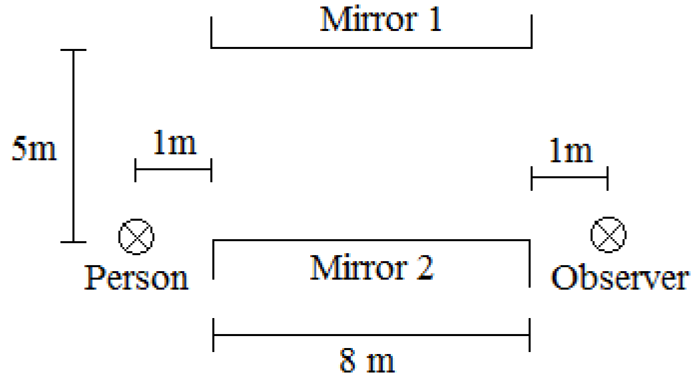
Homework Set 3

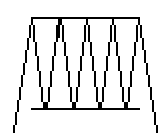
Due Thursday, 07/14

Problem 1

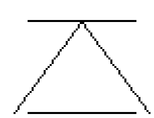
A room contains two parallel wall mirrors, on opposite walls 5 meters apart. The mirrors are 8 meters long. Suppose that one person stands in a doorway, in line with mirror 2 and one meter away from it, and another person (the observer) stands one meter away from the other edge of the mirror. If the observer looks at mirror 1, how many images of the other person does he see?



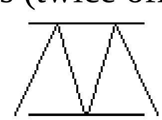
Solution: Each possible path of light between the person and the observer produces an image. There is a distance of 1 meters between the person and the corner of the mirror, so the path with the most possible reflections goes from the person to the edge of mirror 1, a shift of 1 meter. The reflection angle is the same as the incidence angle, so the light gets back to mirror 2 at a point 1 meter from the edge. It goes back to mirror 1 at 2 meters from the edge, then to mirror 2 at 3 meters, mirror 1 at 4 meters, mirror 2 at 5 meters, 1 at 6 meters, 2 at 7 meters, and finally hits mirror 1 at 8 meters and is reflected back to the observer, making a total of 9 reflections:



It is clearly possible to have a path with just 1 reflection (light goes to center of mirror 1, then to the observer):



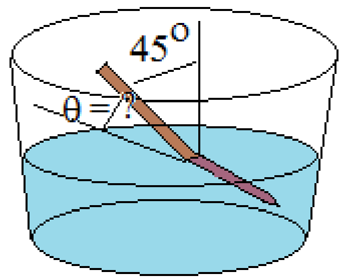
The next possible path has 3 reflections (twice off mirror 1, once off mirror 2):



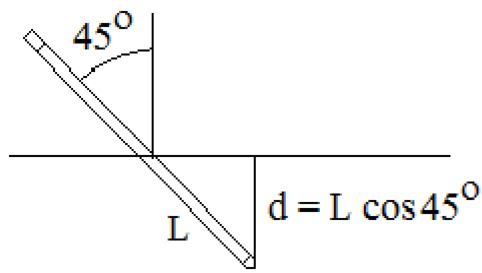
Clearly any odd number of reflections up to 9 is possible. Thus, there are 5 possible paths (with 1, 3, 5, 7 and 9 reflections), and so the observer will see 5 images.

Problem 2

A pencil going through the surface of the water appears to bend slightly where it crosses the surface. The pencil enters the water with an angle of 45 degrees to the vertical. If an observer is looking at the pencil approximately from above, at what angle does the pencil appear to bend (by how many degrees does its image diverge from a straight line)? Assume that the index of refraction of water is 1.33.



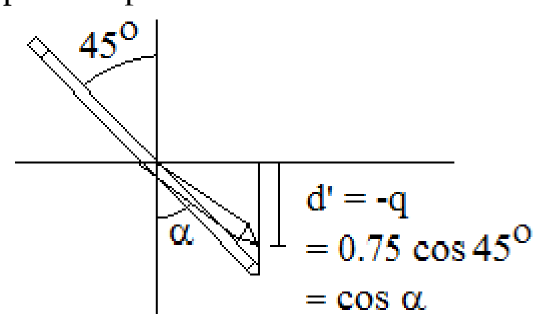
Solution: Suppose a length *L* of pencil is submerged underwater. Then, the depth of the pencil's tip is $L \cos 45^o$.



