# PHYSICS 210A : STATISTICAL PHYSICS
# HW ASSIGNMENT #9 SOLUTIONS

**(1)** Consider a two-state Ising model, with an added quantum dash of flavor. You are invited to investigate the *transverse Ising model*, whose Hamiltonian is written

$$\hat{H} = -J \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - H \sum_i \sigma_i^z \ ,$$

where the $\sigma_i^\alpha$ are Pauli matrices:

$$\sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i \qquad , \qquad \sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i \ .$$

(a) Using the trial density matrix,

$$\varrho_i = \tfrac{1}{2} + \tfrac{1}{2}\, m_x\, \sigma_i^x + \tfrac{1}{2}\, m_z\, \sigma_i^z$$

on each site, compute the mean field free energy $F/N\hat{J}(0) \equiv f(\theta, h, m_x, m_z)$, where $\theta = k_{\rm B}T/\hat{J}(0)$, and $h = H/\hat{J}(0)$. *Hint: Work in an eigenbasis when computing* $\mathsf{Tr}\,(\varrho \ln \varrho)$.

(b) Derive the mean field equations for $m_x$ and $m_z$.

(c) Show that there is always a solution with $m_x = 0$, although it may not be the solution with the lowest free energy. What is $m_z(\theta, h)$ when $m_x = 0$?

(d) Show that $m_z = h$ for all solutions with $m_x \neq 0$.

(e) Show that for $\theta \leq 1$ there is a curve $h = h^*(\theta)$ below which $m_x \neq 0$, and along which $m_x$ vanishes. That is, sketch the mean field phase diagram in the $(\theta, h)$ plane. Is the transition at $h = h^*(\theta)$ first order or second order?

(f) Sketch, on the same plot, the behavior of $m_x(\theta, h)$ and $m_z(\theta, h)$ as functions of the field $h$ for fixed $\theta$. Do this for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$.

Solution :

(a) We have $\mathsf{Tr}\,(\varrho\, \sigma^x) = m_x$ and $\mathsf{Tr}\,(\varrho\, \sigma^z) = m_z$. The eigenvalues of $\varrho$ are $\frac{1}{2}(1 \pm m)$, where $m = (m_x^2 + m_z^2)^{1/2}$. Thus,

$$f(\theta, h, m_x, m_z) = -\tfrac{1}{2} m_x^2 - h m_z + \theta \left[ \frac{1+m}{2} \ln\!\left(\frac{1+m}{2}\right) + \frac{1-m}{2} \ln\!\left(\frac{1-m}{2}\right) \right] .$$

(b) Differentiating with respect to $m_x$ and $m_z$ yields

$$\begin{aligned}\frac{\partial f}{\partial m_x} &= 0 = -m_x + \frac{\theta}{2}\,\ln\!\left(\frac{1+m}{1-m}\right)\cdot\frac{m_x}{m} \\ \frac{\partial f}{\partial m_z} &= 0 = -h + \frac{\theta}{2}\,\ln\!\left(\frac{1+m}{1-m}\right)\cdot\frac{m_z}{m}\ .\end{aligned}$$

Note that we have used the result

$$\frac{\partial m}{\partial m_\mu} = \frac{m_\mu}{m}$$

where $m_\alpha$ is any component of the vector $\boldsymbol{m}$.

(c) If we set $m_x = 0$, the first mean field equation is satisfied. We then have $m_z = m\,\mathrm{sgn}(h)$, and the second mean field equation yields $m_z = \tanh(h/\theta)$. Thus, in this phase we have

$$m_x = 0 \qquad , \qquad m_z = \tanh(h/\theta)\ .$$

(d) When $m_x \neq 0$, we divide the first mean field equation by $m_x$ to obtain the result

$$m = \frac{\theta}{2}\,\ln\!\left(\frac{1+m}{1-m}\right)\ ,$$

which is equivalent to $m = \tanh(m/\theta)$. Plugging this into the second mean field equation, we find $m_z = h$. Thus, when $m_x \neq 0$,

$$m_z = h \qquad , \qquad m_x = \sqrt{m^2 - h^2} \qquad , \qquad m = \tanh(m/\theta)\ .$$

Note that the length of the magnetization vector, $m$, is purely a function of the temperature $\theta$ in this phase and thus does not change as $h$ is varied when $\theta$ is kept fixed. What does change is the canting angle of $\boldsymbol{m}$, which is $\alpha = \tan^{-1}(h/m)$ with respect to the $\hat{\boldsymbol{z}}$ axis.

