Homework 1

Solutions

1. The proposed solution $x(t) = x_i \cos\omega t + \left(\frac{v_i}{\omega}\right)\sin\omega t$

implies velocity $v = \frac{dx}{dt} = -x_i\omega \sin\omega t + v_i \cos\omega t$

and acceleration

$$a = \frac{dv}{dt} = -x_i\omega^2 \cos\omega t - v_i\omega \sin\omega t = -\omega^2\left(x_i \cos\omega t + \left(\frac{v_i}{\omega}\right)\sin\omega t\right) = -\omega^2 x$$

(a) The acceleration being a negative constant times position means we do have SHM, and its angular frequency is ω. At $t = 0$ the equations reduce to $x = x_i$ and $v = v_i$ so they satisfy all the requirements.

(b) $$v^2 - ax = \left(-x_i\omega \sin\omega t + v_i \cos\omega t\right)^2 - \left(-x_i\omega^2 \cos\omega t - v_i \sin\omega t\right)\left(x_i \cos\omega t + \left(\frac{v_i}{\omega}\right)\sin\omega t\right)$$

$$\begin{aligned} v^2 - ax = {} & x_i^2\omega^2 \sin^2\omega t - 2x_i v_i\omega \sin\omega t \cos\omega t + v_i^2 \cos^2\omega t \\ & + x_i^2\omega^2 \cos^2\omega t + x_i v_i\omega \cos\omega t \sin\omega t + x_i v_i\omega \sin\omega t \cos\omega t + v_i^2 \sin^2\omega t = x_i^2\omega^2 + v_i^2 \end{aligned}$$

So this expression is constant in time. On one hand, it must keep its original value $v_i^2 - a_i x_i$. On the other hand, if we evaluate it at a turning point where $v = 0$ and $x = A$, it is $A^2\omega^2 + 0^2 = A^2\omega^2$. Thus it is proved.

2. (a) $E = \frac{1}{2}kA^2$, so if $A' = 2A$, $E' = \frac{1}{2}k(A')^2 = \frac{1}{2}k(2A)^2 = 4E$

Therefore $\boxed{E \text{ increases by factor of } 4}$.

(b) $v_{\max} = \sqrt{\frac{k}{m}}A$, so if A is doubled, $\boxed{v_{\max} \text{ is doubled}}$.

(c) $a_{\max} = \frac{k}{m}A$, so if A is doubled, $\boxed{a_{\max} \text{ also doubles}}$.

(d) $T = 2\pi\sqrt{\frac{m}{k}}$ is independent of A, so $\boxed{\text{the period is unchanged}}$.

3. Using the simple harmonic motion model:

$$A = r\theta = 1\text{ m } 15°\frac{\pi}{180°} = 0.262\text{ m}$$

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\text{ m/s}^2}{1\text{ m}}} = 3.13\text{ rad/s}$$

(a) $v_{\max} = A\omega = 0.262 \text{ m } 3.13/\text{s} = \boxed{0.820 \text{ m/s}}$

(b) $a_{\max} = A\omega^2 = 0.262 \text{ m}(3.13/\text{s})^2 = 2.57 \text{ m/s}^2$

$a_{\tan} = r\alpha \qquad \alpha = \dfrac{a_{\tan}}{r} = \dfrac{2.57 \text{ m/s}^2}{1 \text{ m}} = \boxed{2.57 \text{ rad/s}^2}$

(c) $F = ma = 0.25 \text{ kg } 2.57 \text{ m/s}^2 = \boxed{0.641 \text{ N}}$

More precisely,

(a) $mgh = \dfrac{1}{2}mv_{\max}^2$ and $h = L(1-\cos\theta)$

$\therefore v_{\max} = \sqrt{2gL(1-\cos\theta)} = \boxed{0.817 \text{ m/s}}$

(b) $I\alpha = mgL\sin\theta$

$\alpha_{\max} = \dfrac{mgL\sin\theta}{mL^2} = \dfrac{g}{L}\sin\theta_i = \boxed{2.54 \text{ rad/s}^2}$

(c) $F_{\max} = mg\sin\theta_i = 0.250(9.80)(\sin 15.0°) = \boxed{0.634 \text{ N}}$

4. (a) The parallel-axis theorem:

$$I = I_{\text{CM}} + Md^2 = \frac{1}{12}ML^2 + Md^2 = \frac{1}{12}M(1.00 \text{ m})^2 + M(1.00 \text{ m})^2$$

$$= M\left(\frac{13}{12} \text{ m}^2\right)$$

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{M(13 \text{ m}^2)}{12Mg(1.00 \text{ m})}} = 2\pi\sqrt{\frac{13 \text{ m}}{12(9.80 \text{ m/s}^2)}} = \boxed{2.09 \text{ s}}$$

(b) For the simple pendulum

$T = 2\pi\sqrt{\dfrac{1.00 \text{ m}}{9.80 \text{ m/s}^2}} = 2.01 \text{ s}$ $\qquad$ difference $= \dfrac{2.09 \text{ s} - 2.01 \text{ s}}{2.01 \text{ s}} = \boxed{4.08\%}$

