

Chapter 12

Equilibrium and Elasticity



Tokyo Tower: tallest structure in the world

12.1 *Physics and Equilibrium*

- What allows structures to be stable in spite of forces acting on them?
- Under what conditions do objects deform?

12.2 Equilibrium

The two requirements for the state of **equilibrium** are:

1. The net force on the structure vanishes (the linear momentum of its center of mass is constant).
2. The net torque about any point vanishes (the angular momentum about its center of mass, or about any other point, is also constant).

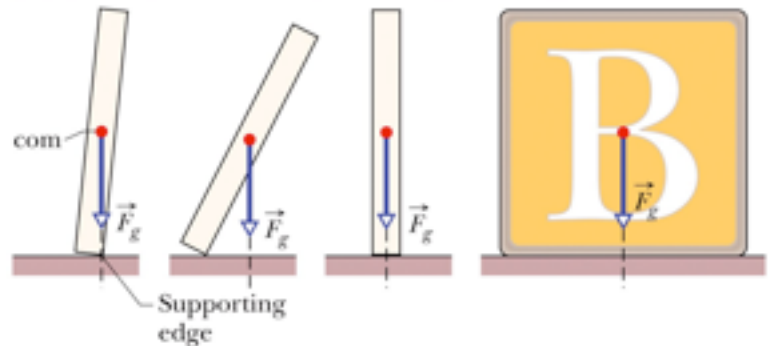
The balancing rock of Fig. 12-1 is an example of an object that is in **static equilibrium**. That is, in this situation the linear and angular momenta in the above requirements vanish;



Fig. 12-1 A balancing rock. Although its perch seems precarious, the rock is in static equilibrium. (*Symon Lobsang/ Photis/ Jupiter Images Corp.*)

12.3 The Requirements of Equilibrium

To tip the block, the center of mass must pass over the supporting edge.



1. The vector sum of all the external forces that act on the body must be zero.
2. The vector sum of all external torques that act on the body, measured about *any* possible point, must also be zero.



$$\vec{F}_{\text{net}} = 0 \quad (\text{balance of forces}).$$



$$\vec{\tau}_{\text{net}} = 0 \quad (\text{balance of torques}).$$



Balance of forces

$$F_{\text{net},x} = 0$$

$$F_{\text{net},y} = 0$$

$$F_{\text{net},z} = 0$$

Balance of torques

$$\tau_{\text{net},x} = 0$$

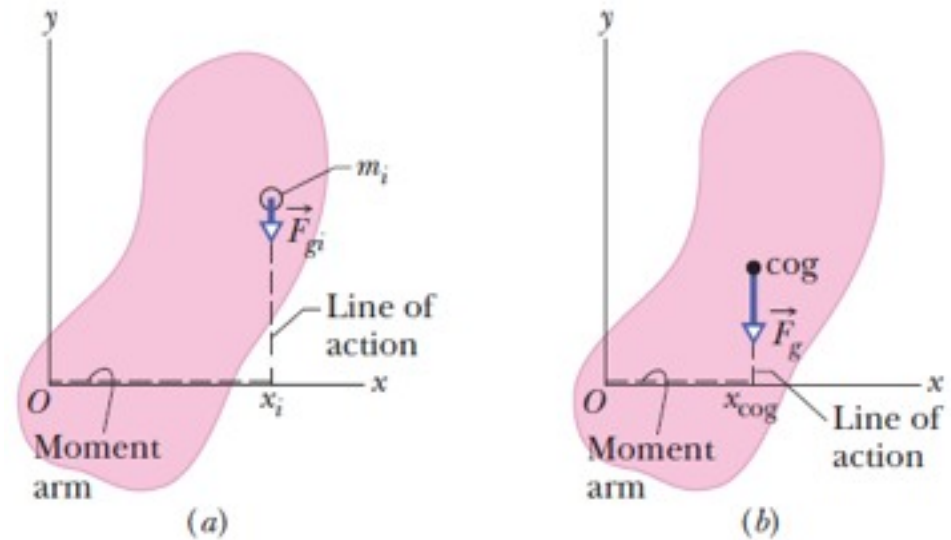
$$\tau_{\text{net},y} = 0$$

$$\tau_{\text{net},z} = 0$$

12.4 The Center of Gravity

We have been calculating the effect of gravity on a rigid body as if the force were acting at a single point, the com.

We can always find some point on which the weight acts giving the same effect as the distributed gravitational force:



The gravitational force \vec{F}_g on a body effectively acts at a single point, called the **center of gravity (cog)** of the body.

If \vec{g} is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

Proof:

define cog by requiring that $\vec{\tau} = \vec{r}_{\text{cog}} \times \vec{F}_g$

$$\vec{\tau} = \int \vec{r} \times (dm \vec{g}) = \left(\int \vec{r} dm \right) \times \vec{g} = (M \vec{r}_{\text{com}}) \times \vec{g} = \vec{r}_{\text{com}} \times (M \vec{g})$$

which shows that cog = com (provided the gravitational acceleration is uniform!)

Example, static equilibrium:

In Fig. 12-5a, a ladder of length $L = 12$ m and mass $m = 45$ kg leans against a slick wall (that is, there is no friction between the ladder and the wall). The ladder's upper end is at height $h = 9.3$ m above the pavement on which the lower end is supported (the pavement is not frictionless). The ladder's center of mass is $L/3$ from the lower end, along the length of the ladder. A firefighter of mass $M = 72$ kg climbs the ladder until her center of mass is $L/2$ from the lower end. What then are the magnitudes of the forces on the ladder from the wall and the pavement?

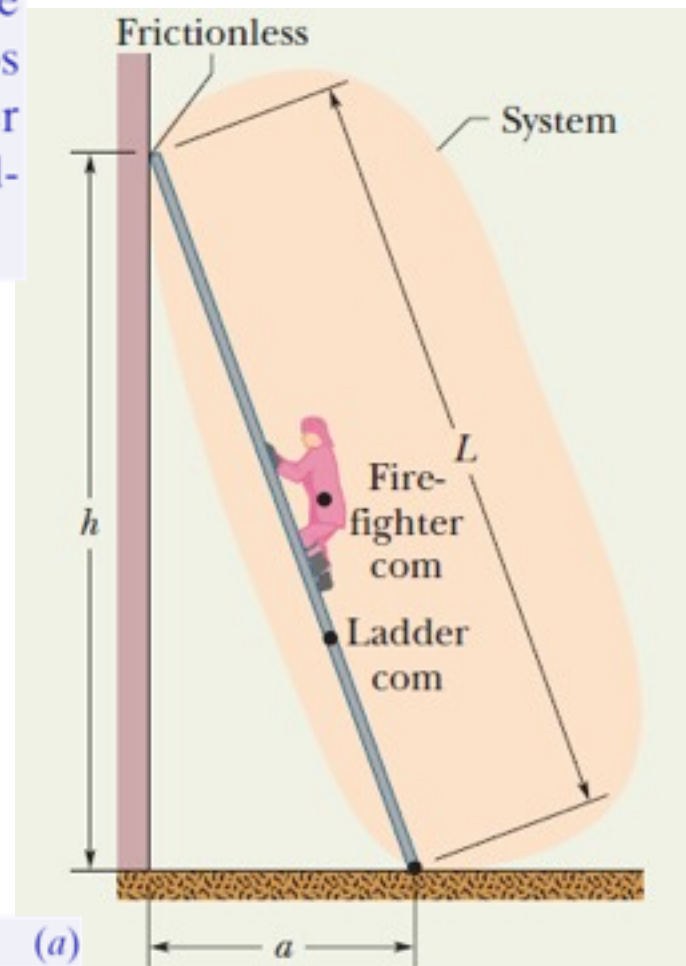


Fig. 12-5 (a)

