Homework Set 5

Due Thursday, 07/28

1. Time dilation and length contraction

Problem 1: A spaceship travels from Earth to a star 100 light-years away. The ship is moving at a constant speed of 0.85*c*.

(a) How long does the trip take according to an observer on Earth?

(b) How long does the trip take according to an observer on board the spaceship?

(c) What is the distance from Earth to the star, according to the observer on the spaceship?

Solution:

(a) According to the observer on Earth, the ship moves at a speed of 0.85*c* = 0.85 light-years per year. The time for the trip is therefore 100ly / 0.85 ly/yr = 118 yrs.

(b) The time according to the observer on Earth is larger than the proper time measured by an observer on the spaceship due to time dilation:

$$t = \gamma t_0 \qquad t_0 = \frac{t}{\gamma} = t\sqrt{1 - v^2/c^2} = 118yrs\sqrt{1 - 0.85^2} = 62yrs$$

The time for an observer on the spaceship is thus only 62 years.

(c) The observer on the spaceship moves at a speed of 0.85*c* relative to the star and to Earth, and takes 62 years to get there. Therefore, according to him, the distance from Earth to the star is (62 yr) x (0.85 ly / yr) = 53 light-years. One could use length contraction to get the same result.

Problem 2: Unstable particles are produced in a nuclear reaction in a laboratory. The particles have an average lifetime of 1.0 x 10^{-10} seconds when at rest. The particles travel an average distance of 5*cm* from the point of production before decaying, according to an observer at rest in the laboratory. What is the speed of the particles relative to the stationary observer?

Solution: The lifetime of the particles is longer in the frame where they are moving. The range is therefore given by

$$r = v\tau = \gamma v \tau_0 = \frac{v}{\sqrt{1 - v^2/c^2}}\tau_0$$

Solve for the speed *v*:

$$\frac{v^2}{1-v^2/c^2} = \frac{r^2}{\tau_0^2} \qquad v^2\left(1+\frac{r^2}{c^2\tau_0^2}\right) = \frac{r^2}{\tau_0^2}$$

$$v = \frac{r/\tau_0}{\sqrt{1+r^2/c^2\tau_0^2}} = \frac{0.05m/10^{-10}s}{\sqrt{1+(0.05m/(3\times10^8 m/s\times10^{-10}s))^2}} =$$
$$2.57\times10^8 m/s = 0.858c$$

Problem 3: Suppose that at t = 0, Bob synchronizes his watch with Alice, and gets on a high-speed train that travels around and around the Earth at a speed of 500*km/h*. If the train never stops, how long would Bob need to stay on it before his watch was off from Alice's by one second, due to time dilation? Express the answer in years.

Solution: Alice sees Bob's clock run slower; the amount of time that passes for Alice is bigger by a factor of

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-((500km/h)(1000m/km)/(3600s/h))^2/(3\times10^8 m/s)^2}} =$$
$$= 1 + 1.07\times10^{-13}$$

Thus Alice's watch accumulates an extra 10^{-13} seconds or so for every second Bob is on the train. Thus it would take 10^{13} seconds for the watches to be off by a second. In years, this is

$$10^{13}s/(365d/yr\times86400s/d) = 320,000yrs!$$

Thus the high-speed train is not a very practical time machine. It would take over 300,000 years to generate a time difference of 1 second between someone on board the train and an inertial observer.

2. Velocity addition

Problem 4: Alice observes a particle moving with a speed of 0.8*c* in the *x* direction, while Bob observes the same particle moving with a speed of 0.5*c* in the *x* direction. How fast is Bob moving relative to Alice?

Solution: Let v be the speed of Bob relative to Alice, v_B be the speed of the particle relative to Bob, and v_A be the speed of the particle relative to Alice. Then, by the velocity addition equation,

$$v_A = \frac{v+v_B}{1+vv_B/c^2}$$

We know v_A and v_B, and need to solve for v:

$$v_A(1 + vv_B/c^2) = v + v_B$$

$$v_A - v_B = v(1 - v_Av_B/c^2)$$

$$v = \frac{v_A - v_B}{1 - v_Av_B/c^2} = \frac{0.8c - 0.5c}{1 - 0.8 \times 0.5} = 0.5c$$

Bob is thus moving at a speed of $0.5c$ relative to Alice.

Problem 5: An observer sees two spaceships, *A* and *B*, coming towards him from opposite directions with an equal speed. If the crew of ship *A* sees ship *B* approaching ship *A* at a speed of $0.9c$, what is the speed of the two spaceships relative to the observer?

