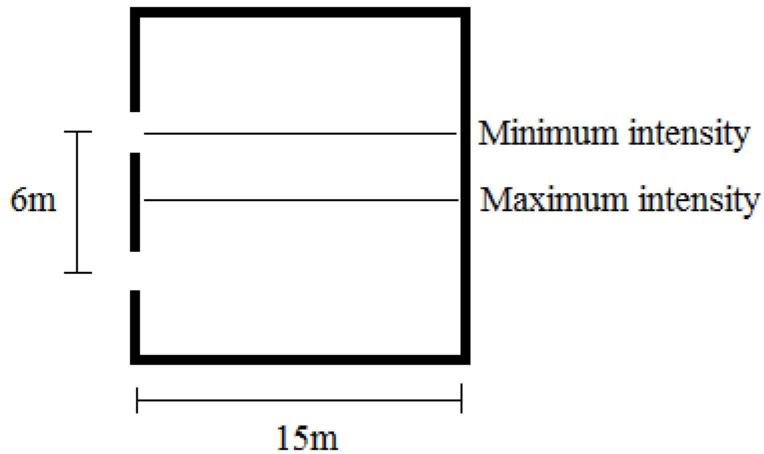
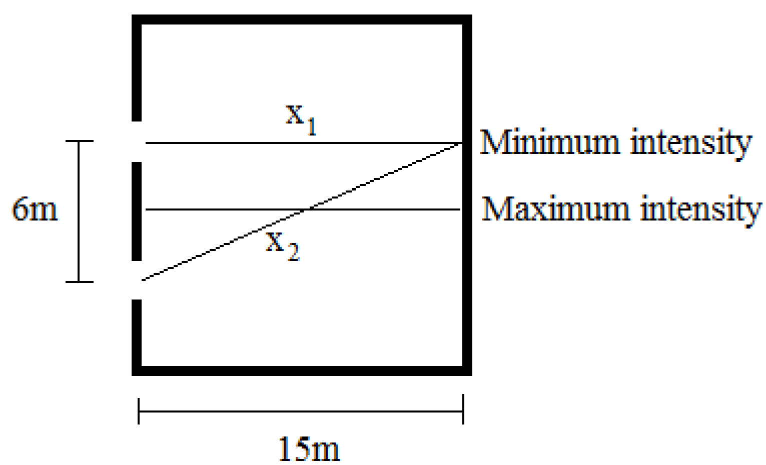
Homework Set 4

Due Wednesday, 07/20

Problem 1: A room has two open windows on the same wall, at a distance of 4 meters apart. A loud car horn is sounded outside. An observer along the opposite wall, 15 meters away, can hear the sound loudly when he is directly opposite the point between the two windows, but if he moves to the point directly across from one of the windows, the sound becomes much quieter, growing louder again as the observer moves farther away from the center. What is the main frequency of the car horn?

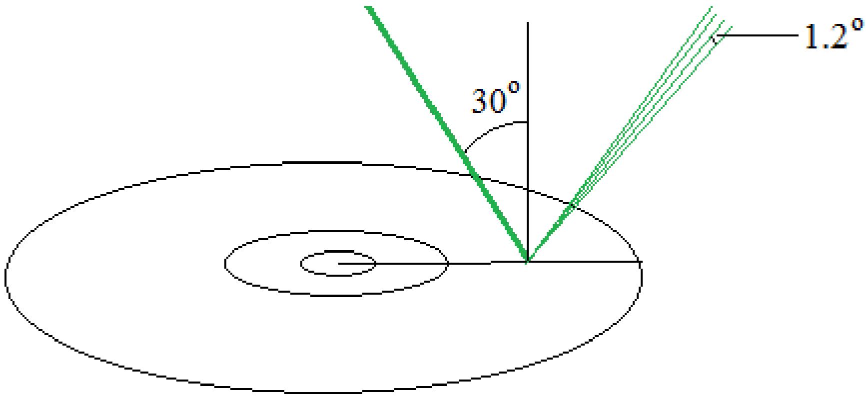


Solution: The point directly across from the window is the first point where destructive interference occurs. The path length difference there is $1/2\lambda$.