Example, static equilibrium:

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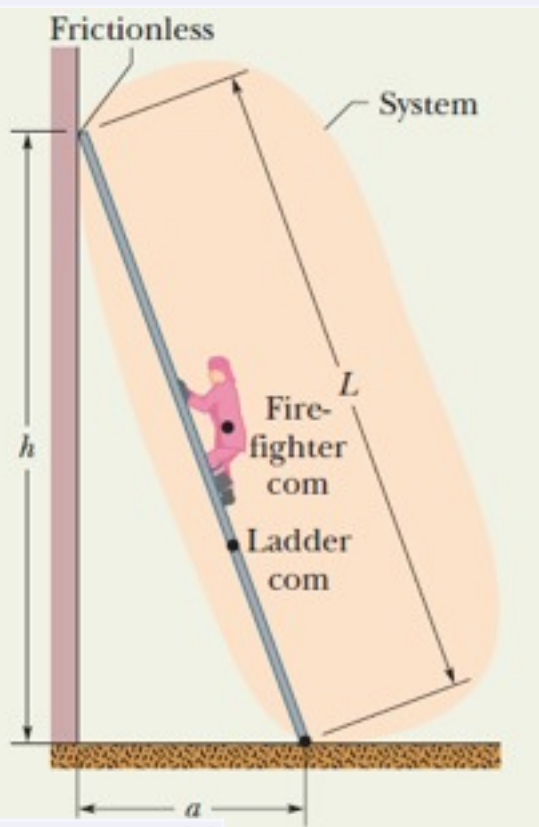


Fig. 12-5 (a)

$$\tau_{\text{net},z} = 0$$



choose z-axis out of the plane of the slide through bottom of ladder

$$-(h)(F_w) + (a/2)(Mg) + (a/3)(mg) + (0)(F_{px}) + (0)(F_{py}) = 0. \quad (12-18)$$

$$a = \sqrt{L^2 - h^2} = 7.58 \text{ m.}$$

$$\begin{aligned} F_w &= \frac{ga(M/2 + m/3)}{h} \\ &= \frac{(9.8 \text{ m/s}^2)(7.58 \text{ m})(72/2 \text{ kg} + 45/3 \text{ kg})}{9.3 \text{ m}} \\ &= 407 \text{ N} \approx 410 \text{ N.} \end{aligned} \quad (\text{Answer})$$

$$F_{\text{net},x} = 0 \quad \Rightarrow$$

$$F_w - F_{px} = 0,$$

$$F_{px} = F_w = 410 \text{ N.}$$

$$F_{\text{net},y} = 0 \quad \Rightarrow$$

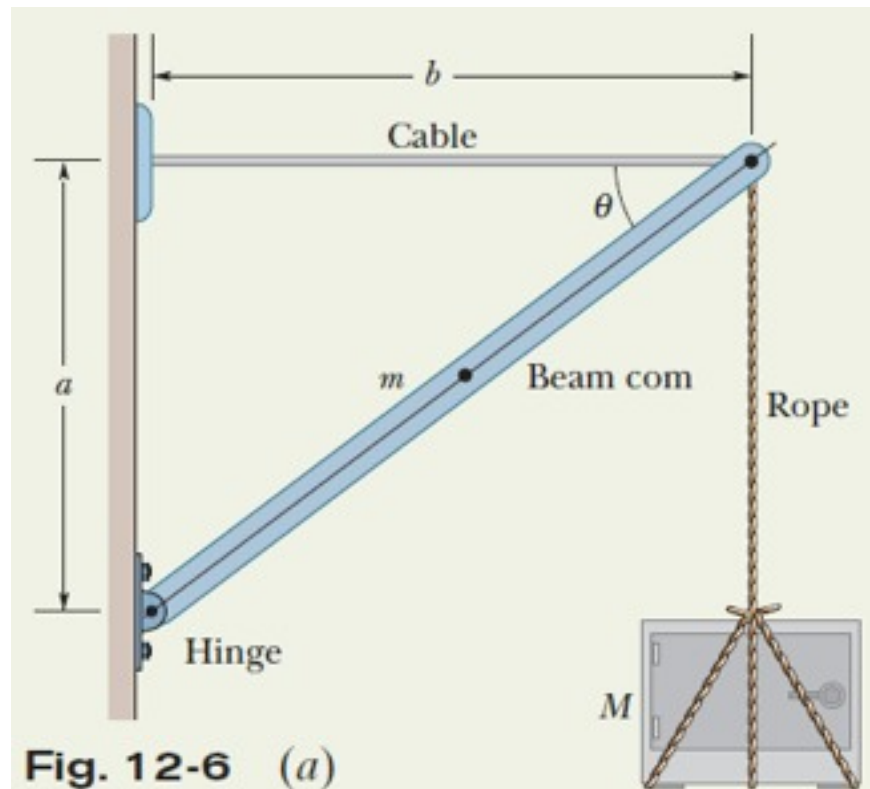
$$F_{py} - Mg - mg = 0,$$

$$\begin{aligned} F_{py} &= (M + m)g = (72 \text{ kg} + 45 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 1146.6 \text{ N} \approx 1100 \text{ N.} \end{aligned}$$

Example, static equilibrium:

Figure 12-6a shows a safe (mass $M = 430$ kg), hanging by a rope (negligible mass) from a boom ($a = 1.9$ m and $b = 2.5$ m) that consists of a uniform hinged beam ($m = 85$ kg) and horizontal cable (negligible mass).

(a) What is the tension T_c in the cable? In other words, what is the magnitude of the force \vec{T}_c on the beam from the cable?



