Chapter 11
Rolling, Torque, and Angular Momentum

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11.2 Rolling as Translation and Rotation Combined

Although the center of the rolling disk moves in a straight line parallel to the surface, a point on the rim certainly does not.

The point on the rolling disk rim in contact with the surface is momentarily at rest.

This motion can be studied by treating it as a combination of translation of the center of mass and rotation of the rest of the object around that center.
11.2 Rolling without slipping ("smooth rolling")

\[ s = \theta R, \quad v_{\text{com}} = \omega R \quad \text{(smooth rolling motion)} \]

\[ a_{\text{com}} = \alpha R \quad \text{(smooth rolling motion)} \]
A wheel rolls horizontally without sliding while accelerating with linear acceleration $a_{com}$. A static frictional force $f_s$ acts on the wheel at $P$, opposing its tendency to slide.

If the wheel does slide when the net force acts on it, the frictional force that acts at $P$ is a kinetic frictional force, $f_k$. The motion then is no longer rolling without slipping ("smooth rolling"), and the above relations do not apply to the motion.

\[ s = \theta R, \quad v_{com} = \omega R \]

\[ a_{com} = \alpha R \] (smooth rolling motion)
Fig. 11-4  Rolling motion of a wheel as a combination of purely rotational motion and purely translational motion. (a) The purely rotational motion: All points on the wheel move with the same angular speed $\omega$. Points on the outside edge of the wheel all move with the same linear speed $v = v_{com}$. The linear velocities $\vec{v}$ of two such points, at top ($T$) and bottom ($P$) of the wheel, are shown. (b) The purely translational motion: All points on the wheel move to the right with the same linear velocity $\vec{v}_{com}$. (c) The rolling motion of the wheel is the combination of (a) and (b).
11.3 The Kinetic Energy of Rolling

If we view the rolling as pure rotation about an axis through $P$, then

$$K = \frac{1}{2} I_p \omega^2,$$

($\omega$ is the angular speed of the wheel and $I_p$ is the rotational inertia of the wheel about the axis through $P$).

Using the parallel-axis theorem ($I_p = I_{\text{com}} + Mh^2$),

$$I_p = I_{\text{com}} + MR^2$$

($M$ is the mass of the wheel, $I_{\text{com}}$ is its rotational inertia about an axis through its center of mass, and $R$ is the wheel’s radius, at a perpendicular distance $h = R$).

Using the relation $v_{\text{com}} = \omega R$, we get:

$$K = \frac{1}{2} I_{\text{com}} \omega^2 + \frac{1}{2} M v_{\text{com}}^2.$$

A rolling object, therefore has two types of kinetic energy:
1. a rotational kinetic energy due to its rotation about its center of mass ($= \frac{1}{2} I_{\text{com}} \omega^2$),
2. and a translational kinetic energy due to translation of its center of mass ($= \frac{1}{2} M v_{\text{com}}^2$).
11.4: The Forces of Rolling. Example: Rolling Down a Ramp

A round uniform body of radius $R$ rolls down a ramp. The forces that act on it are the gravitational force $F_g$, a normal force $F_N$, and a frictional force $f_s$ pointing up the ramp.

The torque due to $f_s$ determines the angular acceleration around the com.

Forces $F_g \sin \theta$ and $f_s$ determine the linear acceleration down the ramp.

Forces $F_N$ and $F_g \cos \theta$ merely balance.

\[
F_{\text{net},x} = ma_x \quad \rightarrow \quad f_s - Mg \sin \theta = Ma_{\text{com},x}
\]

\[
\tau_{\text{net}} = I\alpha \quad \rightarrow \quad Rf_s = I_{\text{com}}\alpha
\]

\[
a_{\text{com}} = \alpha R \quad \rightarrow \quad f_s = -I_{\text{com}}\frac{a_{\text{com},x}}{R^2}
\]

\[
a_{\text{com},x} = -\frac{g \sin \theta}{1 + I_{\text{com}}/MR^2}
\]
A uniform ball, of mass $M = 6.00 \text{ kg}$ and radius $R$, rolls smoothly from rest down a ramp at angle $\theta = 30.0^\circ$.

a) The ball descends a vertical height $h = 1.20 \text{ m}$ to reach the bottom of the ramp. What is its speed at the bottom?