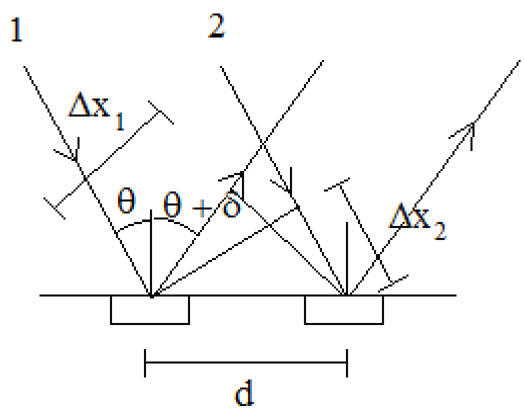


From the diagram above, the path lengths from the two windows are $x_1 = 15m$ and $x_2 = (6^2+15^2)^{1/2}m = 16.16m$. The path length difference is half the wavelength, and is equal to $1.16m$. Thus the wavelength is $2.32m$. This corresponds to a frequency of $f = v / \lambda = 330 m/s / 2.32m = 142hz$.

Problem 2: Laser light of wavelength l = 520*nm* is incident on a CD at an angle of 30 degrees to the normal, as shown in the diagram. The reflected light is projected onto a screen. The diffraction fringes are found to lie in directions 1.2 degrees apart. What is the separation between the tracks on the CD? Hint: the laser light arriving at the different tracks already has a path length difference from the laser, in addition to the path length difference to the screen.



Solution: The CD has a bunch of parallel tracks that reflect light and act as a diffraction grating. Since the diffraction grating produces the same diffraction angles as a double slit with the same spacing, let us look at a pair of nearby tracks and see what is the path length difference for the light reflected off these two tracks:

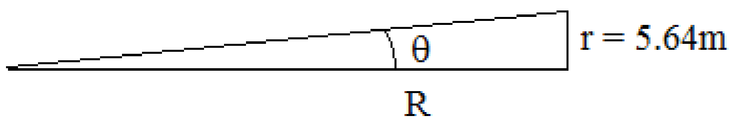


Suppose the light comes in at an angle of $\theta = 30^o$, and we look at a point on the screen in the direction $\phi = \theta + \delta$. Note that we use this angle because some light will be diffracted away by an angle δ from the central maximum given by the law of reflection. Beam 1 accumulates an additional path length $\Delta x_1 = d \sin(\theta + \delta)$ after reflection, while beam 2 has an additional path length $\Delta x_2 = d \sin \theta$ prior to reflection. The path length difference is $\Delta x_1 - \Delta x_2 = d(\sin(\theta+\delta) - \sin \theta) = n\lambda$ for constructive interference. The $n = 1$ diffraction maximum is separated from the central maximum by $\delta = 1.2^o$, so

$$d = \frac{520nm}{\sin(30° + 1.2°) - \sin(30°)} = 28.9\mu m$$

Problem 3: In the year 2265, United Earth spaceships are equipped with 100GW laser cannons, firing infra-red laser beams with a wavelength of $1.0\mu m$. The laser beams have a diameter of 25*cm* when they emerge from the laser cannon. Aliens from Alpha Centauri are invading. The alien spaceships have shields that can withstand infra-red light intensities of up to 1GW / m^2. What is the effective range of the laser cannons against the alien spaceships?

Solution:
The laser beam from the cannon spreads due to diffraction. Its power is 100GW, so the maximum cross-sectional area of the beam should be 100m^2 to give an intensity of 1 GW/m^2. Since $A = \pi r^2$, this gives a beam radius of $r = 5.64m$.