(b)

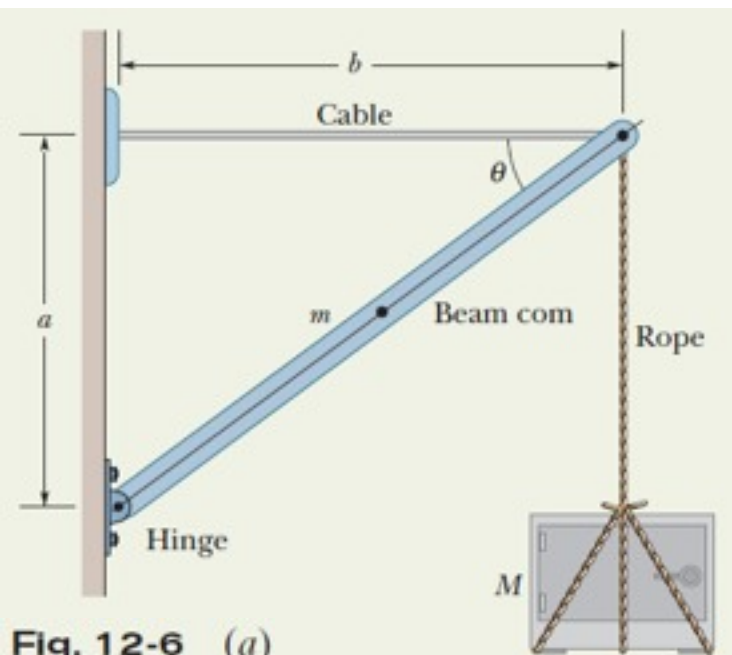
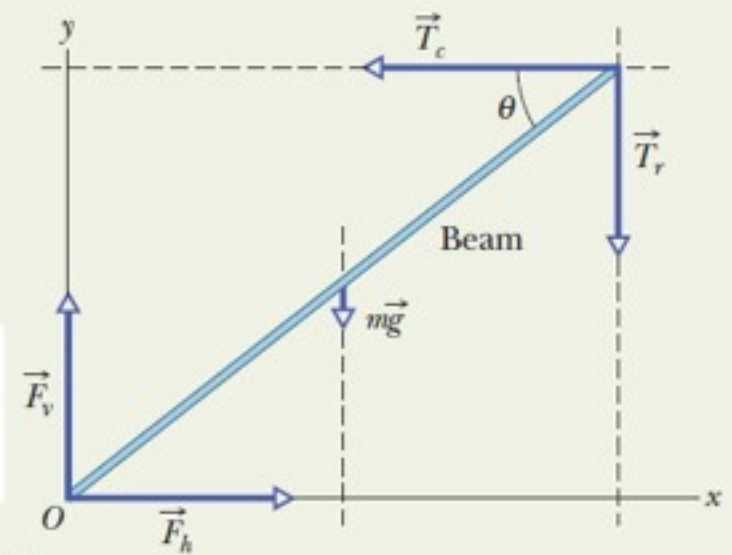


Fig. 12-6 (a)



(b)

Figure 12-6a shows a safe (mass $M = 430$ kg), hanging by a rope (negligible mass) from a boom ($a = 1.9$ m and $b = 2.5$ m) that consists of a uniform hinged beam ($m = 85$ kg) and horizontal cable (negligible mass).

(a) What is the tension T_c in the cable? In other words, what is the magnitude of the force \vec{T}_c on the beam from the cable?

Calculations: Let us start with Eq. 12-9 ($\tau_{\text{net},z} = 0$). Note that we are asked for the magnitude of force \vec{T}_c and not of forces \vec{F}_h and \vec{F}_v acting at the hinge, at point O . To eliminate \vec{F}_h and \vec{F}_v from the torque calculation, we should calculate torques about an axis that is perpendicular to the figure at point O . Then \vec{F}_h and \vec{F}_v will have moment arms of zero. The lines of action for \vec{T}_c , \vec{T}_r , and $m\vec{g}$ are dashed in Fig. 12-6b. The corresponding moment arms are a , b , and $b/2$.

Writing torques in the form of $r_{\perp}F$ and using our rule about signs for torques, the balancing equation $\tau_{\text{net},z} = 0$ becomes

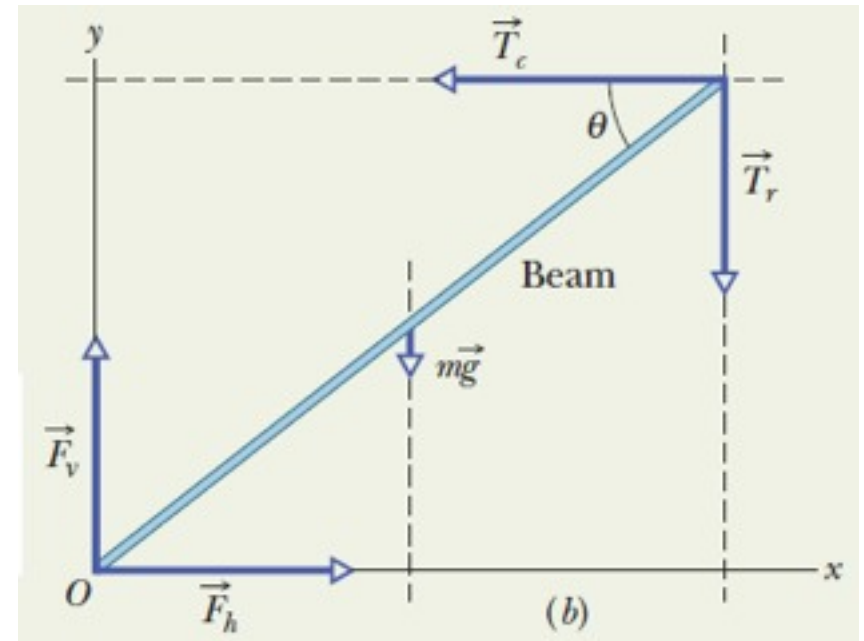
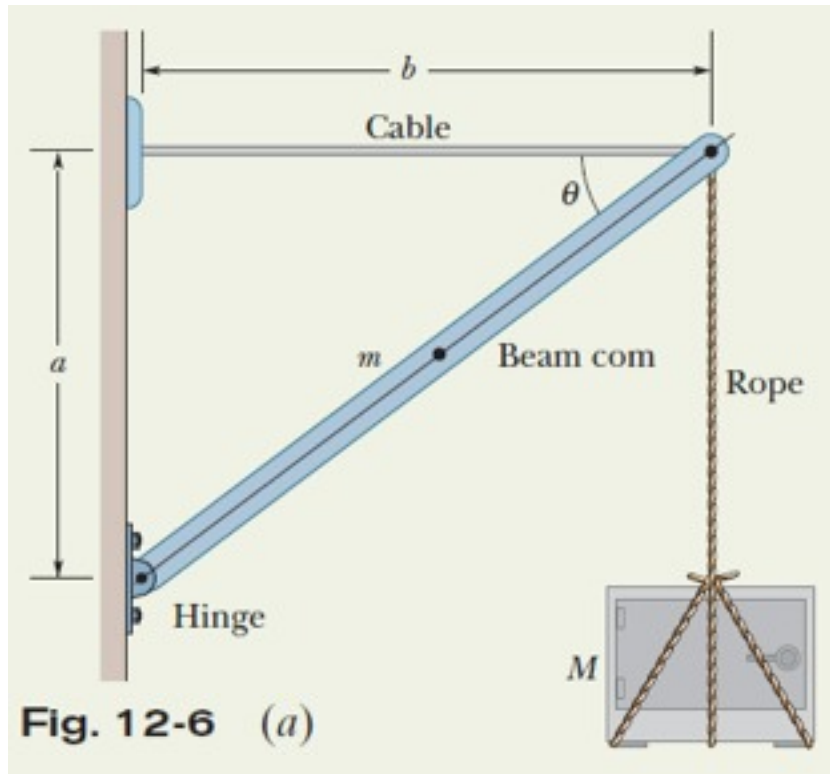
$$(a)(T_c) - (b)(T_r) - (\frac{1}{2}b)(mg) = 0. \quad (12-19)$$

Substituting Mg for T_r and solving for T_c , we find that

$$\begin{aligned} T_c &= \frac{gb(M + \frac{1}{2}m)}{a} \\ &= \frac{(9.8 \text{ m/s}^2)(2.5 \text{ m})(430 \text{ kg} + 85/2 \text{ kg})}{1.9 \text{ m}} \\ &= 6093 \text{ N} \approx 6100 \text{ N}. \end{aligned} \quad (\text{Answer})$$

Example, static equilibrium, cont.:

(b) Find the magnitude F of the net force on the beam from the hinge.