5. The total energy is $E = \dfrac{1}{2}mv^2 + \dfrac{1}{2}kx^2$

Taking the time-derivative, $\dfrac{dE}{dt} = mv\dfrac{d^2x}{dt^2} + kxv$

Use Equation 12.28: $\dfrac{md^2x}{dt^2} = -kx - bv$

$$\frac{dE}{dt} = v(-kx - bv) + kvx$$

Thus, $\boxed{\frac{dE}{dt} = -bv^2 < 0}$

6. $F_0 \cos\omega t - kx = m\dfrac{d^2x}{dt^2}$ $\qquad \omega_0 = \sqrt{\dfrac{k}{m}}$ (1)

$x = A\cos(\omega t + \phi)$ (2)

$\dfrac{dx}{dt} = -A\omega \sin(\omega t + \phi)$ (3)

$\dfrac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$ (4)

Substitute (2) and (4) into (1): $F_0 \cos\omega t - kA\cos(\omega t + \phi) = m\left(-A\omega^2\right)\cos(\omega t + \phi)$

Solve for the amplitude: $\left(kA - mA\omega^2\right)\cos(\omega t + \phi) = F_0 \cos\omega t$

These will be equal, provided only that ϕ must be zero and $kA - mA\omega^2 = F_0$

Thus, $A = \dfrac{F_0/m}{(k/m) - \omega^2}$

7. (a) Total energy $= \dfrac{1}{2}kA^2 = \dfrac{1}{2}(100\ \text{N/m})(0.200\ \text{m})^2 = 2.00\ \text{J}$

At equilibrium, the total energy is:

$$\frac{1}{2}(m_1 + m_2)v^2 = \frac{1}{2}(16.0\ \text{kg})v^2 = (8.00\ \text{kg})v^2.$$

Therefore,

$$(8.00\ \text{kg})v^2 = 2.00\ \text{J}, \text{ and } v = \boxed{0.500\ \text{m/s}}.$$

This is the speed of m_1 and m_2 at the equilibrium point. Beyond this point, the mass m_2 moves with the constant speed of 0.500 m/s while mass m_1 starts to slow down due to the restoring force of the spring.

(b) The energy of the m_1 -spring system at equilibrium is:

$$\frac{1}{2}m_1v^2 = \frac{1}{2}(9.00\ \text{kg})(0.500\ \text{m/s})^2 = 1.125\ \text{J}.$$

This is also equal to $\dfrac{1}{2}k(A')^2$, where A' is the amplitude of the m_1 -spring system.

Therefore,

$$\frac{1}{2}(100)(A')^2 = 1.125 \text{ or } A' = 0.150\ \text{m}.$$

The period of the m_1-spring system is $T = 2\pi\sqrt{\frac{m_1}{k}} = 1.885 \text{ s}$

and it takes $\frac{1}{4}T = 0.471 \text{ s}$ after it passes the equilibrium point for the spring to become fully stretched the first time. The distance separating m_1 and m_2 at this time is:

$$D = v\left(\frac{T}{4}\right) - A' = 0.500 \text{ m/s}(0.471 \text{ s}) - 0.150 \text{ m} = 0.085\,6 = \boxed{8.56 \text{ cm}}.$$

8. $y = (0.020\,0 \text{ m})\sin(2.11x - 3.62t)$ in SI units $\qquad A = \boxed{2.00 \text{ cm}}$

$k = 2.11 \text{ rad/m}$ $\qquad \lambda = \frac{2\pi}{k} = \boxed{2.98 \text{ m}}$

$\omega = 3.62 \text{ rad/s}$ $\qquad f = \frac{\omega}{2\pi} = \boxed{0.576 \text{ Hz}}$

$v = f\lambda = \frac{\omega}{2\pi}\frac{2\pi}{k} = \frac{3.62}{2.11} = \boxed{1.72 \text{ m/s}}$

9. Since μ is constant, $\mu = \frac{T_2}{v_2^2} = \frac{T_1}{v_1^2}$ and

$$T_2 = \left(\frac{v_2}{v_1}\right)^2 T_1 = \left(\frac{30.0 \text{ m/s}}{20.0 \text{ m/s}}\right)^2 (6.00 \text{ N}) = \boxed{13.5 \text{ N}}.$$

10. (a) If the end is fixed, there is inversion of the pulse upon reflection. Thus, when they meet, they cancel and the amplitude is $\boxed{\text{zero}}$.

(b) If the end is free, there is no inversion on reflection. When they meet, the amplitude is $2A = 2(0.150 \text{ m}) = \boxed{0.300 \text{ m}}$.