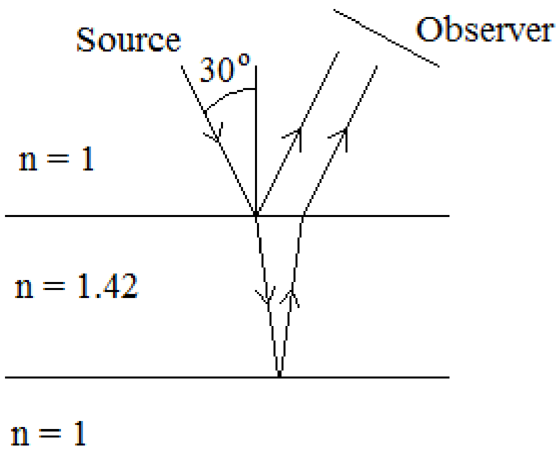
This beam radius is reached at the effective range *R*. The effective range is related to the diffraction angle θ by $\theta = r / R$ (this is true because θ is a small angle). But the diffraction angle is $\theta = \lambda / a$, where λ is the wavelength of light and *a* is the aperture size of the beam. Therefore,

$$\theta = \frac{\lambda}{a} = \frac{r}{R}$$

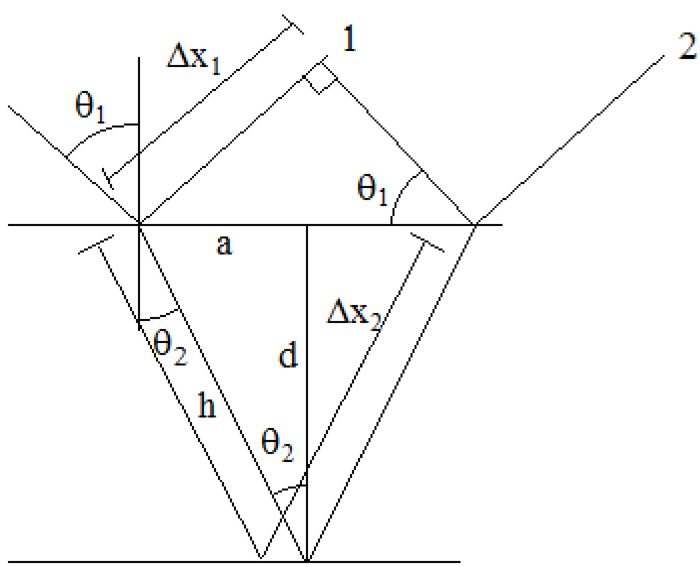
$$R = \frac{a}{\lambda} r = \frac{0.25m}{10^{-6}m} \times 5.64m = 1.41 \times 10^6 m = 1410km$$

The effective range of this laser cannon against the alien spaceships is about 1400 kilometers.

Problem 4: A thin film with thickness d = 400nm has an index of refraction n = 1.42. Light is incident on the film at an angle of 30 degrees with respect to the normal. What wavelengths of visible light in the reflection are enhanced by constructive interference?



Solution:
The two light paths shown in the diagram undergo different phase shifts due to reflection from different interfaces, and different distances traveled. A close-up of the diagram looks like this:



Beam 1 undergoes a phase change of π during reflection, and travels an additional distance $\Delta x_1 = 2a \sin \theta_1 = 2d \tan \theta_2 \sin \theta_1$. The total phase change due to these things is

$$\Delta\phi_1 = \pi + \frac{2\pi\Delta x_1}{\lambda_0} = \pi + \frac{4\pi d \sin\theta_1 \tan\theta_2}{\lambda_0}$$

Beam 2 doesn't undergo a phase change during reflection, and travels an additional distance $\Delta x_2 = 2h = 2d / \cos\theta_2$. Since the wavelength in the medium is $\lambda = \lambda_0 / n$, beam 2 undergoes a phase change of

$$\Delta\phi_2 = \frac{2\pi\Delta x_2}{\lambda} = \frac{4\pi nd}{\lambda_0 \cos\theta_2}$$

For constructive interference, the difference in phase between the two beams should be a multiple of 2π:

$$\Delta\phi_2 - \Delta\phi_1 = \frac{4\pi nd}{\lambda_0 \cos\theta_2} - \pi - \frac{4\pi d \sin\theta_1 \tan\theta_2}{\lambda_0} = 2\pi j$$

$$\frac{4d}{\lambda_0}\left(\frac{n}{\cos\theta_2} - \sin\theta_1 \tan\theta_2\right) = 2j + 1$$

$$\lambda_0 = \frac{4d}{2j+1}\left(\frac{n}{\cos\theta_2} - \sin\theta_1 \tan\theta_2\right)$$