Example, static equilibrium, cont.:

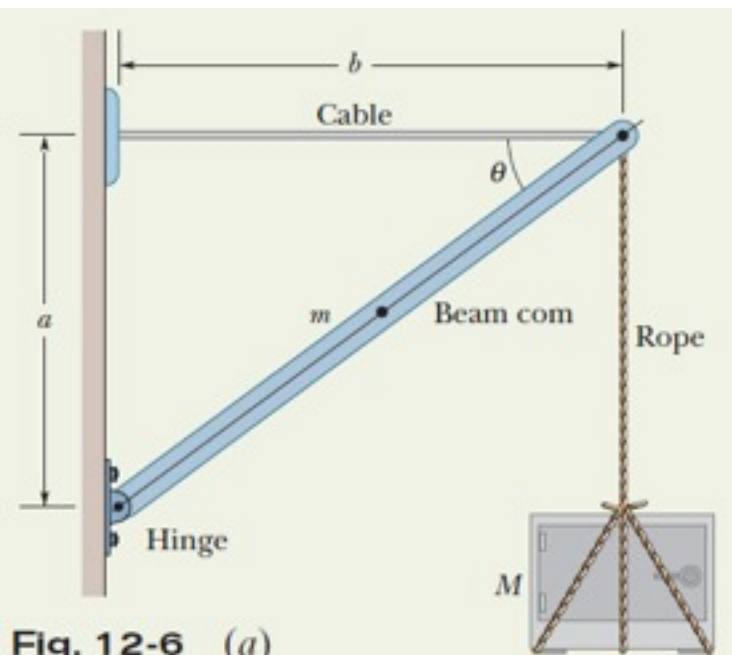
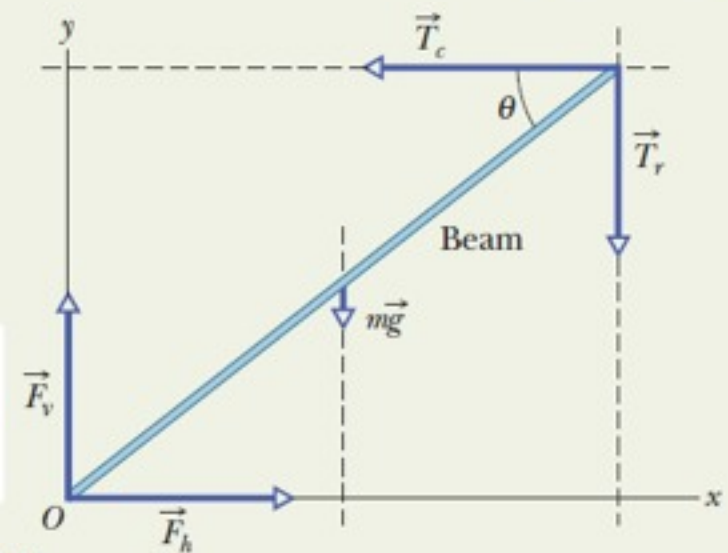


Fig. 12-6 (a)



(b)

(b) Find the magnitude F of the net force on the beam from the hinge.

Calculations: For the horizontal balance, we write $F_{\text{net},x} = 0$ as

$$F_h - T_c = 0, \quad (12-20)$$

and so

$$F_h = T_c = 6093 \text{ N.}$$

For the vertical balance, we write $F_{\text{net},y} = 0$ as

$$F_v - mg - T_r = 0.$$

Substituting Mg for T_r and solving for F_v , we find that

$$\begin{aligned} F_v &= (m + M)g = (85 \text{ kg} + 430 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 5047 \text{ N.} \end{aligned}$$

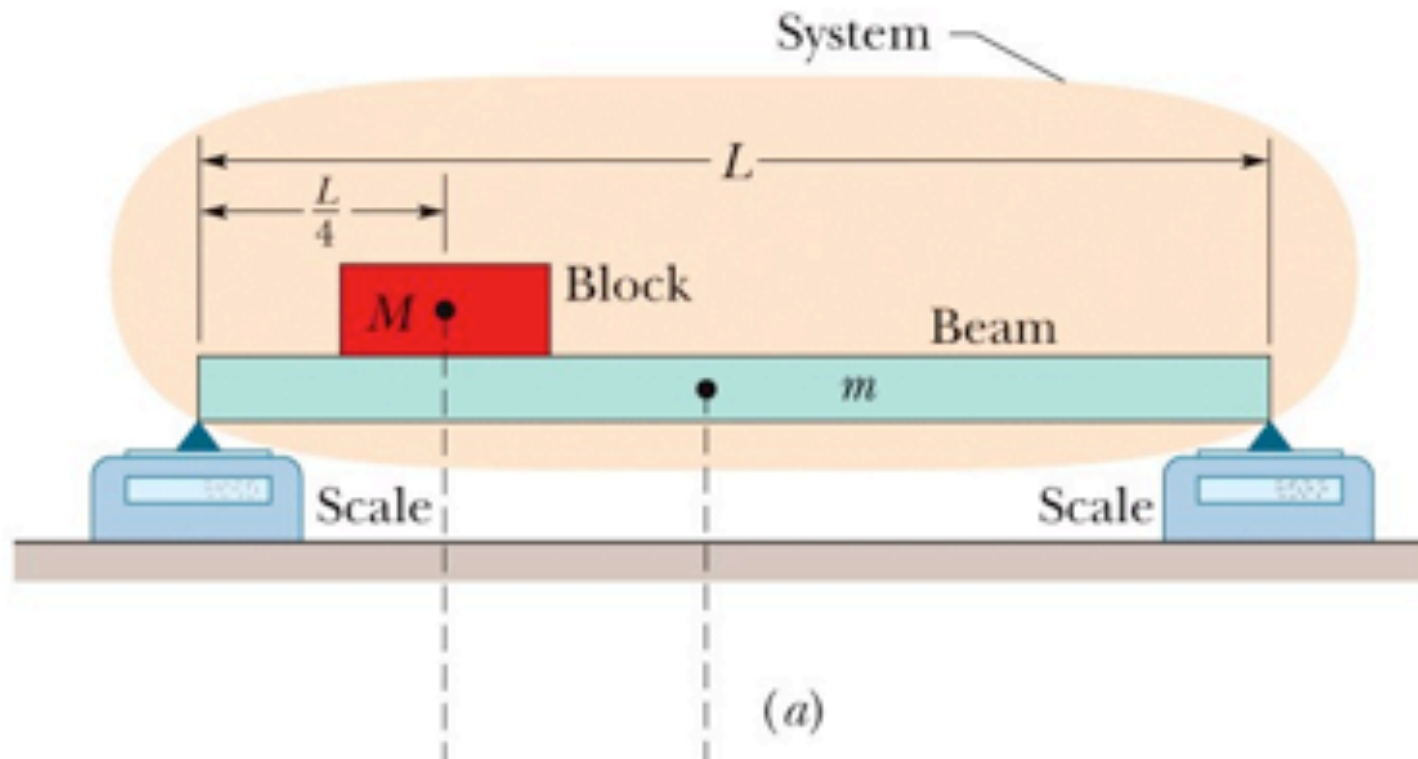
From the Pythagorean theorem, we now have

