1 Principle of relativity, Galilean transformations, Michelson-Morley experiment

"When you sit with a nice girl for two hours, it seems like two minutes. When you sit on a hot stove for two minutes, it seems like two hours that's relativity." ∼*Albert Einstein*

1.1 Galilean principle of relativity

The idea behind the principle of relativity is very simple: *inertial reference frames are indistinguishable from each other.*

Let us understand what this really means. First, what is an inertial reference frame? Let us first recall what a reference frame is in general. A reference frame is an object (or a set of objects) relative to which the positions of all the other objects are determined, and a clock (ideally clocks distributed everywhere) to determine time. We will usually refer to the reference frame by the name of the object it is connected to. For example, "the frame of the earth" is the frame in which positions are measured relative to some fixed point of the earth, with some clock (or clocks) that are at rest with respect to the earth. To be able to do quantitative calculations one usually assigns a coordinate system with the origin fixed at the given object, then the position of every other object is simply given by three coordinates. *Inertial reference frames* are those in which Newton's first law is satisfied, i.e. frames in which free objects retain their uniform motion or stay at rest. It is clear, that not every reference frame will do the job, consider as an example an accelerating bus. If you put a ball at rest on the ground it will start rolling back. On the other hand, any frame that is moving uniformly with respect to an inertial frame is also inertial, because if you have a free object uniformly moving in the inertial frame it will be moving uniformly (perhaps, with a different velocity) with respect to the other one too. The frame of the earth is not ideally inertial (because of the rotation of the earth around its axis and the sun) but for most of practical purposes it is very nearly inertial, and we will assume it is inertial throughout this course. For convenience, we will sometimes refer to "inertial frames" as just "frames", unless otherwise stated.

Now, what does the *indistinguishability* mean? Consider a train moving at a constant velocity relative to the earth. We now have two frames: the frame of the earth and the train, and they are clearly distinguishable. For example, a person sitting inside the train at rest is actually moving with respect to the earth. But now suppose we shut all the windows of the train and do not allow any contact between the inside and outside of the train. Will the person sitting inside the train be able to tell if the train is moving? Is there any experiment she can perform to determine the motion of the train? The principle of relativity says **no way**, this is the sense in which the two frames are indistinguishable! Everything looks the same inside the moving train as outside. If you set up the same experiment with the same initial conditions in two different inertial frames you will get the exact same outcome.

The principle of relativity was first formulated by *Galileo Galilei* in 1602 in his "Dialogue Concerning the Two Chief World Systems". He nicely described it as follows[1]:

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same

[1] Source: Wikipedia

spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.

In a more formal way, the principle of relativity says: *the physical laws must have the same form in all inertial frames*. Now, what the heck is a *physical law*? Remember, the main task of physics is to determine the evolution of different systems from given initial conditions. Physical laws are statements (equations, if you will) that do exactly that job. So "a person is moving at $10m/s$" is in no way a physical law, it is just telling the state of the person at some given time. This is good, because a moving person in one inertial frame can be at rest in another, and that would violate the principle of relativity. However, Newton's laws are physical laws (they are called *laws* for a reason). Consider the following problem: "a $2kg$ ball is at rest initially and subject to a constant force of $10N$, how fast will it be moving in 3 seconds?" We use Newton's second law (not worrying about the direction for this problem):

$$F = ma$$

to determine the acceleration: $a = 5m/s^2$, from which the final velocity

$$v_f = v_i + at = 15m/s$$

Now, if we want to get the same answer in all inertial frames given the same initial condition, $F = ma$ better have the same form in all frames! Let us now see that it actually is the case.

Before we can do that, we need to first derive the transformation laws between two inertial frames. We will refer to anything that happens at a given point at a given time as an *event*. The question is, given the coordinates and time of an event in one frame, how do we find the coordinates and time of it in another frame. Consider two frames: S and S', where S' is moving with respect to S in positive x direction with velocity u, fig. 1. We will denote the time and coordinates in frame S by (t, x, y, z) and in frame S' by (t', x', y', z'). It is conventional to call S the "rest frame" or the "laboratory frame", and S' the "moving frame" (however, it is important to keep in mind that rest and motion are relative concepts, in this case with respect to the earth). Also, we will sometimes refer to spatial coordinates and time as just coordinates. In classical mechanics time is assumed to be absolute, it flows the same way in two different frames. So if we choose the 0-moment of time to be the same in both frames, then the times will coincide at any later (or earlier) time. So we have our first transformation law:

$$t = t'$$

We will also assume that the two frames coincide at this 0-moment of time. Then at time t frame S' will have moved a distance ut, and from fig. 1 we can see that:

$$x = x' + ut$$

Also, since the motion is in the x direction, clearly:

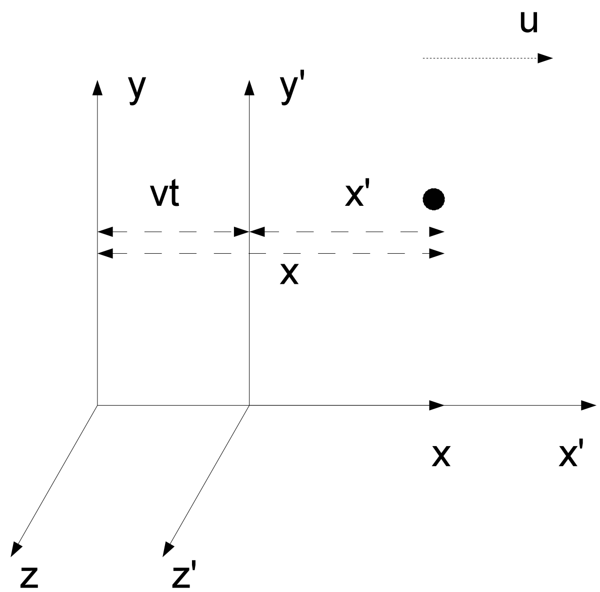

Figure 1: Frame S' is moving with respect to frame S in positive x direction with velocity u.

$$y = y'$$
$$z = z'$$

Putting everything together, we get the *Galilean transformation laws*:

$$\begin{aligned} t &= t' \\ x &= x' + ut' \\ y &= y' \\ z &= z' \end{aligned} \tag{1.1}$$

where we replaced t by t' (they are the same anyway) in the x-transformation equation to get all primed quantities on the right hand side and unprimed ones on the left hand side. It is easy to get the inverse transformation equations from unprimed to primed frame by simply solving (1.1) with respect to the primed coordinates and time. However, we will take a different approach, the use of which will be much better justified when we start dealing with the relativistic counterparts of (1.1). If we regard S' as the rest frame then S will be moving with the same velocity u but in the negative x direction. This means that we can get the inverse transformations by simply exchanging the primed coordinates with the unprimed ones and changing the sign of u. We then get:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned} \tag{1.2}$$

Problem 1 *Verify that solving (1.1) with respect to the primed coordinates gives the same answer (1.2).*

Let us now see how the velocity of a given object can be transformed from the primed to the unprimed frame (and vice-versa). Suppose that a particle is moving with velocity $\vec{v} = (v_x, v_y, v_z)$ as measured in frame S, and $\vec{v'} = (v'_x, v'_y, v'_z)$ as measured in frame S'. Recall, that the velocity projection in a given direction is the time derivative of the position projection in the same direction, in other words:

$$v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt} \tag{1.3}$$

and similarly:

$$v'_x = \frac{dx'}{dt'},\ v'_y = \frac{dy'}{dt'},\ v'_z = \frac{dz'}{dt'} \tag{1.4}$$

so, in order to obtain the relationship between the velocity components in two frames we simply differentiate both sides of (1.1)

$$\begin{aligned} dt &= dt' \\ dx &= dx' + udt' \\ dy &= dy' \\ dz &= dz' \end{aligned} \tag{1.5}$$

and divide the last three equations by dt:

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx'}{dt} + u\frac{dt'}{dt} \\ \frac{dy}{dt} &= \frac{dy'}{dt} \\ \frac{dz}{dt} &= \frac{dz'}{dt} \end{aligned} \tag{1.6}$$

We finally replace dt by dt' on the right hand side of (1.6), since they are equal, to finally get the velocity transformation equations:

$$\begin{aligned} v_x &= v'_x + u \\ v_y &= v'_y \\ v_z &= v'_z \end{aligned} \tag{1.7}$$

This result is nothing but what we would expect to get. The primed frame is moving in the x direction with velocity u so we need to add that velocity u to whatever x component of velocity an object may have to get the x component of velocity in the unprimed frame, while there is no relative motion in y and z directions, so the velocity has the same components in these directions in both frames.