Solution: Let $v = 0.9c$ be the speed of *B* relative to *A*. Let v_A be the speed of the observer relative to *A* and v_B be the speed of the observer relative to *B*. Since the two ships are moving at the same speed relative to the observer, but in opposite directions, $v_B = -v_A$. Then, the velocity addition formula gives

$$v_A = \frac{v + v_B}{1 + vv_B/c^2} = \frac{v - v_A}{1 - vv_A/c^2}$$

Both sides of the equation contain v_A, so we should solve for it:

$$(1 - vv_A/c^2)v_A = v - v_A$$

$$\frac{v}{c^2}v_A^2 - 2v_A + v = 0$$

$$v_A = \frac{c^2}{2v}\left(2 \pm \sqrt{4 - 4\frac{v^2}{c^2}}\right)$$

$$\frac{v_A}{c} = \frac{c}{v}\left(1 \pm \sqrt{1 - \frac{v^2}{c^2}}\right)$$

The plus sign would give a speed faster than c. We must use the minus sign:

$$\frac{v_A}{c} = \frac{1}{0.9}\left(1 - \sqrt{1 - 0.9^2}\right) = 0.627$$

The ships are moving with a speed of $0.627c$ relative to the observer.

3. Lorentz transformation and 4-vectors

Problem 6: Bob is moving in the *x* direction at a speed of 0.8*c* relative to Alice. He detects two simultaneous explosions at $t = 0$. One explosion happens at position $(x, y, z) = (10^6\text{km}, 0, 0)$, while the other happens at $(x, y, z) = (2 \text{ x } 10^6\text{km}, 0, 10^6 \text{ km})$.

(a) According to Alice, the explosions are not simultaneous. How much time passes between them? Use the Lorentz transformation to find the result.

(b) What is the distance between the explosions according to Bob? According to Alice?

Solution: According to Bob, both explosions happen at $t = 0$, so their coordinate 4-vectors are $X_B = (ct, x, y, z) = (0, 1, 0, 0) \text{ x } 10^6\text{km}$ and $(0, 2, 0, 1) \text{ x } 10^6\text{km}$.

The separation between the two events is the coordinates of the second event minus those of the first: $\Delta X_B = (0, 1, 0, 1) \text{ x } 10^6\text{km}$.

(a) Use the Lorentz transformation to find the time component of the separation according to Alice, which gives the separation between the events in time according to her. Relative to Bob, Alice moves in the negative *x* direction with $\beta = 0.8$, so

$$\Delta X_A^0 = \gamma\left(\Delta X_B^0 + \beta \Delta X_B^1\right) = \frac{1}{\sqrt{1-0.8^2}}\left(0 + 0.8 \times 1 \times 10^6 km\right) =$$
$$= 1.33 \times 10^6 km = c\Delta t_A$$

$$\Delta t_A = \frac{1.33 \times 10^6 km}{3 \times 10^5 km/s} = 4.44s$$

Therefore according to Alice, the events occur 4.44 seconds apart.

(b) According to Bob, the spatial part of the separation has components $(1, 0, 1) \text{ x } 10^6\text{km}$. The distance between the two events is just

$$d = \sqrt{1^2 + 0^2 + 1^2} \times 10^6 km = 1.41 \times 10^6 km$$

To find the distance according to Alice, we need to do the Lorentz transformation to find the space components of the separation according to her:

$$\Delta X_A^1 = \gamma\left(\Delta X_B^1 + \beta \Delta X_B^0\right) = \frac{1}{\sqrt{1-0.8^2}}\left(10^6 km + 0.8 \times 0\right) = 1.67 \times 10^6 km$$

$$\Delta X_A^2 = \Delta X_B^2 = 0 \qquad \Delta X_A^3 = \Delta X_B^3 = 1 \times 10^6 km$$

Thus the spatial part of the separation according to Alice is $(1.67, 0, 1) \text{ x } 10^6\text{km}$, and the distance between the two events is

$$d = \sqrt{1.67^2 + 0^2 + 1^2} \times 10^6 km = 1.95 \times 10^6 km$$

Problem 7: Bob is moving in the *x* direction, at a speed of 0.5*c* relative to Alice. He observes a particle moving in the *z* direction with a speed of 0.3*c*.

(a) What is the particle's 4-velocity relative to Bob?

(b) What is the particle's 4-velocity relative to Alice?

(c) How fast is the particle moving relative to Alice?

