PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #1

(1) Consider a system with K possible states $|\,i\,\rangle$, with $i \in \{1, \ldots, K\}$, where the transition rate W_{ij} between any two states is the same, with $W_{ij} = \gamma > 0$.

(a) Find the matrix Γ_{ij} governing the master equation $\dot{P}_i = -\Gamma_{ij}\, P_j$.

(b) Find all the eigenvalues and eigenvectors of Γ. What is the equilibrium distribution?

(c) Now suppose there are $2K$ possible states $|\,i\,\rangle$, with $i \in \{1, \ldots, 2K\}$, and the transition rate matrix is

$$W_{ij} = \begin{cases} \alpha & \text{if} \quad (-1)^{ij} = +1 \\ \beta & \text{if} \quad (-1)^{ij} = -1 \,, \end{cases}$$

with $\alpha, \beta > 0$. Repeat parts (a) and (b) for this system.

(2) A six-sided die is loaded so that the probability to throw a six is twice that of throwing a one. Find the distribution $\{p_n\}$ consistent with maximum entropy, given this constraint.

(3) Consider a three-state system with the following transition rates:

$$W_{12} = 0 \quad , \quad W_{21} = \gamma \quad , \quad W_{23} = 0 \quad , \quad W_{32} = 3\gamma \quad , \quad W_{13} = \gamma \quad , \quad W_{31} = \gamma \,.$$

(a) Find the matrix Γ such that $\dot{P}_i = -\Gamma_{ij} P_j$.

(b) Find the equilibrium distribution P_i^{eq}.

(c) Does this system satisfy detailed balance? Why or why not?

(4) The cumulative grade distributions of six 'old school' (no + or - distinctions) professors from various fields are given in the table below. For each case, compute the entropy of the grade distribution.

Professor	A	B	C	D	F
Landau	1149	2192	1545	718	121
Vermeer	8310	1141	231	56	7
Keynes	3310	4141	3446	1032	642
Noether	1263	1874	988	355	290
Borges	4002	2121	745	109	57
Salk	3318	3875	2921	1011	404
Turing	2800	3199	2977	1209	562

(5) A generalized two-dimensional cat map can be defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & p \\ q & pq+1 \end{pmatrix}}^{M} \begin{pmatrix} x \\ y \end{pmatrix} \ \mathsf{mod}\ \mathbb{Z}^2 \ ,$$

where p and q are integers. Here $x, y \in [0, 1]$ are two real numbers on the unit interval, so $(x, y) \in \mathbb{T}^2$ lives on a two-dimensional torus. The inverse map is

$$M^{-1} = \begin{pmatrix} pq+1 & -p \\ -q & q \end{pmatrix} \ .$$

Note that $\mathsf{det}\, M = 1$.

(a) Consider the action of this map on a pixelated image of size $(lK) \times (lK)$, where $l \sim 10$ and $K \sim 50$. Starting with an initial state in which all the pixels in the left half of the array are "on" and the others are all "off", iterate the image with the generalized cat map, and compute at each state the entropy $S = -\sum_{\boldsymbol{r}} p_{\boldsymbol{r}} \ln p_{\boldsymbol{r}}$, where the sum is over the K^2 different $l \times l$ subblocks, and $p_{\boldsymbol{r}}$ is the probability to find an "on" pixel in subblock $\boldsymbol{r}$. (Take $p = q = 1$ for convenience, though you might want to explore other values).

Now consider a three-dimensional generalization (Chen *et al.*, *Chaos, Solitons, and Fractals* **21**, 749 (2004)), with

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ \mathsf{mod}\ \mathbb{Z}^3 \ ,$$

which is a discrete automorphism of $\mathbb{T}^3$, the three-dimensional torus. Again, we require that both M and M^{-1} have integer coefficients. This can be guaranteed by writing

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p_x \\ 0 & q_x & p_x q_x + 1 \end{pmatrix} \quad , \quad M_y = \begin{pmatrix} 1 & 0 & p_y \\ 0 & 1 & 0 \\ q_y & 0 & p_y q_y + 1 \end{pmatrix} \quad , \quad M_z = \begin{pmatrix} 1 & p_z & 0 \\ q_z & p_z q_z + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and taking $M = M_x M_y M_z$, reminiscent of how we build a general $\mathsf{O}(3)$ rotation from a product of three $\mathsf{O}(2)$ rotations about different axes.

(b) Find M and M^{-1} when $p_x = q_x = p_y = q_y = p_z = q_z = 1$.

(c) Repeat part (a) for this three-dimensional generalized cat map, computing the entropy by summing over the K^3 different $l \times l \times l$ subblocks.

(d) 100 quatloos extra credit if you find a way to show how a three dimensional object (a ball, say) evolves under this map. Is it Poincaré recurrent?

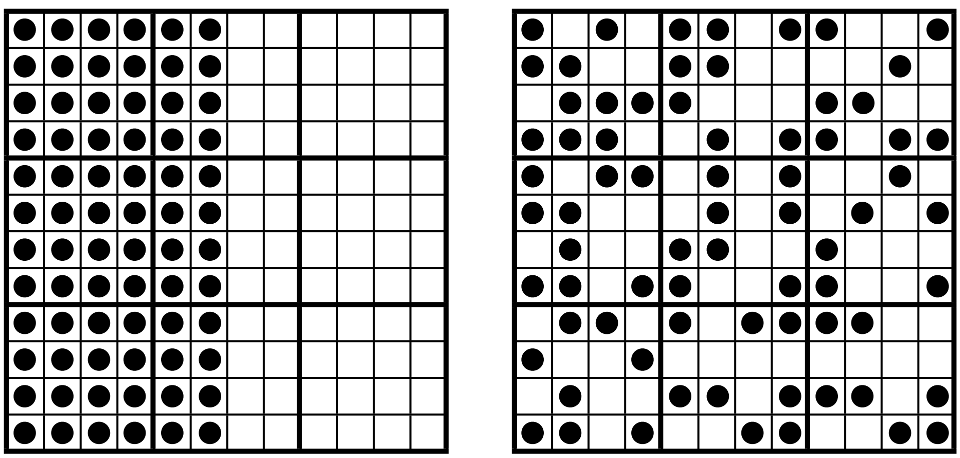

Figure 1: Two-dimensional cat map on a 12×12 square array with $l = 4$ and $K = 3$ shown. Left: initial conditions at $t = 0$. Right: possible conditions at some later time $t > 0$. Within each $l \times l$ cell $\boldsymbol{r}$, the occupation probability $p_{\boldsymbol{r}}$ is computed. The entropy $-p_{\boldsymbol{r}} \log_2 p_{\boldsymbol{r}}$ is then averaged over the K^2 cells.