To get the inverse transformation equations for velocities we use the same old trick of exchanging the primed and unprimed velocities while simultaneously changing the sign of u:

$$\begin{aligned} v'_x &= v_x - u \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned} \tag{1.8}$$

Problem 2 *Derive (1.8) by differentiating (1.2) instead of (1.1).*

Finally, we can differentiate (1.7) to get the transformations of acceleration between two frames (recall that acceleration is the time derivative of velocity). We get:

$$\begin{aligned} dv_x &= dv'_x \\ dv_y &= dv'_y \\ dv_z &= dv'_z \end{aligned} \tag{1.9}$$

Dividing by $dt = dt'$ we get:

$$\begin{aligned} a_x &= a'_x \\ a_y &= a'_y \\ a_z &= a'_z \end{aligned} \tag{1.10}$$

Acceleration is the same in both frames! The force acting on an object does not depend on the reference frame, so if $\vec{F} = m\vec{a}$ is true in one frame then it is true in the other frame: Newton's second law does respect the principle of relativity!

Problem 3 *Explain why Newton's first and third laws also respect the principle of relativity.*

So it is true that the laws of mechanics are invariant in form (this is fancy terminology which is used in literature and simply means "the same in form") in all inertial frames. This means that mechanical experiments cannot in principle tell absolute motion from rest of the train they are performed in. Now the question is, is there **absolutely** no way to tell if the train is moving? Are **all** the laws of physics invariant under Galilean transformations?

1.2 Enter Maxwell

James Clerk Maxwell formulated the laws of electricity and magnetism in a set of equations in his "On Physical Lines of Force" in 1861-1862. And guess what, Maxwell's equations **are not** invariant under Galilean transformations! The first reaction of physicists was that Maxwell's equations must be wrong. After all, Galilean principle of relativity had been around for over two centuries with no contradictions to it whatsoever. Nevertheless, all the experiments were on Maxwell's side, the theory was experimentally proved to be right!

We can explicitly write out the equations, do some boring algebra to transform them from one frame to the other and see that they indeed change their form, but this will not really help

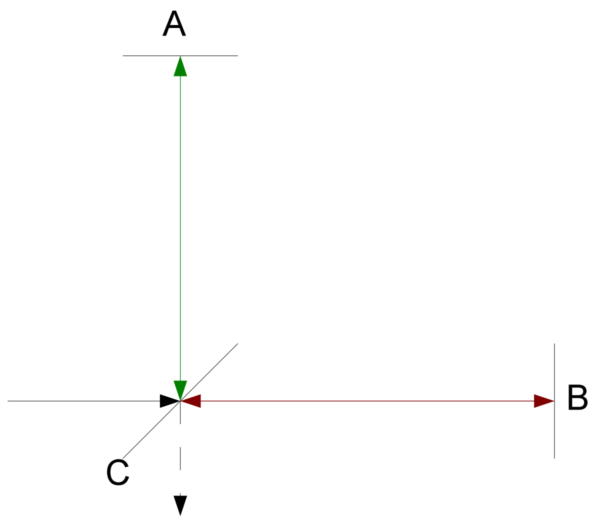

Figure 2: Diagram of Michelson-Morley apparatus at rest.

us understand what is going on (feel free to try it though, it is not as hard as it sounds). We will instead look at some consequences of them, which we can easily visualize. Perhaps the most profound consequence is the existence of electromagnetic waves, light in particular, which propagate at the speed of

$$c = 2.99792458 \times 10^8 m/s \approx 3 \times 10^8 m/s \tag{1.11}$$

called the *speed of light*. Now consider a car moving with velocity u and light coming out from the headlights. If light is moving with speed v with respect to the car then, by Galilean velocity transformation equations (1.7), it is moving with speed $v + u$ with respect to the earth. The question naturally arises, which frame is it where the speed of light is c. It clearly seems that there is one frame where the speed of light is c, the **absolute rest frame**, and the motion of any other frame can in principle be detected[2] by simply measuring the speed of light and comparing to c. Indeed, experimentalists started actively setting up experiments to detect absolute motion. The main difficulty in these experiments is the very big value of c, and very high precision required to detect small velocities compared to it. In fact, the orbital velocity of the earth around the sun is not that small, it is around $30km/s$, so the easiest thing to do was to try and detect the motion of the earth (and there is no need to put anything in motion, the earth is already moving!). One of the most precise experiments was the *Michelson-Morley* experiment performed in 1887, which tried to measure the velocity of the earth by interference effects.

1.3 Michelson-Morley experiment

Let us discuss the main setup of the experiment without going into technical details. The apparatus is schematically depicted in fig. 2. Light beam from a source is split into two perpendicular beams by a half-silvered mirror C, one going horizontally to the mirror B and coming back (depicted in red)[3], and the other going vertically to the mirror A and coming back (depicted in green). Both

[2] Note that light does not use any medium to propagate, which would be dragged with the isolated frame. With sound waves, for example, the absolute motion is impossible to detect because sound waves use air to propagate and air will be moving together with the isolated frame thus leaving the speed of sound unchanged.