Solution:
(a) The 4-velocity according to Bob is

$$U_B = \gamma_B(c, \vec{v}_B) = \frac{1}{\sqrt{1-0.3^2}}(1,0,0,0.5)\,c = (1.048,0,0,0.524)\,c$$

(b) To get the 4-velocity relative to Alice, Lorentz transform. Alice is moving in the negative *x* direction relative to Bob with $\beta = 0.5$, so

$$U_A^0 = \gamma\left(U_B^0 + \beta U_B^1\right) = \frac{1}{\sqrt{1-0.5^2}}(1.048 + 0.5\times 0)\,c = 1.210c$$

$$U_A^1 = \gamma\left(U_B^1 + \beta U_B^0\right) = \frac{1}{\sqrt{1-0.5^2}}(0 + 0.5\times 1.048) = 0.605c$$

$$U_A^2 = U_B^2 = 0 \qquad U_A^3 = U_B^3 = 0.524c$$

Thus the 4-velocity according to Alice is

$$U_A = (1.210, 0.605, 0, 0.524)\,c$$

(c) The first component of the 4-velocity contains a factor of γ according to Alice, so it is easy to solve for v_A:

$$U_A^0 = 1.210c = \gamma_A c = \frac{c}{\sqrt{1-v_A^2/c^2}}$$

$$1 - \frac{v_A^2}{c^2} = \frac{c^2}{(1.210c)^2} \qquad \frac{v_A}{c} = \sqrt{1 - \frac{1}{1.210^2}} = 0.563c$$

The Doppler effect

Problem 8: A driver gets a ticket for running a red light. The driver claims that since he was driving towards the light, the light appeared green to him due to the Doppler effect. How fast would the driver have to be going if his story was true? Assume that red light has a typical wavelength of 650*nm* while green light has a wavelength of 520*nm*. Express the result as a fraction of the speed of light and in miles per hour.

Solution: If the driver moving towards the light at a fraction of the speed of light β, then the frequency of the light is Doppler shifted as follows:

$$f = f_0\sqrt{\frac{1+\beta}{1-\beta}}$$

Since $f = c / \lambda$, we can alter this expression into a change of wavelength as follows:

$$\frac{f}{f_0} = \frac{\lambda_0}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} = \frac{650nm}{520nm} = 1.25$$

Finally, solve for β:

$$\frac{1+\beta}{1-\beta} = 1.25^2 = 1.56 \qquad 1+\beta = 1.56(1-\beta)$$

$$2.56\beta = 0.56 \qquad \beta = 0.220$$

The driver would have to be going at 22% of the speed of light. One mile per hour is approximately 1.6 *km/h* = (1.6 *km/h*) (1000*m / km*) / (3600*s / h*) = 0.44*m/s*, so the driver's speed in miles per hour would be

$$(0.22 \times 3 \times 10^8 m/s)/(0.44(m/s)/mph) = 148 \text{ million } mph!$$

Considering the speed required, the story doesn't sound very likely.

Problem 9:
A radar emits microwave radiation with a wavelength of 20*cm*. The microwaves reflect off an aircraft, which is moving towards the radar with a speed of 400 *m/s*. In the radar device, the reflected waves are made to interfere with the outgoing waves; the slight difference in frequency causes beats, just as for sound waves. What is the beat frequency?

Solution:
The frequency of the waves relative to the aircraft is

$$f_A = f_0\sqrt{\frac{1+\beta}{1-\beta}} = f_0\sqrt{\frac{1+v/c}{1-v/c}}$$

The aircraft reflects the waves at the same frequency in its own reference frame. Relative to the aircraft, it is the radar that is approaching with speed *v*. Thus the frequency of the reflected signal relative to the radar is

$$f_R = f_A\sqrt{\frac{1+v/c}{1-v/c}} = f_0\frac{1+v/c}{1-v/c}$$

The beat frequency is the difference between the outgoing and the reflected frequencies relative to the radar:

$$f_{Beat} = f_R - f_0 = f_0\left(\frac{1+v/c}{1-v/c} - 1\right) = f_0\frac{2v/c}{1-v/c} = f_0\frac{2(400m/s)/(3\times10^8 m/s)}{1-(400m/s)/(3\times10^8 m/s)} =$$

$$= 2.67\times10^{-6} f_0 = 2.67\times10^{-6}\frac{c}{\lambda} = \frac{(2.67\times10^{-6})(3\times10^8 m/s)}{0.20m} = 4000hz$$

Beats are produced at a frequency of 4000 *hz*, a rate that can be easily detected and measured with electronic equipment. Measurement of the beat frequency is a commonly used way to detect tiny fractional differences between very large frequencies, such as those that occur in radars.

Wave-particle duality

Problem 10:
A 650*nm* laser operates at a power of 1.0W. The laser beam is approximately cylindrical, and has a diameter of 1*cm*.

(a) How many photons per second does the laser emit?

(b) How many photons are in a cubic centimeter of the laser beam?