$$\begin{aligned} F &= \sqrt{F_h^2 + F_v^2} \\ &= \sqrt{(6093 \text{ N})^2 + (5047 \text{ N})^2} \approx 7900 \text{ N.} \quad (\text{Answer}) \end{aligned}$$

Note that F is substantially greater than either the combined weights of the safe and the beam, 5000 N, or the tension in the horizontal wire, 6100 N.

Example, static equilibrium:

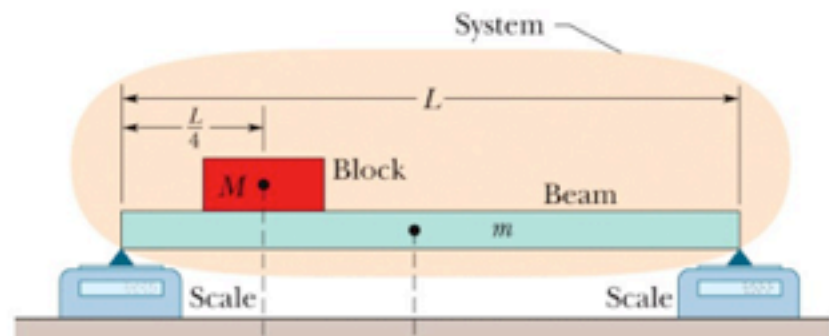
In Fig. 12-7a, a uniform beam, of length L and mass $m = 1.8$ kg, is at rest on two scales. A uniform block, with mass $M = 2.7$ kg, is at rest on the beam, with its center a distance $L/4$ from the beam's left end. What do the scales read?



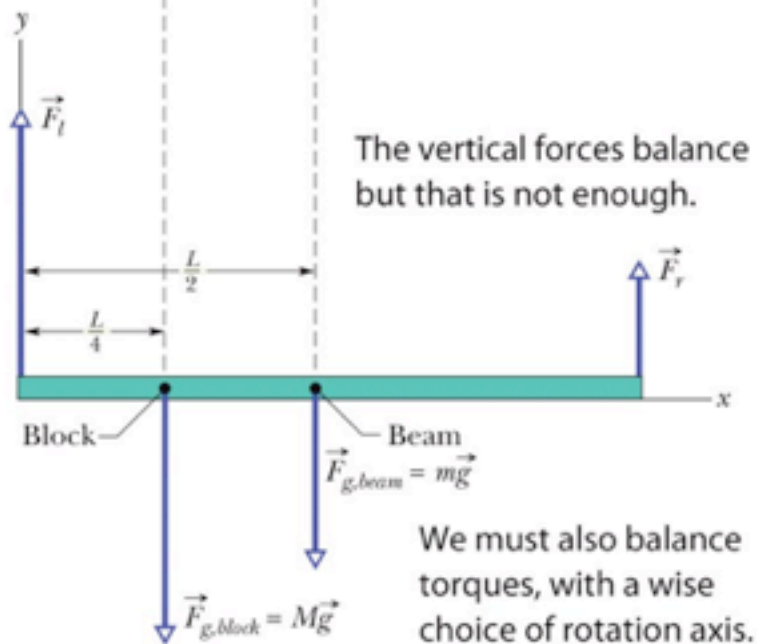
Let's do this for a more general case: the block a distance x from the beam's left end.

Example, static equilibrium:

In Fig. 12-7a, a uniform beam, of length L and mass $m = 1.8$ kg, is at rest on two scales. A uniform block, with mass $M = 2.7$ kg, is at rest on the beam, with its center a distance $L/4$ from the beam's left end. What do the scales read?



(a)



(b)

The vertical forces balance but that is not enough.

We must also balance torques, with a wise choice of rotation axis.

$$(F_{\text{net},x} = 0) \longrightarrow F_l + F_r - Mg - mg = 0.$$

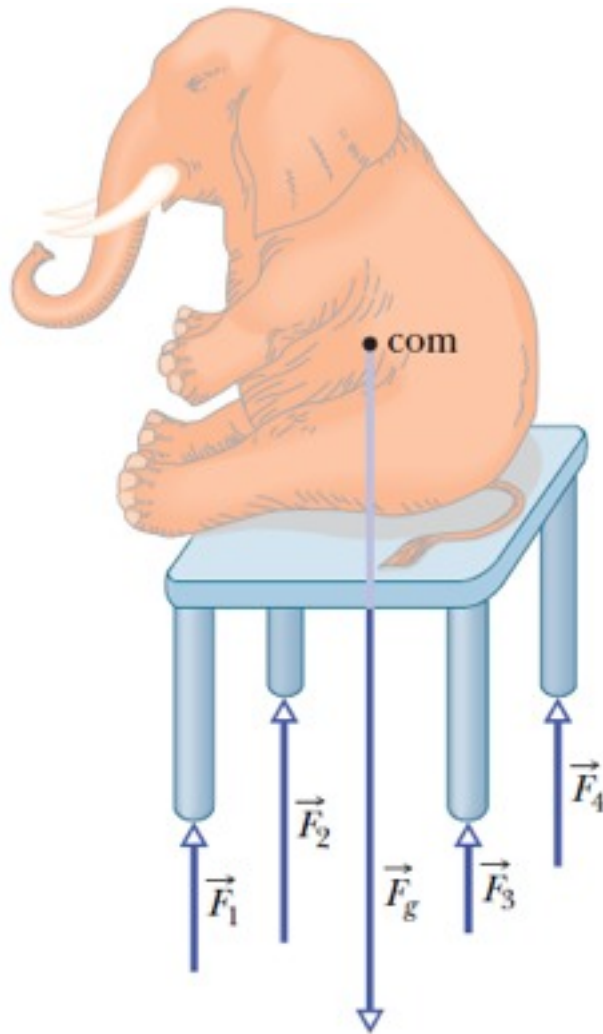
$$\tau_{\text{net},z} = 0$$

$$(0)(F_l) - (L/4)(Mg) - (L/2)(mg) + (L)(F_r) = 0.$$

$$\begin{aligned} F_r &= \frac{1}{4}Mg + \frac{1}{2}mg \\ &= \frac{1}{4}(2.7 \text{ kg})(9.8 \text{ m/s}^2) + \frac{1}{2}(1.8 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 15.44 \text{ N} \approx 15 \text{ N.} \end{aligned} \quad (\text{Answer})$$

$$\begin{aligned} F_l &= (M + m)g - F_r \\ &= (2.7 \text{ kg} + 1.8 \text{ kg})(9.8 \text{ m/s}^2) - 15.44 \text{ N} \\ &= 28.66 \text{ N} \approx 29 \text{ N.} \end{aligned} \quad (\text{Answer})$$

12.6: Indeterminate Structures



Given the weight of the elephant and table, and where the elephant is sitting on the table, what are the four forces on the legs of the table?