**Calculations:** Use conservation of energy (no slipping means no dissipation):

$$K_f + U_f = K_i + U_i$$

$$\left( \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M v_{com}^2 \right) + 0 = 0 + Mgh$$

where $I_{com}$ is the ball’s rotational inertia about an axis through its center of mass, $v_{com}$ is the requested speed at the bottom, and $\omega$ is the angular speed there.

Substituting $v_{com}/R$ for $\omega$, and $2/5 MR^2$ for $I_{com}$:

$$v_{com} = \sqrt{\frac{10}{7}gh} = \sqrt{\frac{10}{7}(9.8 \text{ m/s}^2)(1.20 \text{ m})}$$

$$= 4.10 \text{ m/s.}$$

b) What are the magnitude and direction of the frictional force on the ball as it rolls down the ramp?

**Calculations:** First we need to determine the ball’s acceleration $a_{com,x}$:

$$a_{com,x} = -\frac{g \sin \theta}{1 + \frac{I_{com}}{MR^2}} = -\frac{g \sin \theta}{1 + \frac{2}{5}MR^2/MR^2}$$

$$= -\frac{(9.8 \text{ m/s}^2) \sin 30.0^\circ}{1 + \frac{2}{5}} = -3.50 \text{ m/s}^2.$$ 

We can now solve for the frictional force:

$$f_s = -I_{com} \frac{a_{com,x}}{R^2} = -\frac{2}{5}MR^2 \frac{a_{com,x}}{R^2} = -\frac{2}{5}Ma_{com,x}$$

$$= -\frac{2}{5}(6.00 \text{ kg})(-3.50 \text{ m/s}^2) = 8.40 \text{ N.}$$
11.5: The Yo-Yo

1. Instead of rolling down a ramp at angle $\theta$ with the horizontal, the yo-yo rolls down a string at angle $\theta = 90^\circ$ with the horizontal.

2. Instead of rolling on its outer surface at radius $R$, the yo-yo rolls on an axle of radius $R_0$ (Fig. 11-9a).

3. Instead of being slowed by frictional force $\vec{f}_s$, the yo-yo is slowed by the force $\vec{T}$ on it from the string (Fig. 11-9b).

The analysis would again lead us to Eq. 11-10. Therefore, let us just change the notation in Eq. 11-10 and set $\theta = 90^\circ$ to write the linear acceleration as

$$a_{com} = \frac{-g}{1 + I_{com}/MR_0^2},$$

(11-13)

where $I_{com}$ is the yo-yo’s rotational inertia about its center and $M$ is its mass. A yo-yo has the same downward acceleration when it is climbing back up.

Fig. 11-9  (a) A yo-yo, shown in cross section. The string, of assumed negligible thickness, is wound around an axle of radius $R_0$. (b) A free-body diagram for the falling yo-yo. Only the axle is shown.
Fy = ma_y \quad \Rightarrow \quad T \sin \theta + N - mg = 0

Fx = ma_x \quad \Rightarrow \quad T \cos \theta - f = ma_x

\tau = I \alpha \quad \Rightarrow \quad fr_2 - Tr_1 = I \alpha = Ia_x/r_2

Eliminating f from the last two:

\[ a_x = \frac{T}{m + I/r_2^2} (\cos \theta - r_1/r_2) \]

At small angle yo-y walks right, at large angle yo-yo walks left and at critical angle it will slide once the tension is large enough to exceed the force of static friction!!
The right-hand rule allows us to find the direction of vector $c$. It is perpendicular to both vectors $a$ and $b$, directed as in right hand rule.

The result is a new vector $c$, of magnitude:

$$c = ab \sin \phi,$$

Here $a$ and $b$ are the magnitudes of vectors $a$ and $b$ respectively, and $\phi$ is the smaller of the two angles between $a$ and $b$ vectors.
3.8: \textit{Multiplying Vectors: Vector product in unit-vector notation}

\[ \vec{a} \times \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}) \]

\[ = (a_y b_z - b_y a_z) \hat{i} + (a_z b_x - b_z a_x) \hat{j} + (a_x b_y - b_x a_y) \hat{k}. \]

Note that:

\[ a_x \hat{i} \times b_x \hat{i} = a_x b_x (\hat{i} \times \hat{i}) = 0, \]

And,

\[ a_x \hat{i} \times b_y \hat{j} = a_x b_y (\hat{i} \times \hat{j}) = a_x b_y \hat{k}. \]
Sample Problem: Vector Product

In Fig. 3-20, vector $\vec{a}$ lies in the $xy$ plane, has a magnitude of 18 units and points in a direction $250^\circ$ from the positive direction of the $x$ axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the $z$ axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?