Solution:
(a) Each photon has an energy of

$$E = hf = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34}Js)(3\times10^8 m/s)}{6.50\times10^{-7}m} = 3.06\times10^{-19}J$$

The energy is emitted at a rate of 1W = 1 J/s, so the number of photons per second is

$$\Phi_N = \frac{1.0J/s}{3.06\times10^{-19}J} = 3.27\times10^{18}s^{-1}$$

(b) The photons are emitted through an area $A = \pi r^2$, where r is the radius of the beam. Over a time Δt, $N = \Phi_N \Delta t$ photons are emitted; these photons move a distance $d = c\Delta t$ down the length of the beam. The photons emitted over time Δt thus occupy a cylindrical volume of

$$V = \pi r^2 c\Delta t$$

The number of photons per volume of the beam is thus

$$\frac{N}{V} = \frac{\Phi_N\Delta t}{\pi r^2 c\Delta t} = \frac{\Phi_N}{\pi r^2 c} = \frac{3.27\times10^{18}s^{-1}}{\pi(0.5cm)^2(3\times10^{10}cm/s)} = 1.39\times10^8 cm^{-3}$$

There are 1.39 x 10^8 photons per cubic centimeter of the beam.

Problem 11:
When ultraviolet light with a wavelength of 300*nm* is incident on a metal surface, electrons are emitted via the photoelectric effect. The electrons are stopped by a retarding potential of 1.2 volts.

(a) What is the work function of this metal?

(b) What is the longest wavelength of light that can eject electrons from this metal?

Solution:
(a) The electrons have an energy of 1.2eV, while the photons have an energy of hc / λ = 4.14eV. The difference is the work function of the metal: ϕ = 2.94eV.

(b) In order to eject electrons, the photons must have an energy of at least 2.94eV. Since $E = hc / \lambda$, $\lambda = hc / E$ = 422*nm*. Light with a wavelength longer than 422*nm* will not eject any electrons, as the photons don't have enough energy to overcome the binding energy of the electrons to the metal.

Problem 12:
An electron and a proton both have a kinetic energy of 10^4 eV.

(a) What are the speeds of the electron and the proton?

(b) What are the wavelengths of the electron and the proton?

Solution:
(a) For the electron, the rest energy mc^2 = 0.511MeV, while for the proton, mc^2 = 938MeV. In each case, this is much bigger than the kinetic energy, so the particles are essentially non-relativistic, and we can use the classical relationship between kinetic energy and speed to find the speed:

$$E_K \approx \frac{1}{2}mv^2 = \frac{1}{2}(mc^2)\frac{v^2}{c^2}$$

$$\frac{v}{c} = \sqrt{\frac{2E_K}{mc^2}} = \sqrt{\frac{2 \times 10^4 eV}{5.11 \times 10^5 eV}} = 0.198 \text{ for the electron}$$

$$\frac{v}{c} = \sqrt{\frac{2 \times 10^4 eV}{9.38 \times 10^8 eV}} = 0.0046 \text{ for the proton}$$

The proton is definitely non-relativistic, as its moving at only 0.0046*c*, but you might worry about the electron, which moves at almost 20% the speed of light. You can use the relativistic energy-velocity relationship $E = mc^2 + E_K = \gamma mc^2$ instead; the result will be different by 2% or so, but the classical approximation is still very accurate.

(b) The de Broglie wavelengths are related to the momentum by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\hbar}{p} = \frac{h}{p}$$

We will calculate the momentum from the energy using the classical approximation (if you want better than 2% accuracy, you can use $E^2 = p^2c^2 + m^2c^4$ for the electron instead).

$$E_K \approx \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{p^2c^2}{2mc^2}$$

$$p = \frac{1}{c}\sqrt{2mc^2E_K}$$

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2E_K}} =$$

$$= \frac{(4.14\times10^{-15}eV\,s)(3\times10^8 m/s)}{\sqrt{2(5.11\times10^5 eV)(10^4 eV)}} = 1.23\times10^{-11}m \text{ for the electron}$$

$$\lambda = \frac{(4.14\times10^{-15}eV\,s)(3\times10^8 m/s)}{\sqrt{2(9.38\times10^8 eV)(10^4 eV)}} = 2.87\times10^{-13}m \text{ for the proton}$$

You can see that more massive particles have shorter de Broglie wavelengths.

Problem 13:
An electron microscope is to resolve features as small as 10*nm* across. What is the potential through which the electrons in the microscope must be accelerated?

Solution: We want to achieve a de Broglie wavelength of at 10*nm* or smaller. The corresponding energy is

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{h^2c^2}{2mc^2\lambda^2} =$$

$$= \frac{(4.14\times10^{-15}eV\,s)^2(3\times10^8 m/s)^2}{2(5.11\times10^5 eV)(10^{-8}m)^2} = 0.015eV$$

An energy of only 15 meV is required.

Problem 14:
A beam of electrons is accelerated through a potential of 5000V. The electrons are incident on a crystal with interatomic spacing of 0.15*nm*. What is the angular separation between adjacent diffraction peaks?

