Homework Set 2

Due Wednesday, 07/06

Problem 1

A wave on a string has an amplitude of 1.5cm, a period of 0.12 seconds, and a wavelength of 1.1 meters.

a. What is the wave's frequency, angular frequency, wave number and wave speed?

b. If the string is under tension of 5.0N, what is its mass per unit length, in grams per meter?

c. If the tension in the string changes to 10.0N but the frequency of the wave remains the same, what is its new wavelength?

Solution:

a. The frequency, angular frequency, wave number and wave speed are

$$f = \frac{1}{T} = \frac{1}{0.12s} = 8.3s^{-1} \qquad \omega = 2\pi f = 52s^{-1}$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{1.1m} = 5.7m^{-1} \quad c = \frac{\omega}{k} = 9.2m/s$$

b. We know the wave speed from part (a) and are given the tension, so we can solve for the mass per unit length:

$$c = \sqrt{\frac{F_T}{\mu}} \qquad \mu = \frac{F_T}{c^2} = \frac{5.0N}{(9.2m/s)^2} = 0.060kg/m = 60g/m$$

c. First, determine how the wavelength depends on the tension, if the frequency is held fixed:

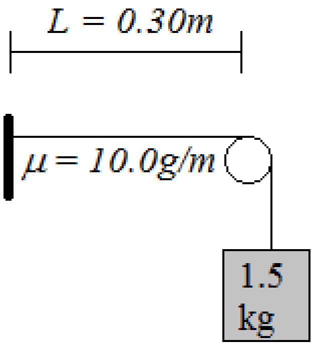
$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega/c} = \frac{c}{f} = \frac{1}{f}\sqrt{\frac{F_T}{\mu}}$$

The wavelength is therefore proportional to the square root of the tension. The tension is doubled, so the new wavelength is

$$\lambda_{new} = \sqrt{2}\lambda_{old} = \sqrt{2} \times 1.1m = 1.56m$$

Problem 2

A string with mass per unit length $\mu = 10.0$g/m is tied down on one side, and runs over a pulley on the other side, 0.30 meters away. The end that runs over the pulley is attached to a 1.5 kilogram mass. Assume that the end of the string that is tied down and the point that runs over the pulley cannot move, but the piece between these points is free to oscillate.



a. What is the string's fundamental frequency?

b. Now this system is placed in an accelerating elevator. The 3rd harmonic frequency is measured to be 204 hz. Is the elevator accelerating up or down? What is its acceleration?

Solution:

a. In the fundamental mode, half a wavelength fits on the string between $x = 0$ and $x = L$. Therefore, the wavelength is 0.60m. The wave speed is

$$c = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{mg}{\mu}} = \sqrt{\frac{1.5kg \times 9.8m/s^2}{0.010kg/m}} = 38.3m/s$$

The frequency of the fundamental mode is then

$$f_1 = \frac{c}{\lambda} = \frac{38.3m/s}{0.60m} = 63.8s^{-1}$$

b. To figure out the acceleration, we need to know the effective gravity inside the elevator. We are given the frequency of the 3rd normal mode, which is equal to 3 times the frequency of the 1st mode. Use this to determine the wave speed:

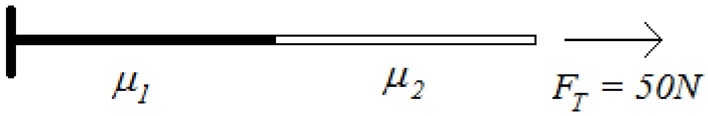
$$f_3 = 3f_1 = \frac{3c}{0.60m} \qquad c = \frac{0.60m \times f_3}{3} = 40.8m/s$$

$$c = \sqrt{\frac{mg_{eff}}{\mu}} \qquad g_{eff} = \frac{\mu c^2}{m} = 11.1m/s^2 = 9.8m/s^2 + 1.3m/s^2$$

Since the apparent acceleration due to gravity inside the elevator is higher than g by $1.3m/s^2$, we know that the elevator is accelerating upward at $1.3m/s^2$.