We know d, n and θ_1 and can calculate θ_2 from Snell's law:

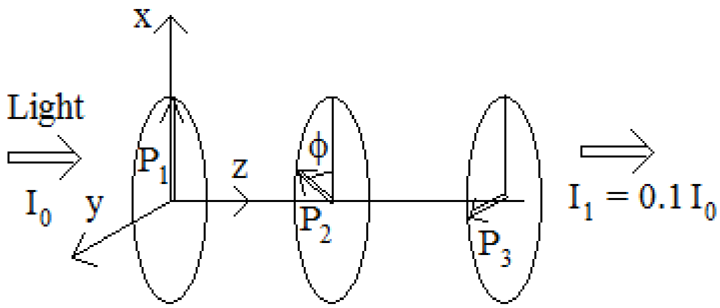
$$d = 400nm$$
$$n = 1.42$$
$$\theta_1 = 30^\circ$$
$$n \sin\theta_2 = \sin\theta_1 \qquad \theta_2 = 20.6^\circ$$

Using this in our equation gives

$$\lambda_0 = \frac{1600nm}{2j+1}\left(\frac{1.42}{0.936} - 0.188\right) = \frac{2126nm}{2j+1}$$

For values of j starting from zero, we get wavelengths of 2126*nm*, 708*nm*, 425*nm*, 303*nm*, etc. Constructive interference will amplify these wavelengths in the reflection and suppress the ones in between. Of these wavelengths, only 708*nm* and 425*nm* are in the visible range.

Problem 5:
Non-polarized light with intensity I_0 travels in the direction of the *z*-axis. The light is incident on a polarizing filter 1 oriented in the *x*-direction. The light then passes through another filter 2, oriented at angle ϕ to the *x*-axis, and then filter 3, oriented in the *y*-direction. The intensity of the emerging light is measured to be 1/10 of I_0. What is the angle ϕ? If there are multiple distinct possible angles, find all of them.



Solution: After the light passes through the first filter, it becomes polarized in the *x* direction, and its intensity becomes $I_0 / 2$. After the light passes through the second filter, it is polarized in the direction of the second filter, and its intensity is $(I_0 / 2)\cos^2\phi$. The third filter is at an angle $\pi / 2 - \phi$ to the second, so after the light passes through the third filter, it is polarized in the *y* direction, and its intensity is
$I_1 = (I_0 / 2)\cos^2\phi\cos^2(\pi / 2 - \phi) = (I_0 / 2)\cos^2\phi\sin^2\phi = (I_0 / 2)\cos^2\phi\,(1 - \cos^2\phi)$. From the problem statement,

$$I_1 = \frac{I_0}{2}\cos^2\phi\left(1-\cos^2\phi\right) = \frac{I_0}{10}$$

$$(\cos^2\phi)^2 - \cos^2\phi + \frac{1}{5} = 0$$

$$\cos^2\phi = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{4}{5}}\right) = 0.276 \quad \text{or} \quad 0.723$$

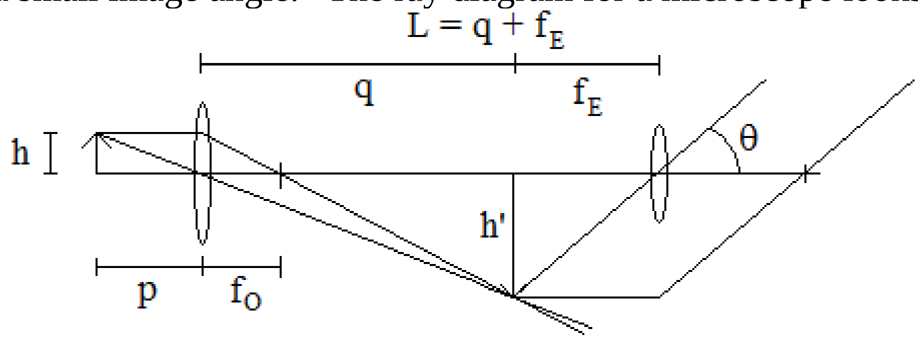
$$\phi = 31.7^\circ \quad \text{or} \quad 58.3^\circ$$

Problem 6:
A microscope has a length of 10.0*cm*. The focal length of the eyepiece is 0.80*cm*. The microscope has objectives that give 300X, 900X and 1800X magnification.