Due to refraction of light through the water, the depth of the tip appears to be less than it actually is. Using the equation for refraction of light by a spherical surface in the limit where *R* goes to infinity, we obtain

$$\frac{n_1}{p} + \frac{n_2}{q} = 0 \qquad q = -\frac{n_2}{n_1}d = -\frac{1}{1.33}d = 0.75d$$

The apparent depth of the pencil tip is thus less than the actual depth:



The angle α made by the image of the pencil with the vertical is $\cos \alpha = 0.75 \cos 45^o$, or $\alpha = 58^o$. The difference between this angle and the actual angle of the pencil is 8^o, so the pencil appears to be bent by 8^o.

Problem 3

A transparent sphere forms an image of the Sun precisely on its surface, on the side opposite to the direction of the Sun. What is the index of refraction of the material of the sphere?

Solution:

The equation relating the image distance to the object distance and the index of refraction for a spherical surface is

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

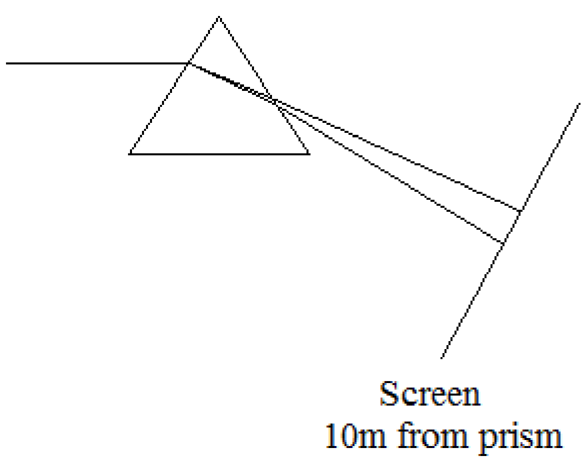
The medium where the light originates is air, so $n_1 = 1.00$. The image is formed on the surface of the sphere, which is a distance of $2R$ from the point where the light is incident. So, $q = 2R$ (positive because the image forms on the opposite side of the interface from the object). Also, the Sun is far away, so $1 / p = 0$:

$$\frac{n_2}{2R} = \frac{n_2 - 1}{R} \qquad n_2 = 2(n_2 - 1) \qquad n_2 = 2$$

Thus, the index of refraction of the sphere material is 2.

Problem 4

A prism is constructed out of a material that has an index of refraction given by $n = 1.388 - 0.006(\lambda / 1000\ nm)$ for visible wavelengths. The cross-section of the prism is an equilateral triangle. A narrow beam of sunlight is passed through this prism as shown below, and continues on to a screen 10 meters away. The screen is perpendicular to the direction of light emerging from the prism.

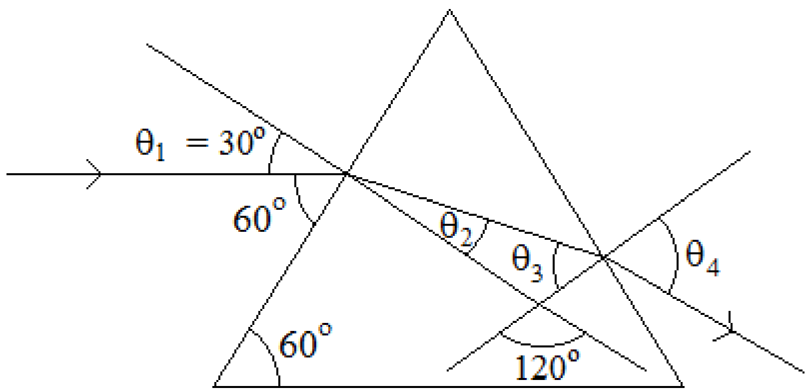


(a) How wide is the image of the visible spectrum on the screen?

(b) If a 2*mm* slit is made in the screen so that the middle of the slit admits a wavelength of 520 *nm*, what is the range of wavelengths that will go through the slit?