Problem 3

Two strings with different masses per unit length are connected together and held at a tension of 50*N*:



An wave with an amplitude of 2.50cm and a frequency of 60.0 hz is incident from the left. The waves on the left-hand string are observed to have a wavelength of 30.0cm, while those on the right-hand string have a wavelength of 20.0cm.

a. What is the mass per unit length, μ_1, of the left string? What is the mass per unit length of the right string?

b. What is the amplitude of the transmitted waves on the right string?

c. What is the transmission coefficient? What is the reflection coefficient?

Solution:

a. We can determine the mass per unit length from the wavelength and the frequency as follows:

$$c = \sqrt{\frac{F_T}{\mu}} = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda \qquad \mu = \frac{F_T}{f^2\lambda^2}$$

$$\mu_1 = \frac{50N}{(60.0s^{-1})^2(0.30m)^2} = 0.154kg/m$$

$$\mu_2 = \frac{50N}{(60.0s^{-1})^2(0.20m)^2} = 0.347kg/m$$

b. Using the relation derived in the lecture, where *A* is the incident amplitude and *C* is the transmitted amplitude,

$$C = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}A = \frac{2 \times \sqrt{0.154kg/m}}{\sqrt{0.154kg/m}+\sqrt{0.347kg/m}} \times 2.50cm = 2.00cm$$

c. The transmission and reflection coefficients were derived in terms of wave speeds. The wave speeds on the two sides are $c_1 = f\lambda_1 = 18\ m/s$ and $c_2 = f\lambda_2 = 12\ m/s$. Then,

$$T = \frac{4c_1c_2}{(c_1+c_2)^2} = 0.96 \qquad R = 1 - T = 0.04$$

This means that 96% of the energy is transmitted by the wave from the left section of string to the right section. Only 4% is reflected.

Problem 4

Suppose that a metal alloy has a density of 7.90g/cm^3. The speed of sound in this metal is measured to be 4.08km/s. We take a piece of this metal and submerge it to the bottom of the Mariana Trench, 10.9km below the surface of the ocean. The metal is compressed by the pressure of the water. By what fraction does the volume of this piece of metal change? Assume that the density of water stays approximately constant at 1g/cm^3.

Solution:

The compression of a material under pressure is determined by its bulk modulus *B*. The bulk modulus is related to the density and the speed of sound:

$$c = \sqrt{\frac{B}{\rho}} \qquad B = \rho c^2 = -\frac{\Delta P}{\Delta V / V}$$

$$\frac{\Delta V}{V} = -\frac{\Delta P}{\rho c^2}$$

Now we need to know the pressure at the bottom of the Mariana Trench. If h = 1.09 x 10^4 meters is the height of the water column, an area A of ocean bottom is overlain by the following mass of water:

$$M = V\rho_W = Ah\rho_W$$

The weight of all this water exerts a pressure on the ocean bottom (recall that pressure is force per area):

$$P = \frac{F}{A} = \frac{Mg}{A} = h\rho_W g = 1.07 \times 10^8 Pa$$

The same pressure is exerted on our piece of metal at the bottom of the trench. As the pressure changes from atmospheric pressure (essentially zero) to this very large value, the fractional change in volume is

$$\frac{\Delta V}{V} = -\frac{\Delta P}{\rho c^2} = -\frac{1.07 \times 10^8 Pa}{7900kg/m^3 \times (4080m/s)^2} = -8.12 \times 10^{-4}$$

The volume of the metal thus decreases by a small but detectable amount, slightly less than one part per thousand.

Problem 5

A loudspeaker at a concert is capable of producing sound with intensity level of 120 decibels at a distance of 5 meters from the speaker.

a. Assuming that the sound is emitted in spherical waves, what is the power of this speaker in watts?

b. What is the sound intensity level, in decibels, 100 meters away from the speaker?

c. At a distance of 5 meters away from the speaker, what is the acoustic energy density in the air (in Joules of acoustic energy per cubic meter of air)? Assume that the speed of sound in air is 331m/s.