(a) What is the focal length of each objective?

(b) The microscope slide is illuminated with a 5-watt light when the lowest-magnification objective is used. If we want the same level of image illumination at the highest magnification, what power light should we use?

Solution: We need to relate the focal length of the objective to the magnification. The magnification given on the objectives is with the microscope adjusted so that the final image is at infinity, compared to a standard eye with a near point of 25*cm*, for objects that produce a small image angle. The ray diagram for a microscope looks like this:



First, relate the angle subtended by the image θ to the focal length of the objective f_O. For a very small object, which produces a small angle, we have

$$\theta \approx \frac{h'}{f_E} \qquad h' = \frac{q}{p}h = \frac{L - f_E}{p}h \qquad p = \frac{1}{1/f_O - 1/(L - f_E)}$$

$$\theta \approx \left(\frac{L - f_E}{f_O} - 1\right)\frac{h}{f_E}$$

The reference angle θ_0 is that produced by the object at the near point without using a microscope. Thus, $\theta_0 = h / p_{NP}$. The angular magnification is

$$m = \frac{\theta}{\theta_0} = \left(\frac{L - f_E}{f_O} - 1\right)\frac{p_{NP}}{f_E}$$

Solving this for the focal length of the objective and using the parameters given for the microscope gives

$$\frac{L - f_E}{f_O} = \frac{f_E}{p_{NP}}m + 1$$

$$f_O = \frac{(L - f_E)}{m(f_E/p_{NP}) + 1} = \frac{9.20cm}{(0.80/25)m + 1} = \frac{9.20cm}{0.032m + 1}$$

Now we just use the given magnifications in this equation. Using m = 300 gives f_O = 0.868*cm*. For m = 900, we get f_O = 0.309*cm*, and for m = 1800, f_O = 0.157*cm*.

For part (b), we realize that for the highest magnification, the image appears to be 6 times bigger than for the lowest magnification, so the area in view is 36 times smaller than at the lowest magnification. We want this area to receive as much light as the field of view did at the lowest magnification, so we must increase the power of the light by a factor of 36 to make up for a smaller area. Therefore, we need 180 watts.

Problem 7:
Suppose that a near-sighted person has a near point of 10.5*cm* and a far point of 56.0*cm*. His lens is 1.90*cm* from the retina.

(a) What lens power should this person use if he is to wear contact lenses?

(b) What lens power should he use if he is to use glasses, which sit 1.50*cm* from the cornea?

(c) If this person's vision is corrected with lenses, what is his new near point?

(d) Between what limits does the power of this person's lens-cornea combination vary?

Solution:
(a) When the person wearing contact lenses looks at objects far away, he should see a virtual image at the far point, so that the eyes can focus on the image properly. The focal length of the lens is given by $f = 1 / (1/p+1/q)$ = 1 / (0 + 1 / -56.0*cm*) = -56.0*cm*. The power of the lens is one over the focal length in meters: P = 1 / -0.56*m* = -1.79D.

(b) With glasses, the image should now be 56.0 - 1.5 = 54.5*cm* from the front of the glasses, so the focal length should be -54.5*cm*. The power is 1 / -0.545*m* = -1.83D.

(c) The answer to this is the distance at which an object forms an image that falls on the eye's near point ($q = -p_{NP}$). This gives p = 1 / (-1/56cm + 1/10.5cm) = 12.9*cm*

(d) When the person is focused on an object at the far point (56.0*cm*), the light is focused on the retina, 1.9*cm* behind the cornea and lens. The focal length of the lens-cornea combination is thus f = 1 / (1/56.0*cm* + 1/1.9*cm*) = 1.838*cm*. The corresponding power is 54.4D.