Solution:

For the red edge of visible light, the wavelength is approximately 750*nm*. Our approximate equation for the index of refraction yields n_{RED} = 1.3835. Now we need to examine the angles at which the light hits the surface of the prism, and apply Snell's law. Let us label the angles as follows:



We know that $\theta_1 = 30^o$, since the light comes in parallel to the bottom face of the prism. The prism is an equilateral triangle, so its angles are 60°, and therefore the light comes in at 60° to the surface of the prism, or 30° to the normal. Next, we can use Snell's law to calculate θ_2: $\sin\theta_2 = (1 / n_{RED}) \sin\theta_1$ or $\theta_2 = 21.19^o$. The normals to the opposite sides of the prism intersect at 120°, so $\theta_3 = 180^o - 120^o - \theta_2 = 38.81^o$. Finally, use Snell's law to determine θ_4: $\sin\theta_4 = n_{RED} \sin\theta_3$, or $\theta_3 = 60.12^o$.

Now repeat the calculation for violet light. Using λ = 390 *nm*, we get n_{VIO} = 1.3857. Repeating the steps in the calculation above, we get $\theta_1 = 30^o$ as before, $\theta_2 = 21.15^o$ by Snell's law, $\theta_3 = 38.85^o$ by geometry, and $\theta_4 = 60.37^o$.

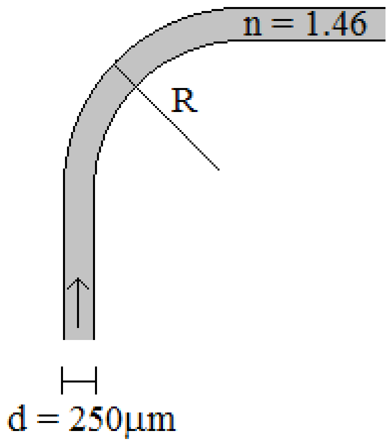
The difference in angle between the emerging red and violet light is $\Delta\theta = 60.37^o - 60.12^o = 0.25^o$. When projected onto a screen 10 meters away, the separation becomes the distance times the angle in radians:

$$\Delta x = 0.25^\circ \frac{\pi}{180} \times 10m = 0.044m = 4.4cm$$

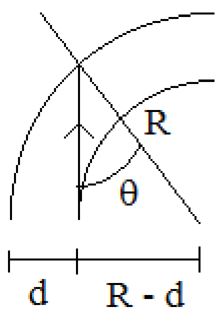
For part (b), the slit extends to 1*mm* off to either side of the center. 1*mm* is 1/44 of the total wavelength range, which is 750 - 390 = 360*nm*, so 1*mm* on the screen corresponds to 8.2*nm* in wavelength. Therefore, the wavelength range admitted through the slit extends 8.2 *nm* to either side of the central wavelength. Thus the range is 511.8 to 528.2 *nm*.

Problem 5

An optical fiber has an index of refraction n = 1.46 and a diameter d = 250 μm. Light travels along the fiber's axis. At one point, the fiber bends with a radius of curvature R. How small can R be without allowing light to escape?



Solution: Light traveling along the inside edge of the cable will strike the surface at the smallest angle to the normal, and is therefore will be the first to escape if R is too small. First, we draw a diagram of this ray of light and determine the angle it will make to the normal for a given R:



From the diagram, tan θ = (R - d) / R.

Now, determine the critical angle for light inside the cable. The critical angle is

$$\sin\theta_C = \frac{n_2}{n_1} = \frac{1.00}{1.42} = 0.704 \qquad \theta_C = 44.8°$$

When the critical angle is reached, we have

$$\tan\theta_C = 0.992 = \frac{R-d}{R} \qquad 0.008R = d$$

$$R = \frac{d}{0.008} = \frac{250\mu m}{0.008} = 31.25mm$$

Thus this cable can be bent with a radius of curvature of 3.125 *cm* before light begins to escape. If it is bent more, light will escape from the cable.

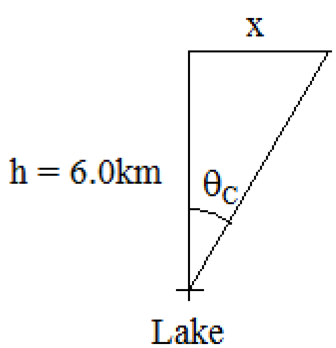
Problem 6
A scuba diver is at the bottom of a lake with a perfectly smooth surface. An airplane is flying past the lake at an altitude of 6.0 kilometers. How close to the scuba diver (in terms of horizontal distance, as on a map) must the path of the plane pass in order for the diver to hear the plane? Assume that the plane is loud enough that the diver will hear it if any sound from it can get to her. Also, assume that the lake is small, so that any sound crossing its surface does so directly above the scuba diver's head. The speed of sound in air is 330 *m/s*, and the speed of sound in water is 1500 *m/s*.