Solution: First we find the wavelength of the electrons. This is

$$\lambda = \frac{hc}{\sqrt{2mc^2E_K}} = 1.73\times10^{-11}m$$

Now the angle of the first minimum away from the central minimum is

$$d \sin\theta = \lambda \qquad \sin\theta = \frac{d}{\lambda} = \frac{1.73 \times 10^{-11} m}{1.5 \times 10^{-10} m} = 0.1153$$

$$\theta = 6.62^\circ$$

Mass, energy and momentum

Problem 15:
How much energy would it take to accelerate an electron to a speed of 0.99*c*? What about 0.999*c*? Express the answers in electron-volts.

Solution:
The kinetic energy is $E_K = (1 - \gamma)mc^2$. For $v = 0.99c$,

$$\gamma = \frac{1}{\sqrt{1 - 0.99^2}} = 7.08$$

Therefore, $E_K = 6.08mc^2$ = 3.11 MeV.

For $v = 0.999c$, $\gamma = 22.4$, and $E_K = 21.4mc^2$ = 10.9 MeV.

Problem 16:
An electron has a kinetic energy of 1MeV relative to Alice. It moves in the positive *x* direction.

(a) Bob is moving in the negative *x* direction (opposite to the direction of the electron's motion) at a speed of 0.5*c*. What is the kinetic energy of the electron relative to Bob?

(b) How fast is the electron moving relative to Alice? Relative to Bob? Obtain both result from the electron's kinetic energy.

(c) Now obtain the speed relative to Bob from the speed relative to Alice using relativistic velocity addition. You should get the same result as in part (b).

Solution:
(a) For this part, we will construct the energy-momentum 4-vector of the electron in Alice's frame, and transform to Bob's frame. There are of course other ways to calculate the energy in Bob's frame. For Alice,

$$E_A = mc^2 + E_{K,A} = 1.511MeV$$

$$E_A^2 = m^2c^4 + p_A^2c^2 \quad p_Ac = \sqrt{E_A^2 - (mc^2)^2} = 1.595MeV$$

$$P_A = \frac{1}{c}(E_A, \vec{p}_Ac) = \frac{1}{c}(1.511, 1.595, 0, 0)\, MeV$$

To get the energy in Bob's frame, Lorentz transform the first component of the energy-momentum 4-vector to Bob's frame:

$$P_B^0 = \frac{E_B}{c} = \gamma\left(P_A^0 + \beta P_A^1\right) = \frac{1}{\sqrt{1-0.5^2}}(1.511 + 0.5 \cdot 1.595)\frac{MeV}{c} = 2.67\frac{MeV}{c}$$

$$E_B = 2.67MeV$$

This is the total energy, which includes a rest energy of 0.511MeV. Thus the electron's kinetic energy in Bob's frame is 2.67 - 0.511 = 2.15MeV.

(b) The speed of the electron relative to Alice and Bob can be obtained from the total energy by using

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \qquad \left(1 - \frac{v^2}{c^2}\right) = \frac{m^2c^4}{E^2}$$

$$\frac{v}{c} = \sqrt{1 - \frac{m^2c^4}{E^2}} = \sqrt{1 - \left(\frac{0.511MeV}{1.511MeV}\right)^2} = 0.941 \text{ for Alice}$$

$$\frac{v}{c} = \sqrt{1 - \left(\frac{0.511MeV}{2.67MeV}\right)} = 0.982 \text{ for Bob}$$

(c) Using velocity addition, the speed of the electron relative to Bob is

$$v_B = \frac{v_A + v}{1 + vv_A/c} = \frac{0.941c + 0.5c}{1 + 0.941 \cdot 0.5} = 0.98c$$

The discrepancy is very small, and due to rounding some of the numbers in the calculation.

Problem 17:
A photon with an energy of E_γ= 25eV collides with an electron at rest, and is reflected back in the direction it came from.

(a) What is the 4-momentum of the electron and of the photon before the collision, in terms of the photon energy E_γ and the electron mass m? Assume the photon is initially traveling in the x-direction.

(b) What is the 4-momentum of the electron and the photon after the collision? What is the kinetic energy of the electron after the collision, in eV? Use energy-momentum conservation, and the mass-energy-momentum relations for the photon and electron.

Solution:
The amount of algebra in this problem has been found to be a bit ridiculous (around 2 pages if done efficiently). In the interest of saving time typing up 2 pages of algebra, I will skip this solution. See other problems involving energy-momentum conservation to see how to go about doing this.

Nuclear and particle physics

Problem 18:
Carbon 14 decays via beta decay.

(a) What isotope does it turn into?