Solution:

a. First, calculate the intensity of the sound waves by solving the relationship between the intensity and the intensity level:

$$\beta = 10log_{10}\left(\frac{I}{10^{-12}W/m^2}\right) \qquad I = 10^{-12}W/m^2 \times 10^{\beta/10} = 1.0W/m^2$$

The power of the speaker is the total acoustic energy flux produced by the speaker. This quantity is equal to the intensity times the area that the energy is moving across, which is the area of a sphere at 5 meters:

$$\Phi = IA = 4\pi r^2 I = 4\pi \times (5.0m)^2 \times 1.0W/m^2 = 314W$$

b. 100 meters away is 20 times as far as 5 meters, so the intensity there is 20^2 = 400 times smaller. The intensity level in decibels is smaller by $10\log_{10}400$=26 db. Thus, the intensity level at 100 meters is 120 db - 26 db = 94 db.

c. The energy density is equal to the intensity divided by the speed of sound. It is

$$\rho_E = \frac{\Phi}{c} = \frac{1.0W}{331m/s} = 3.02 \times 10^{-3}J/m^3$$

Problem 6

Submarine A is following another submarine B. Both are traveling along the same course in the same direction. Submarine A, traveling at a speed of 6.0 knots (1 knot = 1.85km/h) emits a sonar ping at a frequency of 2200 hz. The echo from submarine B returns 22 seconds later, with a frequency of 2185 hz. Assume that the speed of sound in water is 1500 m/s. How far away is submarine B, and how fast is it moving?

Solution: This is a Doppler shift problem with multiple steps. First, we will determine the frequency of the ping relative to the water. Then, we will determine the frequency as observed by submarine B. The echo will be reflected by submarine B at the same frequency as the observed ping. Then, we will determine the frequency of the echo relative to the water, and finally, the frequency of the echo as observed by submarine A.

The frequency of the ping relative to the water is

$$f_{P,W} = f_{P,S}\frac{c}{c - v_A} = f_{P,S}\frac{1500m/s}{1500m/s - 3.08m/s} = 1.00206 f_{P,S}$$

The frequency of the ping detected by the other submarine equal to the frequency of the echo at the source, and is given by

$$f_{P,O} = f_{P,W}\frac{c + v_B}{c} = 1.00206 f_{P,S}\frac{c + v_B}{c} = f_{E,S}$$

The frequency of the echo relative to the water is

$$f_{E,W} = f_{E,S}\frac{c}{c - v_B} = 1.00206 f_{P,S}\frac{c + v_B}{c - v_B}$$

Finally, the frequency of the echo as detected by submarine A is

$$f_{E,O} = f_{E,W}\frac{c + v_A}{c} = 1.00206 f_{P,S}\frac{c + v_B}{c - v_B} \times 1.00205 = 1.00412 f_{P,S}\frac{c + v_B}{c - v_B}$$

Now solve for v_B:

$$(c - v_B) f_{E,O} = 1.00412 (c + v_B) f_{P,S}$$

$$(1.00412 f_{P,S} + f_{E,O}) v_B = (f_{E,O} - 1.00412 f_{P,S}) c$$

$$v_B = \frac{f_{E,O} - 1.00412 f_{P,S}}{f_{E,O} + 1.00412 f_{P,S}} c = -8.21m/s = -29.6km/h = -16.0kts$$

Recall that with our sign convention, a negative number means the object is moving away from the observer. So, submarine B is moving away from submarine A with a speed of 16 knots relative to the water.

The distance to submarine B is R = 1/2 (22 s x 1500 m/s) = 16.5km. There is a small correction due to the fact that the submarines move a bit over 22 seconds.

Problem 7

Two train whistles have identical frequencies of 220 hz. Suppose that one train is at rest, while another is approaching it on a collision course with a constant speed *v.* Both trains are sounding their whistles, and are 100 meters away from each other. An observer on the stationary train hears a beat frequency of 4.0 hz. How many seconds before the trains collide?