Solution:
Since the speed of sound in air is lower than in water, if the angle at which the sound wave comes in makes too large an angle with the normal to the surface of the lake, the sound will be totally reflected, and the diver will not hear it. Sound first enters the water when the direction to the airplane is at a critical angle to the normal:

$$\sin\theta_C = \frac{n_2}{n_1} = \frac{v_1}{v_2} = \frac{331 m/s}{1500 m/s} = 0.221$$

$$\theta_C = 12.7^\circ$$

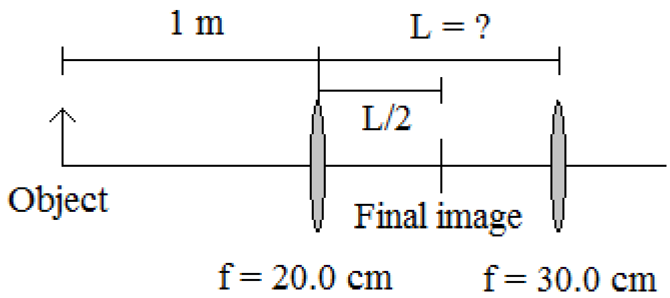
Use this angle to find the horizontal distance to the airplane when the sound first enters the water:



From this diagram, $\tan\theta_C = x / h$ so $x = h \tan\theta_C = 1.36 km$.

Thus if the path of the airplane approaches within 1.36 *km* of the lake (in horizontal distance), the diver will hear the airplane.

Problem 7
An object is placed 1.00 meters in front of a converging lens with a focal length of 20.0 cm. Another converging lens, with a focal length of 30.0 cm, is placed behind the first. The final image forms exactly halfway between the lenses. How far apart are the lenses? Is the image upright or inverted? What is the magnification?



Solution: Let the distance to the object be p_1 = 100.0 *cm* and the distance at which the first lens produces the image be q_1. The second lens is a distance *L* behind the first, so its object distance is $p_2 = L - q_1$ (the minus sign is because q_1 is positive if the first image is to the right of the first lens). Finally, the image distance for the second lens is $q_2 = -L / 2$ (negative because the image is to the left of the lens). We have the equations:

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1} \qquad \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$$

First, solve the first equation for q_1:

$$\frac{1}{q_1} = \frac{1}{f_1} - \frac{1}{p_1} = \frac{1}{20.0cm} - \frac{1}{100cm} \qquad q_1 = 25cm$$

Now plug in our relations from above into the second equation and solve for *L*:

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2} \qquad \frac{1}{L - q_1} - \frac{2}{L} = \frac{1}{f_2}$$

$$f_2 L - 2f_2(L - q_1) = L(L - q_1)$$

$$L^2 + (f_2 - q_1)L - 2f_2 q_1 = 0$$

$$L^2 + (5cm)L - 1500cm^2 = 0$$

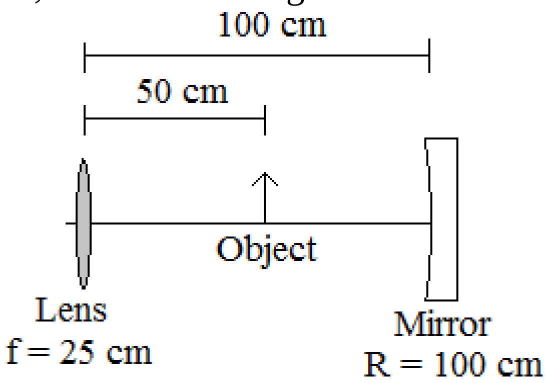
$$L = \frac{-5cm + \sqrt{(5cm)^2 + 6000cm^2}}{2} = 36.3cm$$

Thus the lenses are 36.3*cm* apart. The magnification is the product of the two lenses' magnifications:

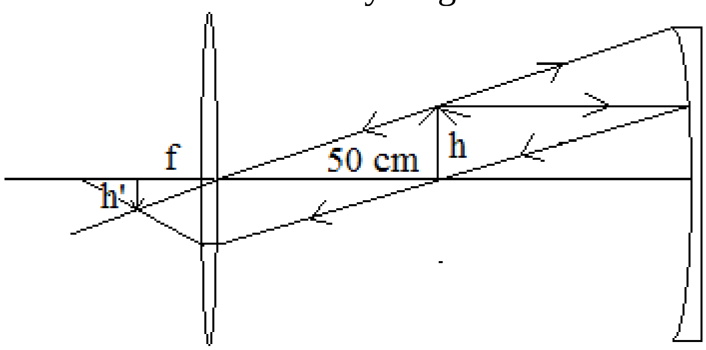
$$M = M_1 M_2 = \left(-\frac{q_1}{p_1}\right)\left(-\frac{q_2}{p_2}\right) = \left(-\frac{25cm}{100cm}\right)\left(\frac{L/2}{L - 25cm}\right) = -0.40$$

Problem 8

A lens with a focal length of 25 cm and a concave mirror with a radius of curvature of 100 cm are placed 1.0 meters apart. An object is placed halfway between the lens and the mirror, 50 cm from each. Locate the final image, determine if it is real or virtual, upright or inverted, and find the magnification.



This problem is a bit less straightforward than it seems. If we attempt to locate the image for the mirror to use as an object for the lens, we find that no image is formed, since the object is located at the focal point. However, the rays reflected by the mirror towards the lens are parallel. Let us draw a ray diagram to see what happens:



Note that we drew the upper ray through the center of the mirror's curvature (which is also in the middle of the lens) so that it gets reflected right back to the center of the lens and passes straight through. The other ray is parallel to the first when it gets back to the lens, and therefore it will cross the first ray at the focal distance after passing through the lens. The image is thus real and inverted, and formed at the lens's focal point.

Now by similar triangles, $h' / f = h / 50\text{cm}$. Thus, $M = -h' / h = -f / 50\text{ cm} = -0.50$. The image is upside-down and a factor of 2 smaller than the object.

Problem 9 (extra credit)
Suppose that you wish to manufacture a "lens" that will focus sound. You have some plastic that has a density of 0.8 g/cm^3 and which decreases in volume by 0.46% when exposed to a pressure of 10^6 Pa. If you make the lens out of this material, should it be concave or convex? Suppose you want the sound to be focused 10 meters away from the lens. If you leave one side of the lens flat, what should be the radius of curvature of the other side?

Solution: First determine the bulk modulus of this material, and the speed of sound in the material:

$$B = -\frac{\Delta P}{\Delta V/V} = -\frac{10^6 N/m^2}{-4.6 \times 10^{-3}} = 2.17 \times 10^8 N/m^2$$

$$c = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.17 \times 10^8 N/m^2}{800 kg/m^3}} = 521 m/s$$

Next, we will determine the radius of curvature of this lens from the lens maker's equation. We will leave the back side of the lens flat, so $1 / R_2 = 0$. Also, the quantity n_2 / n_1 is equal to v_1 / v_2 since the index of refraction is inversely proportional to the wave speed in the medium. The lens maker's equation is then

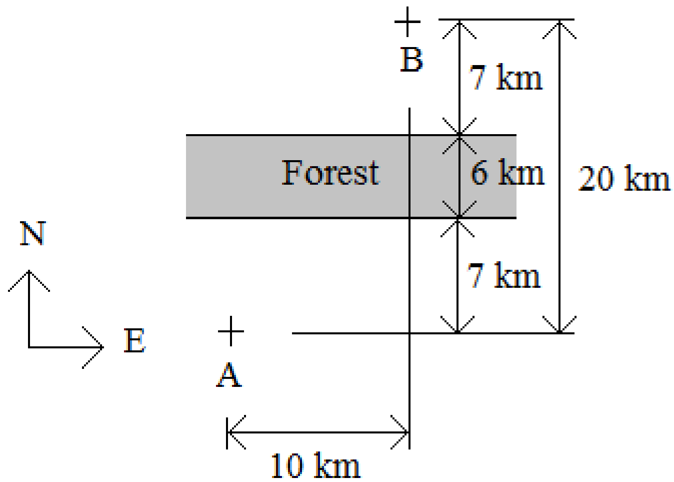
$$\frac{1}{f} = \left(\frac{v_1}{v_2} - 1\right)\frac{1}{R_1}$$

