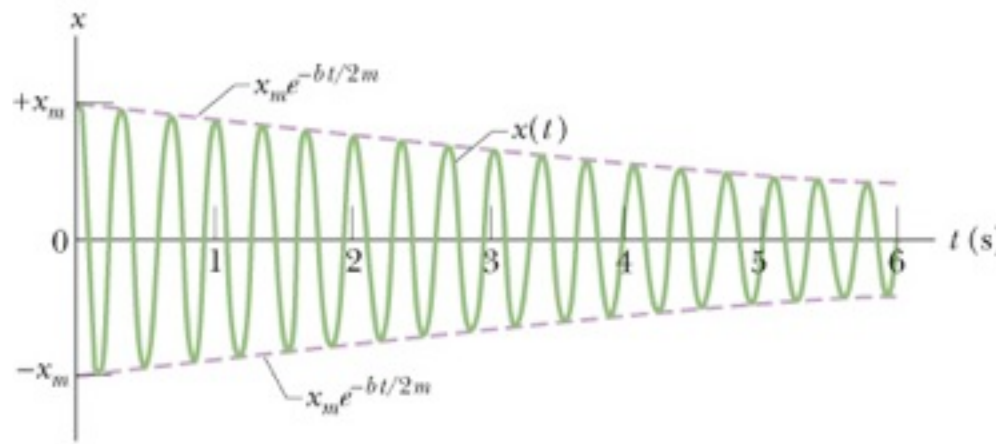
$$m\left[\left\{\left(\frac{b}{2m}\right)^2-\omega'^2\right\}\cos(\omega' t+\phi)-2\omega'\left(\frac{-b}{2m}\right)\sin(\omega' t+\phi)\right]$$
$$+b\left[\left(\frac{-b}{2m}\right)\cos(\omega' t+\phi)-\omega'\sin(\omega' t+\phi)\right]+k\left[\cos(\omega' t+\phi)\right]$$

$$=m\left(\frac{k}{m}-\frac{b^2}{4m^2}-\omega'^2\right)\cos(\omega' t+\phi)$$

Which vanishes provided $\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$

The figure shows the displacement function *x*(*t*) for the damped oscillator described before. The amplitude decreases as $x_m \exp(-bt/2m)$ with time.



The mechanical energy is not constant, some is being lost to thermal energy. If the damping is small

$$E_{\text{mec}}(t) \approx \tfrac{1}{2} k x_m^2 e^{-bt/m}$$

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Note that this does not work if $\frac{k}{m} - \frac{b^2}{4m^2} \leq 0$

When

$$\frac{k}{m} - \frac{b^2}{4m^2} < 0$$

the motion is *over-damped* $x(t) = Ae^{-\gamma_+ t} + Be^{-\gamma_- t}$

and when $\frac{k}{m} - \frac{b^2}{4m^2} = 0$

the motion is *critically-damped* $x(t) = (A + Bt)e^{-\gamma t}$

Exercise: show that when $\frac{k}{m} - \frac{b^2}{4m^2} < 0$

the solution is $x(t) = Ae^{-\gamma_+ t} + Be^{-\gamma_- t}$

and find the expressions for in terms of *k*, *b* and *m*.

Answer:

$$\frac{dx(t)}{dt} = -\gamma_+ Ae^{-\gamma_+ t} - \gamma_- Be^{-\gamma_- t}$$

$$\frac{d^2x(t)}{dt^2} = \gamma_+^2 Ae^{-\gamma_+ t} + \gamma_-^2 Be^{-\gamma_- t}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = (m\gamma_+^2 - b\gamma_+ + k)Ae^{-\gamma_+ t} + (m\gamma_-^2 - b\gamma_- + k)Be^{-\gamma_- t}$$

this vanishes if

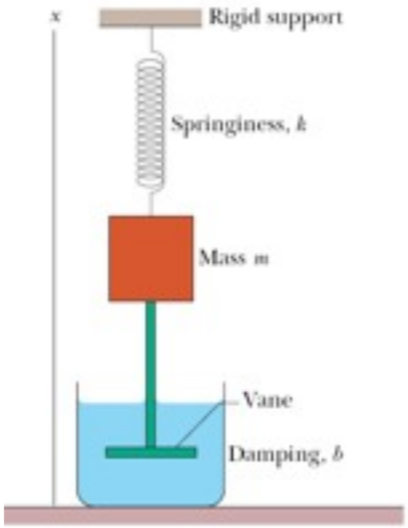
$$m\gamma_\pm^2 - b\gamma_\pm + k = 0$$

Hence:

$$\gamma_\pm = \frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

Example, damped SHM:

For the damped oscillator in the figure, m = 250 g, k = 85 N/m, and b =70 g/s.