When the person is focused on an object at a near point (10.5*cm*), the focal length becomes f = 1 / (1/10.5*cm* + 1/1.9*cm*) = 1.609*cm*. The power is 62.2D. The power of the lens-cornea combination thus varies from 54.4 to 62.2 diopters.

Problem 8:
We want to build a telescope that is designed to directly observe planets orbiting other stars. The light of the star would normally overwhelm the much less intense reflected light from the planet, but if the telescope can separate the images of the star and the planet, the light of the star can be blocked out, allowing the image of the planet to be detected.

(a) We want to be able to detect planets in Earth-like orbits around stars up to 80 light-years away. If the telescope looks for wavelengths of light close to 600*nm*, what is the minimum required diameter of the telescope mirror? Assume that we will put this telescope in space, so the resolution is limited by the diffraction limit, not by the atmosphere.

(b) If we aim the same telescope at the Moon, 400,000 kilometers away, what is the minimum size of the features on the Moon's surface that can be resolved?

(c) Images with this telescope are obtained with a CCD camera, with no eyepiece. The pixels on the CCD are 6.8μm across. We do not want the telescope's resolution to be limited by that of the camera. What is the minimum focal length of the telescope's mirror?

Solution:
(a) The Earth is about 150 million km from the Sun, and we are looking for planets at a similar distance from their star. The largest angular separation between the star and the planet is $\theta = r / R$, where r is the radius of the orbit and R is the distance from the star-planet system to the telescope. We want to at least be able to detect the planet when it is at maximum angular separation from the star.

A light-year is the distance light travels in a year; we can calculate this to be (365 d/yr)(86400 s/d)(3 x 10^5 km/s) = 9.46 x 10^{12} km. At 80 years, the maximum angle between the planet and the star as viewed from Earth is

$$\theta = \frac{1.50 \times 10^8 km}{80 \times 9.46 \times 10^{12} km} = 1.98 \times 10^{-7} rad$$

The diffraction limit must be smaller than this angle:

$$\theta_d = \frac{\lambda}{a} < 1.98 \times 10^{-7}$$

$$a > \frac{6.00 \times 10^{-7} m}{1.98 \times 10^{-7}}$$

$$a > 3m$$

The telescope mirror must thus be at least 3 meters across to distinguish the star and the planet. However, note that at this aperture size, the images of the star and the planet will still overlap considerably, so it will be difficult to separate the light. Also, the planet will only be detectable when it is at the point in its orbit that places it at the largest possible angle from the star as seen from Earth. Increasing the aperture size well above the minimum limit would help alleviate these difficulties. 6 meters would be much better than the minimum value of 3 meters.

(b) The telescope can resolve features 2×10^{-7} radians across (or even smaller, if the bigger mirror is used). This corresponds to a distance on the Moon of

$$h = (2 \times 10^{-7})(3.84 \times 10^{5} km) = 77m$$

(c) In order to not limit the telescope's resolution, the camera must be able to distinguish angles at the diffraction limit. The size of the image at the diffraction limit on the image plane is $h' = f\,\theta_d$. We want h' to be at least $6.8\mu m$, so that the CCD can resolve it. Therefore,