11. $\mu = 30.0 \text{ g/m} = 30.0 \times 10^{-3} \text{ kg/m}$

$\lambda = 1.50 \text{ m}$

$f = 50.0 \text{ Hz}: \quad \omega = 2\pi f = 314 \text{ s}^{-1}$

$2A = 0.150 \text{ m}: \quad A = 7.50 \times 10^{-2} \text{ m}$

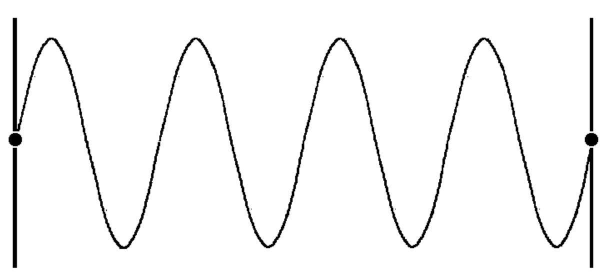

FIG. P13.20

(a) $y = A\sin\left(\frac{2\pi}{\lambda}x - \omega t\right)$

$\boxed{y = (7.50 \times 10^{-2})\sin(4.19x - 314t)}$

(b) $\mathsf{P} = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}(30.0 \times 10^{-3})(314)^2(7.50 \times 10^{-2})^2\left(\frac{314}{4.19}\right) \text{ W}$ $\qquad \boxed{\mathsf{P} = 625 \text{ W}}$

12. The sound speed is $v = 331\ \text{m/s} + 0.6\ \text{m/s}\cdot{}^\circ\text{C}(26^\circ\text{C}) = 347\ \text{m/s}$

(a) Let t represent the time for the echo to return. Then

$$d = \frac{1}{2}vt = \frac{1}{2}347\ \text{m/s}\ 24\times10^{-3}\ \text{s} = \boxed{4.16\ \text{m}}.$$

(b) Let Δt represent the duration of the pulse:

$$\Delta t = \frac{10\lambda}{v} = \frac{10\lambda}{f\lambda} = \frac{10}{f} = \frac{10}{22\times10^{6}\ 1/\text{s}} = \boxed{0.455\ \mu\text{s}}.$$

(c) $L = 10\lambda = \dfrac{10v}{f} = \dfrac{10(347\ \text{m/s})}{22\times10^{6}\ 1/\text{s}} = \boxed{0.158\ \text{mm}}$

13. The maximum speed of the speaker is described by

$$\frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kA^2$$

$$v_{\text{max}} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{20.0\ \text{N/m}}{5.00\ \text{kg}}}(0.500\ \text{m}) = 1.00\ \text{m/s}$$

The frequencies heard by the stationary observer range from

$$f'_{\text{min}} = f\left(\frac{v}{v+v_{\text{max}}}\right) \text{ to } f'_{\text{max}} = f\left(\frac{v}{v-v_{\text{max}}}\right)$$

where v is the speed of sound.

$$f'_{\text{min}} = 440\ \text{Hz}\left(\frac{343\ \text{m/s}}{343\ \text{m/s}+1.00\ \text{m/s}}\right) = \boxed{439\ \text{Hz}}$$

$$f'_{\text{max}} = 440\ \text{Hz}\left(\frac{343\ \text{m/s}}{343\ \text{m/s}-1.00\ \text{m/s}}\right) = \boxed{441\ \text{Hz}}$$

14. At distance x from the bottom, the tension is $T = \left(\dfrac{mxg}{L}\right) + Mg$, so the wave speed is:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{TL}{m}} = \sqrt{xg + \left(\frac{MgL}{m}\right)} = \frac{dx}{dt}.$$

(a) Then $t = \int_0^t dt = \int_0^L \left[xg + \left(\dfrac{MgL}{m}\right)\right]^{-1/2} dx$ $\qquad t = \dfrac{1}{g}\dfrac{\left[xg + (MgL/m)\right]^{1/2}}{\frac{1}{2}}\Bigg|_{x=0}^{x=L}$

$$t = \frac{2}{g}\left[\left(Lg + \frac{MgL}{m}\right)^{1/2} - \left(\frac{MgL}{m}\right)^{1/2}\right] \qquad \boxed{t = 2\sqrt{\frac{L}{g}}\left(\frac{\sqrt{m+M}-\sqrt{M}}{\sqrt{m}}\right)}$$

(b) When $M = 0$, as in the previous problem, $t = 2\sqrt{\frac{L}{g}}\left(\frac{\sqrt{m} - 0}{\sqrt{m}}\right) = \boxed{2\sqrt{\frac{L}{g}}}$

(c) As $m \to 0$ we expand $\sqrt{m + M} = \sqrt{M}\left(1 + \frac{m}{M}\right)^{1/2} = \sqrt{M}\left(1 + \frac{1}{2}\frac{m}{M} - \frac{1}{8}\frac{m^2}{M^2} + \ldots\right)$

to obtain
$$t = 2\sqrt{\frac{L}{g}}\left(\frac{\sqrt{M} + \frac{1}{2}\left(m/\sqrt{M}\right) - \frac{1}{8}\left(m^2/M^{3/2}\right) + \ldots - \sqrt{M}}{\sqrt{m}}\right)$$

$$t \approx 2\sqrt{\frac{L}{g}}\left(\frac{1}{2}\sqrt{\frac{m}{M}}\right) = \boxed{\sqrt{\frac{mL}{Mg}}}$$