(a) What is the period of the motion?

KEY IDEA Because $b \ll \sqrt{km} = 4.6$ kg/s, the period is approximately that of the undamped oscillator.

Calculation: From Eq. 15-13, we then have

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s.} \quad \text{(Answer)}$$

Example, damped SHM, continued:

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

KEY IDEA The amplitude at time t is $x_m e^{-bt/2m}$.

Calculations: The amplitude has the value x_m at $t = 0$. Thus, we must find the value of t for which

$$x_m e^{-bt/2m} = \tfrac{1}{2} x_m.$$

Canceling x_m and taking the natural logarithm of the equation that remains, we have $\ln \frac{1}{2}$ on the right side and

$$\ln(e^{-bt/2m}) = -bt/2m$$

$$t = \frac{-2m \ln \frac{1}{2}}{b} = \frac{-(2)(0.25\ \text{kg})(\ln \frac{1}{2})}{0.070\ \text{kg/s}}$$

$$= 5.0\ \text{s}. \qquad \text{(Answer)}$$

Because $T = 0.34$ s, this is about 15 periods of oscillation.

Example, damped SHM, continued:

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

KEY IDEA From Eq. 15-44, the mechanical energy at time t is $\frac{1}{2}kx_m^2\, e^{-bt/m}$.

Calculations: The mechanical energy has the value $\frac{1}{2}kx_m^2$ at $t = 0$. Thus, we must find the value of t for which

$$\tfrac{1}{2}kx_m^2\, e^{-bt/m} = \tfrac{1}{2}\left(\tfrac{1}{2}kx_m^2\right).$$

If we divide both sides of this equation by $\frac{1}{2}kx_m^2$ and solve for t as we did above, we find

$$t = \frac{-m \ln \frac{1}{2}}{b} = \frac{-(0.25\text{ kg})(\ln \frac{1}{2})}{0.070\text{ kg/s}} = 2.5\text{ s.} \quad \text{(Answer)}$$

15.9: Forced oscillations and Resonance

When the oscillator is subjected to an external force that is periodic, the oscillator will exhibit forced/driven oscillations.

Example: A swing in motion is pushed with a periodic force of angular frequency, ω_d.

There are two frequencies involved in a forced driven oscillator:

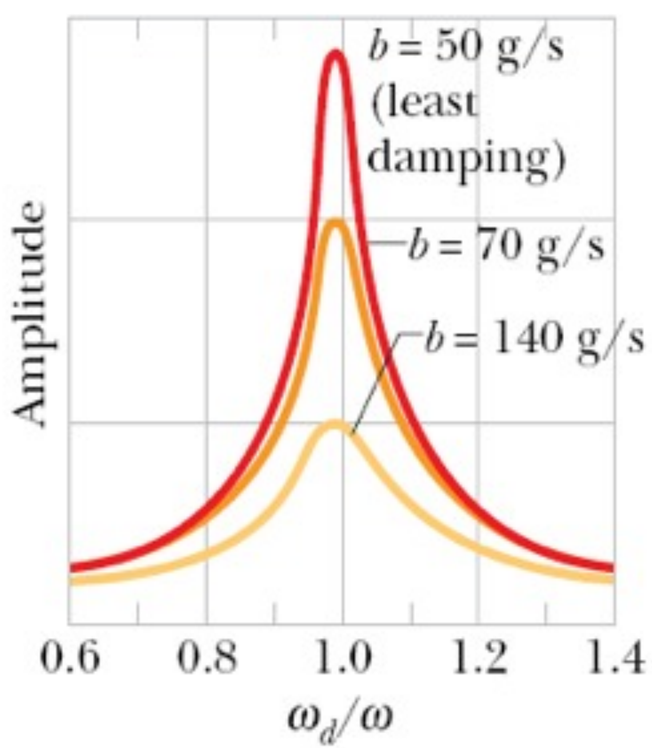
I. ω, the natural angular frequency of the oscillator, without the presence of any external force, and

II. ω_d, the angular frequency of the applied external force.

15.9: Forced oscillations and resonance

<u>Resonance</u> will occur in the forced oscillation if the natural angular frequency, ω, is equal to ω_d.

This is the condition when the velocity amplitude is the largest, and to some extent, also when the displacement amplitude is the largest. The adjoining figure plots displacement amplitude as a function of the ratio of the two frequencies.



Example: Mexico City collapsed in September 1985 when a major earthquake hit the western coast of Mexico. The seismic waves of the earthquake was close to the natural frequency of many buildings