(b) The carbon 14 nucleus has a mass of 14.003241 u, and the nucleus it turns into has a mass of 14.003074 u. Suppose that the electron and the antineutrino are emitted in such a way that the nucleus remains at rest. What are the energies of the electron and the antineutrino? How fast is the electron moving (as a fraction of the speed of light)?

Solution:
(a) Beta decay increases the charge of the nucleus by 1 by converting a neutron to a proton, but it leaves the number of nucleons unchanged. Thus the product of this reaction is nitrogen 14 (which is the most common naturally occurring isotope of nitrogen).

(b) First convert the atomic masses to electron-volts: 1 u = 931.46 MeV / c^2, so the energy decreases by 0.156 MeV. But that's the change in the atomic rest energy; the nitrogen actually has an additional electron, with a rest energy of 0.511 MeV, so the decrease in nuclear rest energy is actually 0.156 + 0.511 = 0.667 MeV. By conservation of energy, this must be equal to the total energy of the electron and the antineutrino.

Conservation of momentum tells us that since the nucleus is at rest both in the initial state, the total momentum must be zero in the final state as well. Since the nucleus remains at rest, the electron and the antineutrino momenta must add up to zero. Thus we have the following equations for the energy and momentum conservation:

$$E_e + E_\nu = \Delta E = 0.667 MeV$$
$$\vec{p}_e = -\vec{p}_\nu$$

We can choose the *x*-axis so that the electron moves in the positive *x* direction with momentum of magnitude p_e, while the neutrino moves in the negative *x* direction with momentum of magnitude p_ν. Then the second equation becomes

$$p_e = p_\nu$$

We have 2 equations and 4 unknowns, so we supplement the equations with energy-momentum relationships for the electron and the antineutrino. The antineutrino is nearly massless, while the electron has mass *m*:

$$E_\nu = p_\nu c \qquad E_e^2 = p_e^2 c^2 + m^2 c^4$$

Plug this into the equation for the conservation of momentum:

$$p_e = p_\nu$$
$$p_e^2 c^2 = p_\nu^2 c^2$$
$$E_e^2 - m^2 c^4 = E_\nu^2$$

Plug this into the equation for conservation of energy as follows:

$$E_e + E_\nu = \Delta E$$
$$E_\nu = \Delta E - E_e$$
$$E_\nu^2 = \Delta E^2 + E_e^2 - 2E_e \Delta E = E_e^2 - m^2 c^4$$
$$\Delta E \left(2E_e - \Delta E\right) = m^2 c^4$$
$$E_e = \frac{1}{2}\left(\frac{m^2 c^4}{\Delta E} + \Delta E\right) = \frac{1}{2}\left(\frac{(0.511)^2}{0.667} + 0.667\right) MeV = 0.529 MeV$$

This is the total energy of the electron; the kinetic energy is 0.529 - 0.511MeV = 18 keV

The energy of the antineutrino is

$$E_\nu = \Delta E - E_e = 0.667 MeV - 0.529 MeV = 0.138 MeV = 138 keV$$

This is all kinetic energy, since the neutrino has no mass and thus no rest energy.

The electron's speed can be calculated from the energy as follows

$$E = \gamma m c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$
$$\frac{v}{c} = \sqrt{1 - \frac{m^2 c^4}{E^2}} = 0.259$$

Problem 19:
Plutonium-239 has a half-life of approximately 24,000 years. It decays to uranium-235 via alpha decay. The isotope mass of Pu-239 is 239.052156 u, the isotope mass of uranium-235 is 235.0439299 u, and that of the alpha particle is 4.002602 u.

(a) How many decays per second spontaneously occur in 1 kilogram of Pu-239?

(b) How much energy per second is released by all these decays?

(c) The heat radiated away by an object with temperature T is given by $P = \sigma A(T^4 - T_0^4)$, where T is the temperature of the object in kelvins, T_0 is the ambient temperature, σ is Stefan-Boltzmann constant ($\sigma = 5.67 \times 10^{-8} Wm^{-2}K^{-4}$) and A is the surface area of the object. Plutonium has a density of 19.82g cm^{-3}. If the 1kg piece of plutonium is shaped like a sphere and all the heat from the decays is radiated away, what is the temperature of the surface of the sphere? Assume the ambient temperature is 273K.

Solution:
(a) The number of atoms in 1 *kg* of plutonium-239 is equal to

$$N = \frac{(1000g)(6.022 \times 10^{23} mol^{-1})}{239.05 g/mol} = 2.52 \times 10^{24} \text{ atoms}$$

(Instead of using the method with Avogadro's number from chemistry, we could simply convert the atomic mass unit to kilograms, and divide 1*kg* by that mass).