Sweep $\vec{a}$ into $\vec{b}$.

This is the resulting vector, perpendicular to both $\vec{a}$ and $\vec{b}$.

Fig. 3-20 Vector $\vec{c}$ (in the $xy$ plane) is the vector (or cross) product of vectors $\vec{a}$ and $\vec{b}$. 
Sample Problem: Vector Product

In Fig. 3-20, vector \( \vec{a} \) lies in the xy plane, has a magnitude of 18 units and points in a direction 250° from the positive direction of the x axis. Also, vector \( \vec{b} \) has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product \( \vec{c} = \vec{a} \times \vec{b} \)?

\[ \begin{align*}
\text{Sweep} & \; \vec{a} \; \text{into} \; \vec{b}. \\
\end{align*} \]

This is the resulting vector, perpendicular to both \( \vec{a} \) and \( \vec{b} \).

\[ \begin{align*}
\text{Fig. 3-20} \; \text{Vector} \; \vec{c} \; (\text{in the xy plane}) & \; \text{is the vector (or cross) product of vectors} \; \vec{a} \; \text{and} \; \vec{b}. \\
\end{align*} \]

Calculations: For the magnitude we write

\[ c = ab \sin \phi = (18)(12)(\sin 90°) = 216. \quad \text{Answer} \]

To determine the direction in Fig. 3-20, imagine placing the fingers of your right hand around a line perpendicular to the plane of \( \vec{a} \) and \( \vec{b} \) (the line on which \( \vec{c} \) is shown) such that your fingers sweep \( \vec{a} \) into \( \vec{b} \). Your outstretched thumb then gives the direction of \( \vec{c} \). Thus, as shown in the figure, \( \vec{c} \) lies in the xy plane. Because its direction is perpendicular to the direction of \( \vec{a} \) (a cross product always gives a perpendicular vector), it is at an angle of

\[ 250° - 90° = 160° \quad \text{Answer} \]

from the positive direction of the x axis.

KEY IDEA

When we have two vectors in magnitude-angle notation, we find the magnitude of their cross product with Eq. 3-27 and the direction of their cross product with the right-hand rule of Fig. 3-19.
Sample Problem: Vector product, unit vector notation

If \( \vec{a} = 3\hat{i} - 4\hat{j} \) and \( \vec{b} = -2\hat{i} + 3\hat{k} \), what is \( \vec{c} = \vec{a} \times \vec{b} \)?

**KEY IDEA**

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

**Calculations:** Here we write

\[
\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})
\]

\[
= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i}) + (-4\hat{j}) \times 3\hat{k}.
\]

We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle \( \phi \) between the two vectors being crossed is 0. For the other terms, \( \phi \) is 90°. We find

\[
\vec{c} = -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i}
\]

\[
= -12\hat{i} - 9\hat{j} - 8\hat{k}.
\]

(Answer)

This vector \( \vec{c} \) is perpendicular to both \( \vec{a} \) and \( \vec{b} \), a fact you can check by showing that \( \vec{c} \cdot \vec{a} = 0 \) and \( \vec{c} \cdot \vec{b} = 0 \); that is, there is no component of \( \vec{c} \) along the direction of either \( \vec{a} \) or \( \vec{b} \).
11.6: Torque Revisited

\[ \vec{\tau} = \vec{r} \times \vec{F} \]  
(torque defined).