We can't tell if all we use is the conditions for static equilibrium: there are more unknowns than equations:

Since all the forces are in the vertical direction (z -axis) there are four unknowns. There is one force equation (z -component), and two torque equations (x - and y -components).

We call this an **indeterminate** problem.

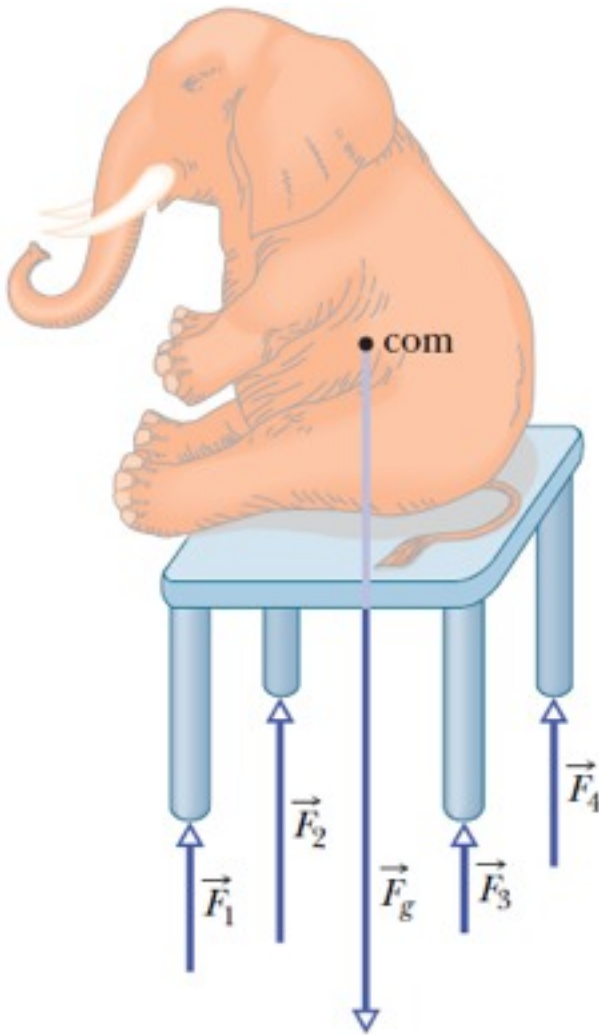
But we know that if we put four scales under the legs of the table they will read a definite force on each leg. So there must be additional data that allows one to determine these forces.

What went wrong?

We assumed the bodies involved are perfectly rigid. This is an idealization.

Once we understand how bodies deform under the action of applied forces we will be able to fix our inability to compute in cases of indeterminate structures.

Need to learn about *elasticity*.

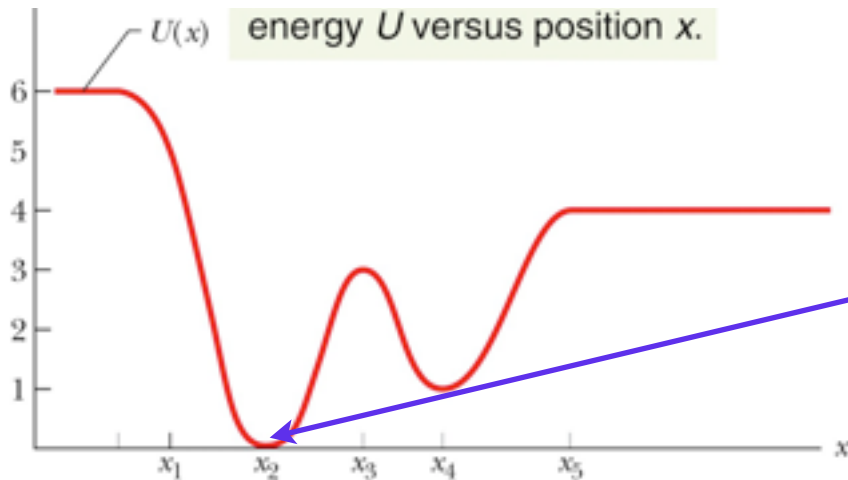
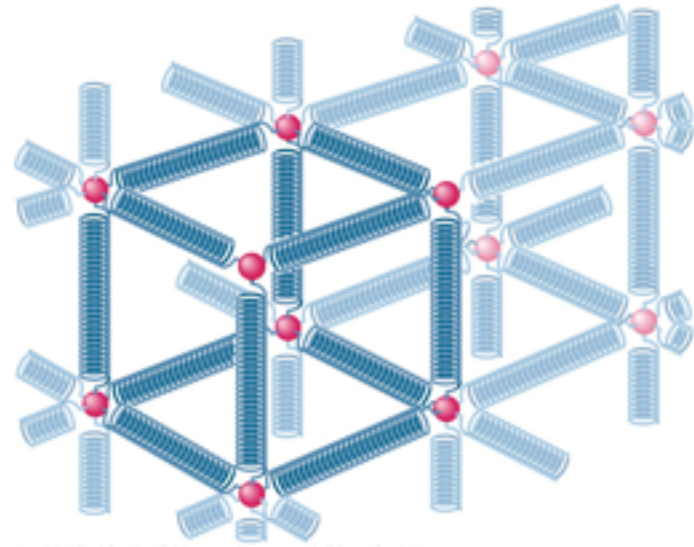


Will not cover the rest in class

12.7: Elasticity

Solid:

- Many atoms
- Arranged in regular lattice
- Each atom held in place by forces
- Equilibrium
- Like interatomic springs



Near equilibrium position potential looks like parabola, which is potential of spring:

$$U(x) \approx \frac{1}{2}k(x - x_2)^2$$

12.7: Elasticity

A **stress** is defined as deforming force per unit area, which produces a **strain**, or unit deformation.

Stress and strain are proportional to each other. The constant of proportionality is called a **modulus of elasticity**.