(e) The two solutions coincide when $m = h$, hence

$$h = \tanh(h/\theta) \qquad \Longrightarrow \qquad \theta^*(h) = \frac{2h}{\ln\left(\frac{1+h}{1-h}\right)}\ .$$

Inverting the above transcendental equation yields $h^*(\theta)$. The component $m_x$, which serves as the order parameter for this system, vanishes smoothly at $\theta = \theta_{\mathrm{c}}(h)$. The transition is therefore second order.

(f) See Fig. 1.

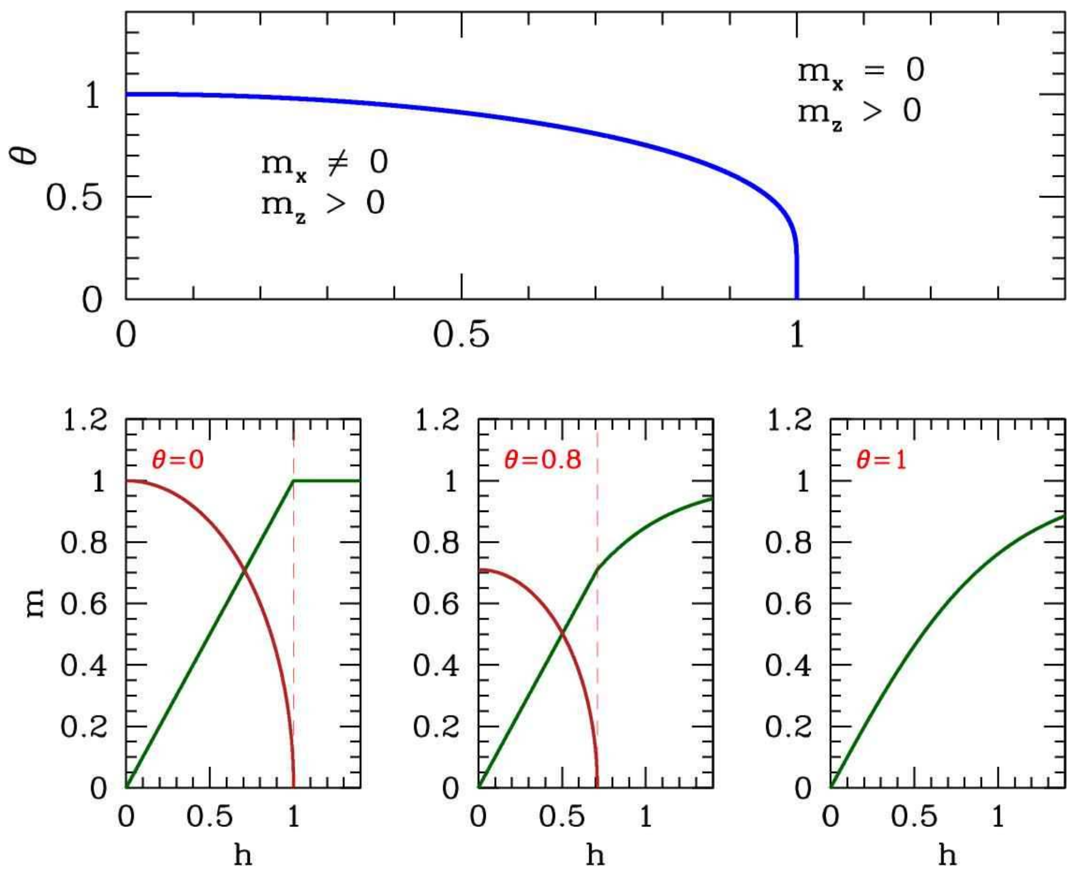


Figure 1: Solution to the mean field equations for problem 2. Top panel: phase diagram. The region within the thick blue line is a canted phase, where $m_x \neq 0$ and $m_z = h > 0$; outside this region the moment is aligned along $\hat{\boldsymbol{z}}$ and $m_x = 0$ with $m_z = \tanh(h/\theta)$.