Fig. 11-10 (a) A force \( \vec{F} \), lying in an \( x-y \) plane, acts on a particle at point \( A \). (b) This force produces a torque \( \vec{\tau} = \vec{r} \times \vec{F} \) on the particle with respect to the origin \( O \). By the right-hand rule for vector (cross) products, the torque vector points in the positive direction of \( z \). Its magnitude is given by \( rF \) in (b) and by \( rF \) in (c).
• Refers to a particular choice of origin (that is, a choice of coordinate system)
• Does not involve a real axis of rotation
• Can be used for a single particle
• Can be used for a non-rigid collection of particles (not necessarily a rigid body)

We will see later how this relates to our previous discussion of torque for a rigid body about a fixed axis of rotation.
For a system (a collection of particles):

\[ \vec{\tau}_{\text{net}} = \vec{\tau}_{\text{net},1} + \vec{\tau}_{\text{net},2} + \cdots + \vec{\tau}_{\text{net},n} \]

\[ = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i} \]

Do internal forces cancel (as they did for total net force)?
Consider pair 12 only:

\[ \vec{\tau}_{\text{net},1} + \vec{\tau}_{\text{net},2} = \vec{r}_1 \times (\vec{F}_{\text{ext},1} + \vec{F}_{12}) + \vec{r}_2 \times (\vec{F}_{\text{ext},2} + \vec{F}_{21}) \]

\[ = \vec{\tau}_{\text{ext},1} + \vec{\tau}_{\text{ext},2} + \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} \]

\[ = \vec{\tau}_{\text{ext},1} + \vec{\tau}_{\text{ext},2} + \vec{r}_1 \times (-\vec{F}_{21}) + \vec{r}_2 \times \vec{F}_{21} \]

\[ = \vec{\tau}_{\text{ext},1} + \vec{\tau}_{\text{ext},2} + (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{21} \]

\[ = \vec{\tau}_{\text{ext},1} + \vec{\tau}_{\text{ext},2} \]

The last line is correct if \( F_{21} \) is parallel to \( \vec{r}_2 - \vec{r}_1 \). This is a vector from 1 to 2, and often (but not always) the force between 1 and 2 is along the line joining them (strong form of Newton’s 3rd Law)
Sample problem

In the figure three forces, each of magnitude 2.0 N, act on a particle.

The particle is in the $xz$ plane at point $A$ given by position vector $r$, where $r = 3.0$ m and $\theta = 30^\circ$. Force $F_1$ is parallel to the $x$ axis, force $F_2$ is parallel to the $z$ axis and force $F_3$ is parallel to the $y$ axis.

What is the torque, about the origin $O$, due to each force?
Calculations: We use the formula for torque as a cross product. We first determine the magnitude of the torque and then its direction. To compute the magnitude we need the angle \( \theta \) between the direction of \( \mathbf{r} \) and the direction of each force. To this end shift the force vectors, each in turn, so that their tails are at the origin. The figures to the left, which are direct views of the \( xz \) plane, show the shifted force vectors \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) (the third one is in the next slide).

Now, we find the magnitudes of the torques to be

\[
\tau_1 = rF_1 \sin \phi_1 = (3.0 \text{ m})(2.0 \text{ N})(\sin 150^\circ) = 3.0 \text{ N} \cdot \text{m}, \\
\tau_2 = rF_2 \sin \phi_2 = (3.0 \text{ m})(2.0 \text{ N})(\sin 120^\circ) = 5.2 \text{ N} \cdot \text{m},
\]

The direction is done by hand demonstration in class: since the two vectors are in the \( xz \) plane the cross product, always perpendicular to the plane of the vectors, must be along the \( y \) axis. The right hand rule gives \( \tau_1 \) along \(-y\) while \( \tau_2 \) is along \(+y\).
The symbol $\otimes$ means $\vec{F}_3$ is directed into the page. If it were directed out of the page it would be represented by the symbol $\ominus$.

\[
t_3 = rF_3 \sin \phi_3 = (3.0 \text{ m})(2.0 \text{ N})(\sin 90^\circ)
\]
\[= 6.0 \text{ N} \cdot \text{m.} \quad \text{(Answer)}
\]
11.7 Angular Momentum

\[ \ell = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \]  
(angular momentum defined),

magnitude:

\[ \ell = rmv \sin \phi = rp_\perp = rmv_\perp = r_\perp p = r_\perp mv \]

- Definition for a single particle.
- Refers to specific choice of origin.
11.8: Newton’s 2nd Law in Angular Form