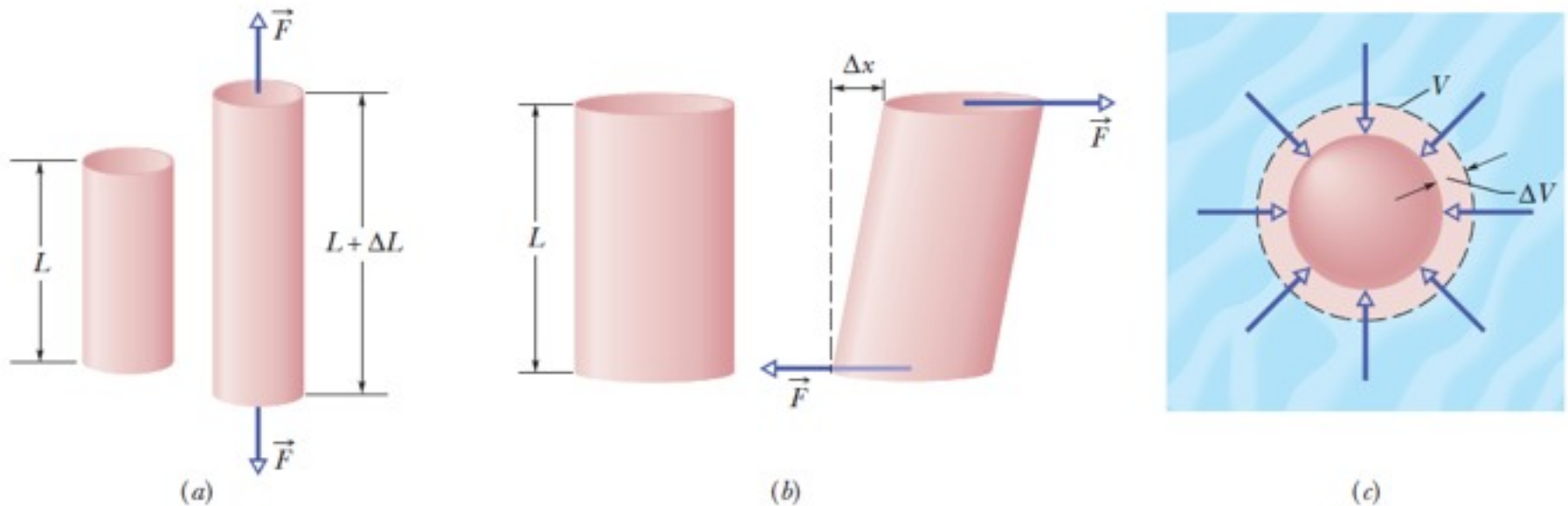


Fig. 12-10 (a) A cylinder subject to *tensile stress* stretches by an amount ΔL . (b) A cylinder subject to *shearing stress* deforms by an amount Δx , somewhat like a pack of playing cards would. (c) A solid sphere subject to uniform *hydraulic stress* from a fluid shrinks in volume by an amount ΔV . All the deformations shown are greatly exaggerated.

12.7: Elasticity: Tension and Compression

For simple tension or compression, the stress on the object is defined as F/A , where F is the magnitude of the force applied perpendicularly to an area A on the object.

The strain, or unit deformation, is then the dimensionless quantity $\Delta L/L$, the fractional change in a length of the specimen.

The modulus for tensile and compressive stresses is called the **Young's modulus** and is represented in engineering practice by the symbol E .

$$\frac{F}{A} = E \frac{\Delta L}{L}.$$

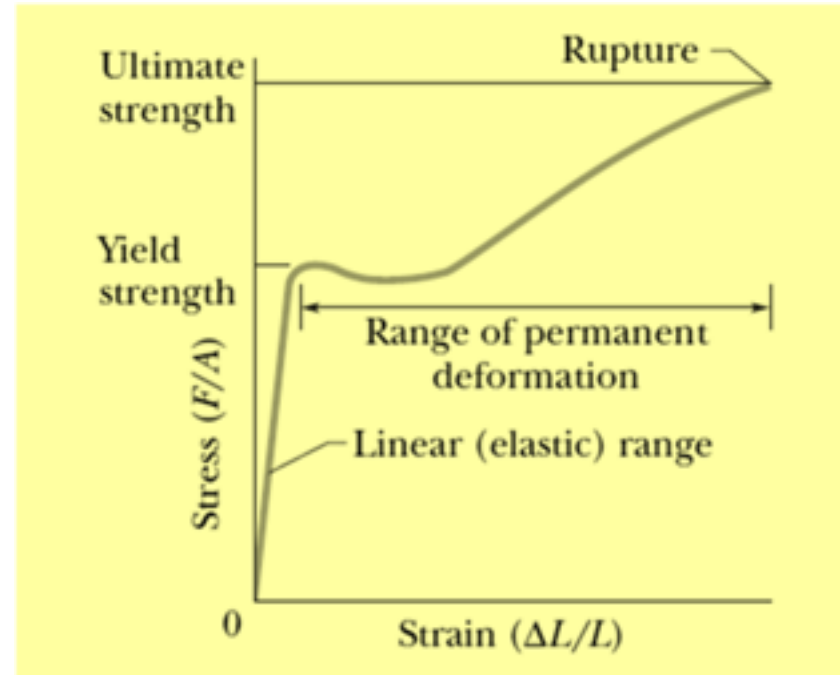
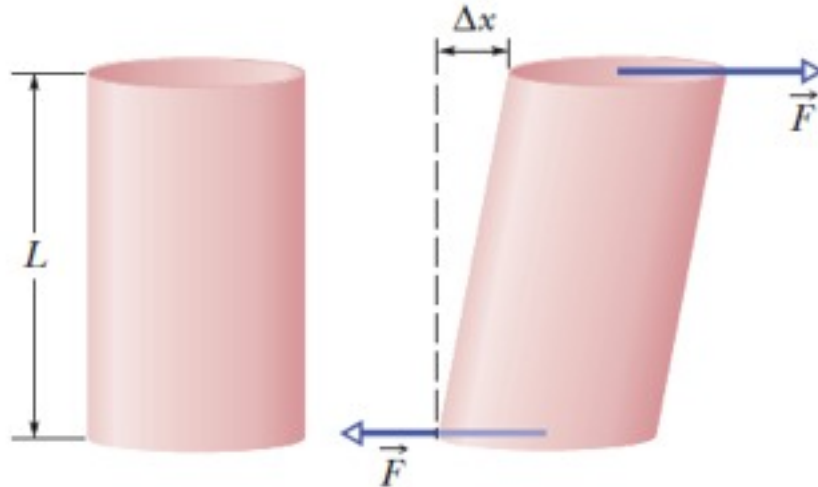


Fig. 12-12 A stress–strain curve for a steel test specimen. The specimen deforms permanently when the stress is equal to the yield strength of the specimen's material. It ruptures when the stress is equal to the ultimate strength of the material.