Solve for R_1:

$$R_1 = \left(\frac{v_1}{v_2} - 1\right) f = \left(\frac{331 m/s}{521 m/s} - 1\right) \times 10 m = -3.65 m$$

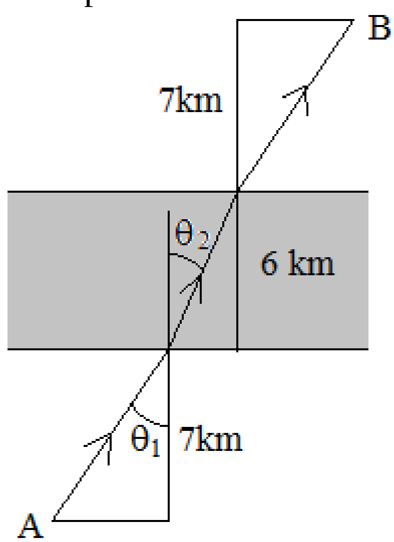
The radius of curvature should be -3.65m. Since the radius of curvature for the front side of the lens is negative, this means the lens is concave.

Problem 10 (extra credit)
A hiker is traveling from point A to point B on the map below. Point B is 20 kilometers north and 10 kilometers east of point A. There is a strip of forest between the hiker and his destination, spanning from 7 km to 13 km north of point A. The hiker's speed on open terrain is 5 km/h, and in the forest, 3 km/h.



If the hiker wishes to get to point B in the least possible time, what path should he take? How many kilometers to the east of his position should he enter the forest, and where should he exit it? How much time will it take the hiker to get to his destination? How much time would it take if the hiker simply went in a straight line from A to B?

Solution: Since the hiker is trying to minimize the time to get from point A to point B, he will behave in exactly the same way as a beam of light refracted by a pane of glass. The speed of light in the forest is 3/5 that in the open, so if the open terrain is considered to have an index of refraction of 1, then the forest has an index of refraction of 5/3 = 1.67. Draw a diagram of the hiker's path and label the angles:



Snell's law (which will minimize the time) tells us that sin $\theta_1 = n$ sin θ_2. Also, the total horizontal displacement of the hiker is Δx = 2(7km) tan θ_1 + (6km) tan θ_2 = 10km. Express tan θ in terms of sin θ and plug into the second equation:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$(14km)\frac{1.67\sin\theta_2}{\sqrt{1-1.67^2 sin^2\theta_2}} + (6km)\frac{\sin\theta_2}{\sqrt{1-sin^2\theta_2}} = 10km$$

This equation has only one variable to solve for: sin θ_2. Using a graphing calculator, any computer program with a numerical equation solver, or a trial, error and correction method, we obtain the solution sin θ_2= 0.3007. Using Snell's law, sin θ_1= 0.5012.

This means the hiker enters the forest at the point x_1 = (7km) *tan* θ_1 = 4.05*km* to the east of his initial position. He leaves the forest at $x_2 = x_1$ + (6km) *tan* θ_2 = 5.95*km* to the east of his original position.

The distance traveled by the hiker through open terrain is 2 x (7*km*) / cos θ_1 =16.18*km*, while the distance traveled through the forest is (6*km*) / cos θ_2 = 6.30*km*. The total time is therefore

t = (16.18*km*) / (5 *km/h*) + (6.30*km*) / (3 *km/h*) = 5.336*h*

If the hiker had simply gone straight, his distance through open terrain would be (from similar triangles)

$$2\times\frac{7}{20}\sqrt{20^2+10^2} = 15.65km$$

His distance through the forest would be

$$\frac{6}{20}\sqrt{20^2+10^2} = 6.71km$$

The amount of time for the straight-line path is

t = (15.65*km*) / (5 *km/h*) + (6.71*km*) / (3 *km/h*) = 5.367*h*

The amount of time saved by taking the optimal path is 0.031 hours, or 1.86 minutes. So, the hiker doesn't really save that much time, only 2 minutes from an approximately 5 hour 20 minute hike. Calculating the optimal path would probably take longer than this.