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

\[ \vec{\ell} = m(\vec{r} \times \vec{v}), \quad \frac{d\vec{\ell}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right) \]

\[ = m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}). \]

\[ = m(\vec{r} \times \vec{a}) = \vec{r} \times m\vec{a}. \]

\[ = \vec{r} \times \vec{F}_{\text{net}} = \sum(\vec{r} \times \vec{F}). \]

\[ \vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \quad \text{(single particle)} \]

This is the rotational analogue of \[ \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \]}
Sample problem: Torque, Penguin Fall

In Fig. 11-14, a penguin of mass $m$ falls from rest at point $A$, a horizontal distance $D$ from the origin $O$ of an $xyz$ coordinate system. (The positive direction of the $z$ axis is directly outward from the plane of the figure.)

(a) What is the angular momentum $\vec{\ell}$ of the falling penguin about $O$?
Sample problem: Torque, Penguin Fall

Calculations: The magnitude of \( \ell \) can be found by using

\[
\ell = r_\perp p = r_\perp mv
\]

The perpendicular distance between \( O \) and an extension of vector \( p \) is the given distance \( D \). The speed of an object that has fallen from rest for a time \( t \) is \( v = gt \). Therefore,

\[
\ell = r_\perp mv = Dmg t.
\]

To find the direction of we use the right-hand rule for the vector product, and find that the direction is into the plane of the figure. The vector changes with time in magnitude only; its direction remains unchanged.

(b) About the origin \( O \), what is the torque on the penguin due to the gravitational force?

Calculations: \( \tau = r_\perp F \) \( \rightarrow \tau = DF_g = Dmg \)

Using the right-hand rule for the vector product we find that the direction of \( \tau \) is the negative direction of the \( z \) axis, the same as \( \ell \).
11.9: The Angular Momentum of a System of Particles

The total angular momentum $\mathbf{L}$ of the system is the (vector) sum of the angular momenta $\mathbf{l}$ of the individual particles (here with label $i$):

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \cdots + \mathbf{l}_n = \sum_{i=1}^{n} \mathbf{l}_i.$$ 

With time, the angular momenta of individual particles may change because of interactions between the particles or with the outside.

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} \frac{d\mathbf{l}_i}{dt} = \sum_{i=1}^{n} \mathbf{\tau}_{\text{net},i}.$$ 

Therefore, the net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\mathbf{L}$.

$$\mathbf{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt} \quad \text{(system of particles)}$$
Sample problem: Angular Momentum

Figure 11-13 shows an overhead view of two particles moving at constant momentum along horizontal paths. Particle 1, with momentum magnitude $p_1 = 5.0 \text{ kg} \cdot \text{m/s}$, has position vector $\vec{r}_1$ and will pass 2.0 m from point $O$. Particle 2, with momentum magnitude $p_2 = 2.0 \text{ kg} \cdot \text{m/s}$, has position vector $\vec{r}_2$ and will pass 4.0 m from point $O$. What are the magnitude and direction of the net angular momentum $\vec{L}$ about point $O$ of the two-particle system?

**KEY IDEA**

To find $\vec{L}$, we can first find the individual angular momenta $\vec{\ell}_1$ and $\vec{\ell}_2$ and then add them. To evaluate their magnitudes, we can use any one of Eqs. 11-18 through 11-21. However, Eq. 11-21 is easiest, because we are given the perpendicular distances $r_{1\perp} = 2.0 \text{ m}$ and $r_{2\perp} = 4.0 \text{ m}$ and the momentum magnitudes $p_1$ and $p_2$.

**Calculations:** For particle 1, Eq. 11-21 yields

$$\ell_1 = r_{1\perp} p_1 = (2.0 \text{ m})(5.0 \text{ kg} \cdot \text{m/s}) = 10 \text{ kg} \cdot \text{m}^2/\text{s}.$$