The chance that each atom will decay in 1 second is 1*s*/(24000y x 365 d/y x 86400 s/d), so the number of decays per second is

$$r = \frac{2.52 \times 10^{24}}{24000 \cdot 365 \cdot 86400 s} = 3.33 \times 10^{12} s^{-1}$$

(b) First, we need to compute the energy released per decay. Here, the amount of electrons is the same for the helium plus the uranium as for the plutonium, since we are not changing the charge of the isotope. Thus the difference in the total isotope mass is entirely due to change of mass of the nuclei, not because the neutral atoms have different numbers of electrons. This gives an amount of energy released

$$\Delta E = (239.052156u - 235.0439299u - 4.002602u)\, c^2 \times 931.46 (MeV/c^2)/u =$$
$$= 5.24 MeV = (5.24 MeV)(1.602 \times 10^{-13} J/MeV) = 8.39 \times 10^{-13} J$$

The energy released by all the collisions per second gives the power produced by the radioactive decay:

$$P = rE = (3.33 \times 10^{12} s^{-1})(8.39 \times 10^{-13} J) = 2.80 W$$

(c) A kilogram of plutonium would have a volume of 1000 g / 19.82 g/cm^3 = 50.5cm^3. For a sphere, we have

$$V = \frac{4}{3}\pi R^3 \qquad R = \left(\frac{3V}{4\pi}\right)^{1/3} = 2.29cm$$
$$A = 4\pi R^2 = 66.0cm^2 = 6.60 \times 10^{-3}m^2$$

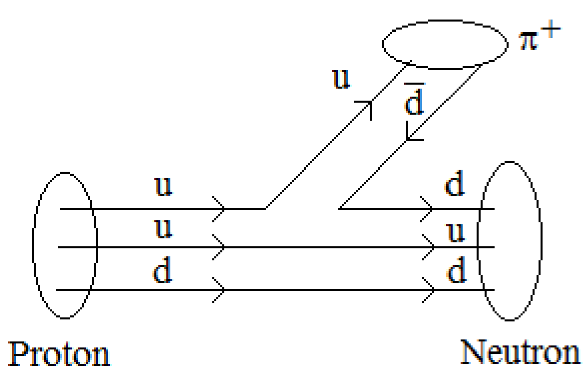
Solve for the surface temperature of the piece of plutonium:

$$P = \sigma A\left(T^4 - T_0^4\right)$$
$$T^4 - T_0^4 = \frac{P}{\sigma A}$$
$$T = \left(\frac{P}{\sigma A} + T_0^4\right) = \left(\frac{2.80W}{(5.67 \times 10^{-8}Wm^{-2}K^{-4})(6.60 \times 10^{-3}m^2)} + (273K)^4\right) =$$
$$= 338K = 65°C$$

The piece of plutonium would thus be quite warm, almost uncomfortably hot to the touch.

Problem 20:
Suppose that a collision causes a quark to begin accelerating away from a proton. Two particles are formed as a result, one of which is a neutron. What is the other particle?

Solution:
A proton can be turned into a neutron by taking out a *u* quark and replacing it with a *d* quark. Thus, it was a *u* quark that got kicked out of the proton, and since unbound quarks do not exist, a *d* quark-antiquark pair formed to allow it to escape. The *d* quark remained with the former proton, turning it into a neutron, while the *d* antiquark went with the *u* quark to form a charged pion. The *u* quark has a charge of +2/3, while the *d* antiquark has a charge of +1/3; the other particle thus has charge +1 and is a positively charged pion, π^+:

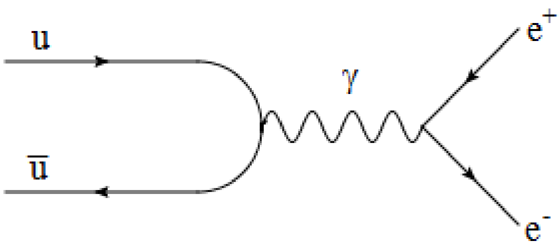


Problem 21:
A neutral pion consists of a quark-aniquark pair, both of the same type. A charged pion consists of a pair of different types (for example, up quark and down antiquark). Pions are lighter than any particles apart from photons, electrons and neutrinos.

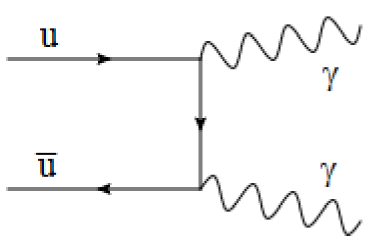
(a) How does a neutral pion decay? How does a charged pion decay? Draw Feynman diagrams for the decay processes.

(b) Which would you predict would have a longer lifetime, a charged pion or a neutral pion? Why?