[3] Note that these colors have nothing to do with the color of the light, both beams have the **same wavelength**. The colors are chosen for illustrational purposes only.

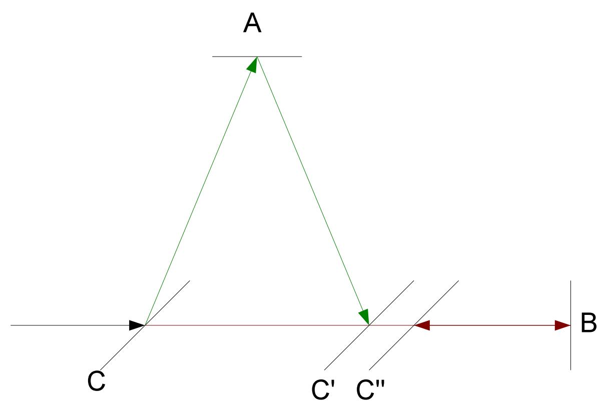

Figure 3: Diagram of Michelson-Morley apparatus moving to the right.

mirrors are placed at the same distance L from C, so if the apparatus is at rest, as depicted in fig. 2. then upon arrival both beams will have travelled the same distance $2L$, and they will both take the same time $2L/c$. However, they will arrive back at C at slightly different times if the apparatus is in motion, as depicted in fig. 3. Let us calculate the time difference assuming that the apparatus is moving to the right with velocity u. Suppose the "red" beam takes time t_1 to reach the mirror B, and t_2 to come back to C, and the "grean" beam takes time t_3 to reach the mirror A, which will be the same as coming back from A to C (this can be clearly seen in fig. 3). By the time light reaches B, B will have moved a distance ut_1 so light will have to travel a distance $L + ut_1$ instead of L during time t_1, which implies:

$$\begin{aligned} ct_1 &= L + ut_1 \\ t_1 &= \frac{L}{c-u} \end{aligned} \tag{1.12}$$

On the other hand, on the way back, the mirror C will have moved a distance ut_2 closer (to the final position C''), so light will need to travel only a distance $L - ut_2$ during time t_2, implying:

$$\begin{aligned} ct_2 &= L - ut_2 \\ t_2 &= \frac{L}{c+u} \end{aligned} \tag{1.13}$$

Both equations (1.12) and (1.13) can also be understood based on Galilean velocity transformation equations (1.7). Indeed, light traveling forward will be moving with velocity $c-u$ with respect to the apparatus, and light traveling back will be moving with velocity $c+u$ with respect to the apparatus. Adding (1.12) and (1.13) gives the total time for the "red" beam:

$$t_B = t_1 + t_2 = \frac{2Lc}{c^2 - u^2} = \frac{2L/c}{1 - u^2/c^2} \tag{1.14}$$

Let us now find t_3. By the time light reaches mirror A it will have moved a distance ut_3. Light itself will have travelled a distance ct_3, which in this case becomes the hypotenuse of a right-angled triangle with edges ut_3 and L. So

$$\begin{aligned} (ct_3)^2 &= L^2 + (ut_3)^2 \\ t_3 &= \frac{L}{\sqrt{c^2 - u^2}} \end{aligned} \tag{1.15}$$

the total time to A and back of the "green" beam is then

$$t_A = 2t_3 = \frac{2L}{\sqrt{c^2 - u^2}} = \frac{2L/c}{\sqrt{1 - u^2/c^2}} \tag{1.16}$$

We can see that the two times t_A and t_B differ by a factor of $\sqrt{1 - u^2/c^2}$, t_A is slightly smaller. Although this difference is small, the apparatus was well capable of measuring it. The high precision of the apparatus comes from the fact that interference between two light beams can measure differences in paths of the order of the wavelength of light, which is very small.

Problem 4 *Assume that visible light of wavelength* $400nm$ *was used in the Michelson-Morley experiment, and the apparatus is capable of measuring difference in paths of two beams as small as that wavelength. Assuming* $L = 100m$ *calculate the smallest possible value of velocity* u *that the experiment can measure. Can the velocity of earth* $30km/s$ *be measured by this setup?*

The experiment gave a negative result! The velocity of earth could not be detected. What would this mean? The principle of relativity would not give up. Any experiment that tried to measure the absolute motion completely failed. In the same way, any experiment that tried to rule out Maxwell's theory completely failed. This was a big mystery for physicist of the end of nineteenth century.