To find the direction of vector $\vec{\ell}_1$, we use Eq. 11-18 and the right-hand rule for vector products. For $\vec{r}_1 \times \vec{p}_1$, the vector product is out of the page, perpendicular to the plane of Fig. 11-13. This is the positive direction, consistent with the counterclockwise rotation of the particle’s position vector $\vec{r}_1$ around $O$ as particle 1 moves. Thus, the angular momentum vector for particle 1 is

$$\ell_1 = +10 \text{ kg} \cdot \text{m}^2/\text{s}.$$

Similarly, the magnitude of $\vec{\ell}_2$ is

$$\ell_2 = r_{2\perp} p_2 = (4.0 \text{ m})(2.0 \text{ kg} \cdot \text{m/s}) = 8.0 \text{ kg} \cdot \text{m}^2/\text{s},$$

and the vector product $\vec{r}_2 \times \vec{p}_2$ is into the page, which is the negative direction, consistent with the clockwise rotation of $\vec{r}_2$ around $O$ as particle 2 moves. Thus, the angular momentum vector for particle 2 is

$$\ell_2 = -8.0 \text{ kg} \cdot \text{m}^2/\text{s}.$$

The net angular momentum for the two-particle system is

$$L = \ell_1 + \ell_2 = +10 \text{ kg} \cdot \text{m}^2/\text{s} + (-8.0 \text{ kg} \cdot \text{m}^2/\text{s}) = +2.0 \text{ kg} \cdot \text{m}^2/\text{s}.$$  

(Answer)

The plus sign means that the system’s net angular momentum about point $O$ is out of the page.
11.10: Angular Momentum of a Rigid Body Rotating About a Fixed Axis

a) A rigid body rotates about a $z$ axis with angular speed $\omega$. A mass element of mass $\Delta m_i$ within the body moves about the $z$ axis in a circle with radius $r_{\perp i}$. The mass element has linear momentum $p_i$ and it is located relative to the origin $O$ (on rotational axis) by position vector $r_i$. Here the mass element is shown when $r_{\perp i}$ is parallel to the $x$ axis.

b) The angular momentum $\ell_i$, with respect to $O$, of the mass element in (a). The $z$ component $(\ell_i)_z$ is also shown.

\[
\ell_i = (r_i)(p_i)(\sin 90^\circ) = (r_i)(\Delta m_i v_i)
\]
\[
(\ell_i)_z = \ell_i \sin \theta = (r_i \sin \theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i.
\]
\[
L_z = \sum_{i=1}^{n} (\ell_i)_z = \sum_{i=1}^{n} \Delta m_i v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}
\]
\[
= \omega \left( \sum_{i=1}^{n} \Delta m_i r_{\perp i}^2 \right).
\]

$L = I \omega$ (rigid body, fixed axis).

note: dropped subscript $z$, must remember this equation is for component of $L$ along axis.
### 11.10: More Corresponding Variables and Relations for Translational and Rotational Motion

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<tr>
<td>Conservation law (^d) (\vec{P} = ) a constant</td>
<td>Conservation law (^d) (\vec{L} = ) a constant</td>
</tr>
</tbody>
</table>

\(^a\)See also Table 10-3.

\(^b\)For systems of particles, including rigid bodies.

\(^c\)For a rigid body about a fixed axis, with \(L\) being the component along that axis.

\(^d\)For a closed, isolated system.
11.11: Conservation of Angular Momentum

If the net external torque acting on a system is zero, the angular momentum $L$ of the system remains constant, no matter what changes take place within the system.

$\vec{L} = \text{a constant} \quad \text{(isolated system)}$

$\vec{L}_i = \vec{L}_f \quad \text{(isolated system)}$

(Really only for isolated system with internal forces that satisfy strong form of Newton’s 3rd Law)
11.11: Conservation of Angular Momentum

If the component of the net external torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

(a) The student has a relatively large rotational inertia about the rotation axis and a relatively small angular speed.

(b) By decreasing his rotational inertia, the student automatically increases his angular speed. The angular momentum of the rotating system remains unchanged.
Its angular momentum is now $-L_{wh}$. The inversion results in the student, the stool, and the wheel’s center rotating together as a composite rigid body about the stool’s rotation axis, with rotational inertia $I_b = 6.8 \text{ kg m}^2$. With what angular speed $\omega_b$ and in what direction does the composite body rotate after the inversion of the wheel?