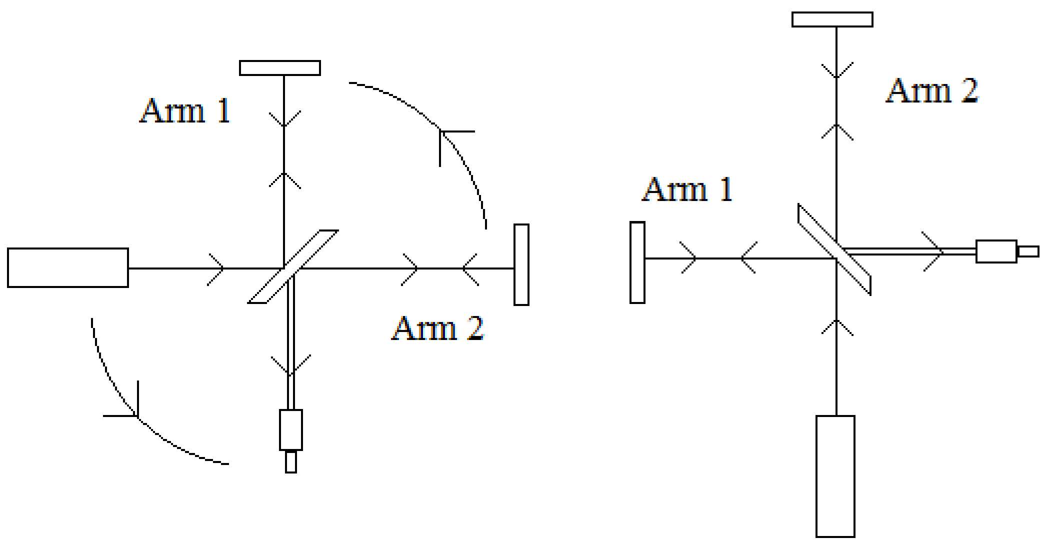
$$f > \frac{6.8 \times 10^{-6} m}{2 \times 10^{-7}} \qquad f > 34m$$

The focal length of the telescope's mirror must be more than 34 meters to allow the camera to fully resolve the image.

Problem 9: (Extra Credit)
This problem asks you to analyze a key experiment that established that the speed of light is the same for all observers, and for light coming from all directions.

Suppose that Einstein was wrong, and the speed of light did in fact depend on the observer. Suppose that there is a specific reference frame in which the speed of light is the same in all directions. If an observer moves with respect to that frame, then if not for Einstein's relativity, he would see light coming towards him from the direction he's moving in go faster than c, while light catching up from behind would go slower than c.

Now suppose that the observer carries a Michelson interferometer with him. The interferometer has arms 25 meters long, and uses 500*nm* light. The observer begins with arm 1 of the interferometer aligned in the direction of the observer's motion, and slowly rotates the apparatus until arm 2 points in the direction of his motion:



If the speed of light depends on the direction light is coming from, then as the interferometer is rotated, fringes would go by. Suppose the observer does the experiment in January. No fringes go by. The observer concludes that in his current frame, the speed of light is the same in all directions.

Now the observer repeats this experiment six months later, when the Earth's orbital motion is carrying him in the opposite direction and he is thus moving relative to his previous frame. If the speed of light depended on the observer's motion, how many fringes would go by as the interferometer was rotated?

Solution:

Note that the physics in the solution are completely wrong. The ether model has been disproved (in part by this experiment), but to get the predictions for the experiment, we will pretend that it is correct. If it was correct, then the speed of light would be the same in all directions relative to the ether, but not for observers moving through the ether.

Consider first the light traveling from the beam splitter to the end of arm 1, with arm 1 pointing in the direction of the observer's motion. The light travels at speed c relative to the ether, but since the observer is moving at speed v in the direction of arm 1, the light relative to the observer moves at speed $c - v$. The time it takes the light to get to the end of arm 1 is therefore $t = d / (c-v)$, and the number of oscillations the wave undergoes is $ft = fd / (c-v)$.

Now the light comes back. The observer is now moving head-on towards the light, so he should see the light moving at speed $c+v$. The amount of time it takes the light to get back to the beam splitter is therefore $t = d / (c+v)$, and the number of oscillations the wave goes through is $ft = fd / (c+v)$.

Thus the wave traveling up and down arm 1 undergoes a total number of oscillations

$$N_1 = \left(\frac{1}{c-v} + \frac{1}{c+v}\right) fd = \frac{c+v+c-v}{(c+v)(c-v)} fd = \frac{2c}{c^2-v^2} fd$$

The wave traveling up and down arm 2 travels with the same speed in each direction. The amount of time it takes is $2d / c$, and the number of oscillations is undergoes is

$$N_2 = \frac{2fd}{c}$$

The difference in the number of oscillations between the first and the second arm is

$$N_1 - N_2 = 2fd\left(\frac{c}{c^2-v^2} - \frac{1}{c}\right) = 2fd\frac{c^2-(c^2-v^2)}{c(c^2-v^2)} = \frac{2fd}{c}\frac{v^2}{c^2-v^2}$$