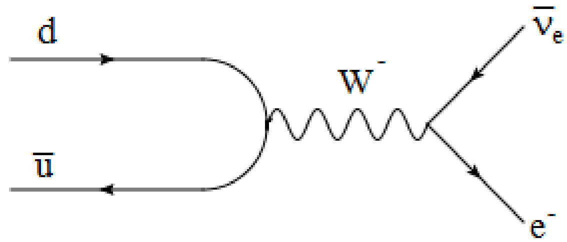
Solution:
(a) Since the quark and antiquark in the neutral pion are of the same type, they can annihilate to form a photon or a *Z* boson. The photon or *Z* boson can then form an electron-positron pair; the Z boson can form either an electron-positron pair or a neutrino-antineutrino pair. However, the probability of the photon being an intermediate is much higher, since the photon is massless, while the Z boson has a huge mass. The dominant reaction from this kind of process is thus



Another, approximately equally likely, possibility is the formation of two photons in the final state (note that you can't have just one photon in the final state because that would violate energy-momentum conservation):



For a charged pion, let's consider a negatively charged one, consisting of a down quark and an up antiquark. The positively charged pion behaves the same way, except with particles and antiparticles reversed. Now, the only thing that the down quark and up antiquark can combine to form is the W^- boson. The W^- boson must in turn produce an electron and an electron antineutrino, since all other possible products would be heavier than the pion. This decay process looks like this:



(b) A neutral pion decays primarily via photons, which cost no energy to produce, being massless. The charged pion must go through the very massive W boson as an intermediate state. The mass of the W boson essentially forms an energy barrier to the pion's decay, and slows it down greatly. This is in fact confirmed experimentally: the charged pions have a lifetime of 2.6×10^{-8} *s* while the neutral pion has a much shorter lifetime of 8.4×10^{-17} *s*. The neutral pion decays about 300 million times faster!

Problem 22:
In some grand unified theories, the proton can decay into a pair of particles, with a very long lifetime.

(a) Based on conservation of charge, energy, momentum and spin, what kind of particles could a proton decay into?

(b) Pick a possible pair of particles. If the proton is at rest before it decays, what will be the energy of each of these particles after the decay?

Solution:

(a) The pair of particles must have a half-integer spin, a net charge of +1, and a combined mass less than that of a proton.

Pions have integer spin (0 or 1) while leptons have half-integer spin. We must have one of each to give a half-integer spin. Electrons, muons, neutrinos and their antiparticles, as well as charged and neutral pions, are lighter than the proton, and are thus candidates for proton decay products.

One possibility is a positron or anti-muon, and a neutral pion. Another possibility is a charged pion with charge +1, and a neutrino or antineutrino of any flavor. Note that the grand unified theory doesn't necessarily respect conservation of lepton number, just the conservation laws given above.

(b) The proton's mass, Mc^2, goes to the energy of the particles in the final state. The momenta of the two particles are equal and opposite, since they must add up to zero (which is the proton's initial momentum). For particle masses m_1 and m_2, the conservation of energy and momentum, as well as the energy-momentum relationships, give the following equations:

$$E_1 + E_2 = Mc^2 \qquad p_1 + p_2 = 0$$

$$E_1^2 = p_1^2c^2 + m_1^2c^4 \qquad E_2^2 = p_2^2c^2 + m_2^2c^4$$

We have four equations and four unknowns.

Plugging the energy-momentum relationships into the second equation (conservation of momentum) gives

$$p_1^2c^2 = p_2^2c^2$$
$$E_1^2 - m_1^2c^4 = E_2^2 - m_2^2c^4$$

Plugging this into the first equation (conservation of energy) gives

$$E_2^2 = \left(Mc^2 - E_1\right)^2 = E_1^2 + m_2^2c^4 - m_1^2c^4$$
$$M^2c^4 - 2Mc^2E_1 = m_2^2c^4 - m_1^2c^4$$

$$E_1 = \frac{M^2c^4 - m_2^2c^4 + m_1^2c^4}{2Mc^2}$$
$$E_2 = \frac{M^2c^4 + m_2^2c^4 - m_1^2c^4}{2Mc^2}$$

Now we need to pick a process and look up some masses. Suppose the proton decays into a positron and a neutral pion. The mass of the positron is 0.511MeV, same as that of an electron. The mass of the neutral pion is easy to look up on Wikipedia, and is equal to 135 MeV. The mass of the proton is 938 MeV. The energy of each product is therefore

$$E_{e+} = \frac{938^2 - 135^2 + 0.511^2}{2 \times 938} = 459MeV$$
$$E_{\pi 0} = \frac{938^2 - 0.511^2 + 135^2}{2 \times 938} = 479MeV$$

The kinetic energy of the positron and the pion is the total energy minus the rest energy. The kinetic energy of the electron is therefore about 458 MeV, while that of the pion is 344 MeV.