Calculations: The conservation of $L_{tot}$ is represented with vectors in Fig. 11-20c. We can also write this conservation in terms of components along a vertical axis as

$$L_{hf} + L_{wh,f} = L_{hi} + L_{wh,i},$$

(11-35)

where $i$ and $f$ refer to the initial state (before inversion of the wheel) and the final state (after inversion). Because inversion of the wheel inverted the angular momentum vector of the wheel’s rotation, we substitute $-L_{wh,i}$ for $L_{wh,f}$. Then, if we set $L_{hi} = 0$ (because the student, the stool, and the wheel’s center were initially at rest), Eq. 11-35 yields

$$L_{hf} = 2L_{wh,i}.$$

Using Eq. 11-31, we next substitute $I_b \omega_b$ for $L_{hf}$ and $I_{wh} \omega_{wh}$ for $L_{wh,i}$ and solve for $\omega_b$, finding

$$\omega_b = \frac{2I_{wh}}{I_b} \omega_{wh} = \frac{(2)(1.2 \text{ kg m}^2)(3.9 \text{ rev/s})}{6.8 \text{ kg m}^2} = 1.4 \text{ rev/s}. \quad \text{(Answer)}$$
In Fig. 11-21, a cockroach with mass $m$ rides on a disk of mass $6.00m$ and radius $R$. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50$ rad/s. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?
In Fig. 11-21, a cockroach with mass $m$ rides on a disk of mass 6.00$m$ and radius $R$. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50 \text{ rad/s}$. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

**Calculations:** We want to find the final angular speed. Our key is to equate the final angular momentum $L_f$ to the initial angular momentum $L_i$, because both involve angular speed. They also involve rotational inertia $I$. So, let’s start by finding the rotational inertia of the system of cockroach and disk before and after the crawl.

The rotational inertia of a disk rotating about its central axis is given by Table 10-2$c$ as $\frac{1}{2}MR^2$. Substituting 6.00$m$ for the mass $M$, our disk here has rotational inertia

$$I_d = 3.00mR^2.$$  \hspace{1cm} (11-36)

(We don’t have values for $m$ and $R$, but we shall continue with physics courage.)

From Eq. 10-33, we know that the rotational inertia of the cockroach (a particle) is equal to $mr^2$. Substituting the cockroach’s initial radius ($r = 0.800R$) and final radius ($r = R$), we find that its initial rotational inertia about the rotation axis is

$$I_{ci} = 0.64mR^2$$ \hspace{1cm} (11-37)

and its final rotational inertia about the rotation axis is

$$I_{cf} = mR^2.$$ \hspace{1cm} (11-38)

So, the cockroach–disk system initially has the rotational inertia

$$I_i = I_d + I_{ci} = 3.64mR^2,$$ \hspace{1cm} (11-39)

and finally has the rotational inertia

$$I_f = I_d + I_{cf} = 4.00mR^2.$$ \hspace{1cm} (11-40)

Next, we use Eq. 11-31 ($L = I\omega$) to write the fact that the system’s final angular momentum $L_f$ is equal to the system’s initial angular momentum $L_i$:

$$I_f\omega_f = I_i\omega_i$$

or

$$4.00mR^2\omega_f = 3.64mR^2(1.50 \text{ rad/s}).$$

After canceling the unknowns $m$ and $R$, we come to

$$\omega_f = 1.37 \text{ rad/s}.$$ \hspace{1cm} (Answer)

Note that the angular speed decreased because part of the mass moved outward from the rotation axis, thus increasing the rotational inertia of the system.
11.12: Precession of a Gyroscope

(a) A non-spinning gyroscope falls by rotating in an $xz$ plane because of torque $\tau$.

$$\tau = \frac{d\vec{L}}{dt} = Mgr \sin 90^\circ = Mgr$$

(b) Precession: A rapidly spinning gyroscope, with angular momentum, $L$, precesses around the $z$ axis. Its precessional motion is in the $xy$ plane.

$$d\vec{L} = \tau \, dt$$

(c) The change in angular momentum, $dL/dt$, leads to a rotation of $L$ about $O$.

$$dL = \tau \, dt = Mgr \, dt.$$  
$$d\phi = \frac{dL}{L} = \frac{Mgr \, dt}{I\omega}.$$  
$$\Omega = \frac{d\phi}{dt},$$  
$$\Omega = \frac{Mgr}{I\omega} \quad \text{(precession rate)}$$