When the two arms are exchanged, this difference is reversed, so the number of fringes that go by is twice that amount:

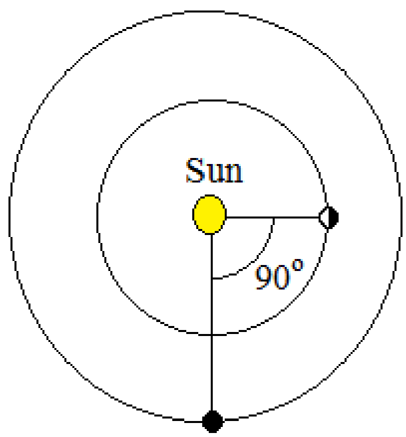
$$N_f = \frac{4fd}{c}\frac{v^2}{c^2-v^2} = \frac{4d}{\lambda}\frac{v^2}{c^2-v^2} = \frac{100m}{5\times 10^{-7}m}\frac{v^2}{c^2-v^2} = 2\times 10^8\frac{v^2}{c^2-v^2}$$

Finally, the Earth's orbital speed can be determined from the fact that the Earth takes one year to complete a whole revolution around the sun. The speed is thus $v = 2\pi R / T = 30\text{km/s}$. If we assume that the first experiment yielded negative results because the Earth happened to be at rest relative to the ether, then six months later, it will be moving in the opposite direction, and thus will move at about 60km/s relative to the ether. Using this for the speed *v* above, we calculate the number of fringes:

$$N_f = 2 \times 10^8 \frac{(60km/s)^2}{(3 \times 10^5 km/s)^2 - (60km/s)^2} = 8$$

Therefore, 8 fringes would go by.

Problem 10: (Extra Credit)
It is said that individuals with particularly good eyesight can see the phases of Venus (that is, they can see that Venus is not a fully illuminated disk, but rather a portion of a disk, just as the Moon is when it is not full). Human pupil size can be up to 9*mm* when fully dark-adapted, the distance from the eye to the retina can be up to 22*mm*, and the distance between individual rods on the retina is about 3μm. Is it possible that an individual with eyes like this, and with perfect vision, can see the phase of Venus when Venus is positioned relative to the Earth as shown in the diagram? Consider both the diffraction limit and the resolution limit of the retina.



Solution: First, we will determine the angular diameter of Venus as seen from the Earth, and then we will see if the eye can resolve this as more than a single point, subject to the diffraction limit and the resolution limit.

The distance to Venus in the diagram forms the hypotenuse of the right triangle. If a_E is the distance of the Earth from the Sun, and a_V is the distance of Venus from the Sun, the distance of Venus from Earth in the diagram is equal to

$$D = \sqrt{a_E^2 + a_V^2} = \sqrt{(1.50 \times 10^8 km)^2 + (1.08 \times 10^8 km)^2} = 1.85 \times 10^8 km$$

The diameter of Venus is 12100*km*, so the angle subtended by Venus is

$$\theta = \frac{1.21 \times 10^4 km}{1.85 \times 10^8 km} = 6.54 \times 10^{-5} rad$$

Now we will determine the diffraction limit and the resolution limit of the eye mentioned. The diffraction limit depends on the wavelength, but the smallest angles can be seen in the shortest wavelengths. The human eye can clearly see down to about 400*nm*, so the diffraction limit is approximately

$$\theta_d = \frac{\lambda}{a} = \frac{4 \times 10^{-7} m}{9 \times 10^{-3} m} = 4.4 \times 10^{-5} rad$$

Venus passes the diffraction limit, but barely. The size of the image of Venus on the retina is

$$h' = f\theta = (22mm)(6.5 \times 10^{-5} rad) = 1.4 \mu m$$

This is a lot smaller than the separation between the rods, and in fact even smaller than the separation between the cones, which can be as small as 2 μm. The separation between light-sensitive cells doesn't vary much from individual to individual. Thus, the eye cannot distinguish Venus as more than a point, due to the resolution limit of the retina. Stories about individuals who can see the phases of Venus are most likely false, unless there are people with eyes 4*cm* in diameter.