12.7: Elasticity: Shearing

In the case of shearing, the stress is also a force per unit area, but the force vector lies in the plane of the area rather than perpendicular to it. The strain is the dimensionless ratio $\Delta x/L$, with the quantities defined as shown in the figure. The corresponding modulus, which is given the symbol G in engineering practice, is called the **shear modulus**.

$$\frac{F}{A} = G \frac{\Delta x}{L}.$$



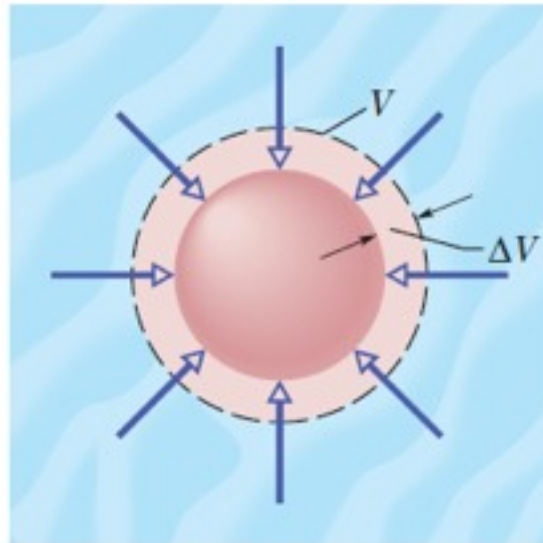
12.7: Elasticity: Hydraulic Stress

In the figure, the stress is the fluid pressure p on the object, where pressure is a force per unit area.

The strain is $\Delta V/V$, where V is the original volume of the specimen and ΔV is the absolute value of the change in volume.

The corresponding modulus, with symbol B , is called the **bulk modulus** of the material. The object is said to be under hydraulic compression, and the pressure can be called the hydraulic stress.

$$p = B \frac{\Delta V}{V}$$



12.7: Elasticity

Table 12-1

Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E (10^9 N/m^2)	Ultimate Strength S_u (10^6 N/m^2)	Yield Strength S_y (10^6 N/m^2)
Steel ^a	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50 ^b	—
Concrete ^c	2320	30	40 ^b	—
Wood ^d	525	13	50 ^b	—
Bone	1900	9 ^b	170 ^b	—
Polystyrene	1050	3	48	—

^aStructural steel (ASTM-A36).

^cHigh strength

^bIn compression.

^dDouglas fir.

Example, elongated rod

One end of a steel rod of radius $R = 9.5$ mm and length $L = 81$ cm is held in a vise. A force of magnitude $F = 62$ kN is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod?

KEY IDEAS

- (1) Because the force is perpendicular to the end face and uniform, the stress is the ratio of the magnitude F of the force to the area A . The ratio is the left side of Eq. 12-23.
- (2) The elongation ΔL is related to the stress and Young's modulus E by Eq. 12-23 ($F/A = E \Delta L/L$).
- (3) Strain is the ratio of the elongation to the initial length L .

Calculations: To find the stress, we write

$$\begin{aligned}\text{stress} &= \frac{F}{A} = \frac{F}{\pi R^2} = \frac{6.2 \times 10^4 \text{ N}}{(\pi)(9.5 \times 10^{-3} \text{ m})^2} \\ &= 2.2 \times 10^8 \text{ N/m}^2. \quad (\text{Answer})\end{aligned}$$

The yield strength for structural steel is 2.5×10^8 N/m², so this rod is dangerously close to its yield strength.

We find the value of Young's modulus for steel in Table 12-1. Then from Eq. 12-23 we find the elongation:

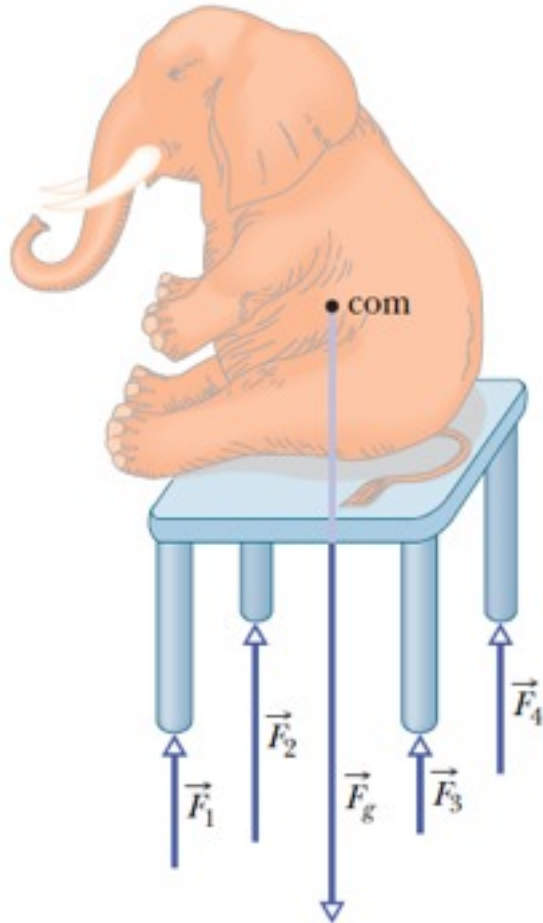
$$\begin{aligned}\Delta L &= \frac{(F/A)L}{E} = \frac{(2.2 \times 10^8 \text{ N/m}^2)(0.81 \text{ m})}{2.0 \times 10^{11} \text{ N/m}^2} \\ &= 8.9 \times 10^{-4} \text{ m} = 0.89 \text{ mm}. \quad (\text{Answer})\end{aligned}$$

For the strain, we have

$$\begin{aligned}\frac{\Delta L}{L} &= \frac{8.9 \times 10^{-4} \text{ m}}{0.81 \text{ m}} \\ &= 1.1 \times 10^{-3} = 0.11\%. \quad (\text{Answer})\end{aligned}$$

Example, wobbly table

A table has three legs that are 1.00 m in length and a fourth leg that is longer by $d = 0.50$ mm, so that the table wobbles slightly. A steel cylinder with mass $M = 290$ kg is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0$ cm²; Young's modulus is $E = 1.3 \times 10^{10}$ N/m². What are the magnitudes of the forces on the legs from the floor?



We take the table plus steel cylinder as our system. The situation is like that in the figure. If the tabletop remains level, the legs must be compressed in the following ways: Each of the short legs must be compressed by the same amount (call it ΔL_3) and thus by the same force of magnitude F_3 . The single long leg must be compressed by a larger amount ΔL_4 and thus by a force with a larger magnitude F_4 .

$$\Delta L_4 = \Delta L_3 + d. \quad \rightarrow \quad \frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d.$$

$$F_{\text{net},y} = 0 \quad \rightarrow \quad 3F_3 + F_4 - Mg = 0,$$

$$\begin{aligned} F_3 &= \frac{Mg}{4} - \frac{dAE}{4L} \\ &= \frac{(290 \text{ kg})(9.8 \text{ m/s}^2)}{4} \\ &\quad - \frac{(5.0 \times 10^{-4} \text{ m})(10^{-4} \text{ m}^2)(1.3 \times 10^{10} \text{ N/m}^2)}{(4)(1.00 \text{ m})} \\ &= 548 \text{ N} \approx 5.5 \times 10^2 \text{ N.} \end{aligned} \quad \text{(Answer)}$$

$$\begin{aligned} F_4 &= Mg - 3F_3 = (290 \text{ kg})(9.8 \text{ m/s}^2) - 3(548 \text{ N}) \\ &\approx 1.2 \text{ kN.} \end{aligned} \quad \text{(Answer)}$$