Solution: The frequency of the approaching train, as heard by the stationary observer, is Doppler shifted. Since the train is approaching, the frequency increases. A beat frequency of 4.0 hz means that the shifted frequency is 4.0hz greater than the frequency of the sound coming from the stationary train. Therefore,

$$f_O = f_S\left(\frac{c}{c - v_S}\right) = f_S + \Delta f$$

$$v_S = c\left(1 - \frac{f_S}{f_S + \Delta f}\right) = 331 m/s\left(1 - \frac{220}{224}\right) = 5.91 m/s$$

If the moving train doesn't stop, the trains will collide in 100 m / 5.91 m/s = 16.9 s.

Problem 8
A flute generates a note at a frequency of 1440 hz. The air temperature is 25ºC.

a. If one were to attempt the same note on the flute at 0ºC, what frequency would be generated? What about at 40ºC?

b. Now suppose that the note is played in an atmosphere of helium at 25ºC. What is the frequency then? (Assume that $B_{HELIUM} \approx B_{AIR}$).

Solution:
Recall that the density of air is inversely proportional to the temperature above absolute zero. At 0ºC, the density of the air is

$$\rho_0 = \frac{T_{25C}}{T_{0C}}\rho_{25C} = \frac{273K + 25K}{273K}\rho_{25C} = 1.092\rho_{25C}$$

Now, the flute's frequency is the frequency of a standing wave within the flute. This frequency depends on the density of air as follows:

$$f_1 = \frac{c}{4\lambda} = \frac{1}{4\lambda}\sqrt{\frac{B}{\rho}}$$

Thus the frequency is inversely proportional to the air density. If the density increases by a factor of 1.092, the frequency becomes

$$f = \frac{1440s^{-1}}{\sqrt{1.092}} = 1378s^{-1}$$

Repeating the calculation for 40ºC, we obtain

$$\rho_{40C} = \frac{273K + 25K}{273K + 40K}\rho_{25C} = 0.952\rho_{25C}$$

$$f = \frac{1440s^{-1}}{\sqrt{0.952}} = 1476hz$$

(b) The density of an ideal gas, at equal temperature and pressure, is proportional to its molecular weight. Air is primarily nitrogen, N_2, with a molecular weight of approximately 28 amu. Helium is a monoatomic gas with a molecular weight of 4 amu. Therefore, the density of helium is approximately 4/28 = 0.143 that of air. Therefore, in helium,

$$f = \frac{1440s^{-1}}{\sqrt{0.143}} = 3808hz$$

Problem 9 (Extra Credit)
Suppose that a type of wave obeys the following variant of the wave equation, known as the *Klein-Gordon equation*:

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} + a^2 y = 0$$

a. Show that a sine wave, $y = A\sin(kx-\omega t)$, satisfies the Klein-Gordon equation.

b. What is the relationship between the angular frequency ω and the wave number k for a Klein-Gordon wave, in terms of the constants c and a?

c. What is the phase speed of Klein-Gordon waves? What is the group speed? Show that although the phase moves faster than c, wave packets always move slower than c.

Solution:
a. Take the derivatives of the sine wave:

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A sin(kx - \omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A sin(kx - \omega t)$$

Plugging y and its derivatives into the Klein-Gordon equation and eliminating the common factor of $A\sin(kx-\omega t)$, we obtain

$$\omega^2 - c^2k^2 - a^2 = 0$$

As long as this relation is satisfied, the sine wave satisfies the Klein-Gordon equation.

b. Solving the relation above for ω,

$$\omega = \sqrt{c^2k^2 + a^2}$$

c. The phase speed is

$$v_p = \frac{\omega}{k} = \frac{1}{k}\sqrt{c^2k^2 + a^2} = \sqrt{c^2 + \frac{a^2}{k^2}}$$

Note that we are using the symbol v_p rather than c for the phase speed, since c is now just a parameter in the equation, not an actual speed of any wave. The phase speed is always greater than c, and approaches c from above as k approaches infinity.

The group speed is

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{c^2k^2 + a^2} = \frac{c^2k}{\sqrt{c^2k^2 + a^2}} = \frac{c}{\sqrt{1 + a^2/c^2k^2}}$$

Since the denominator is always greater than 1, the group speed is always less than c. It approaches c from below as k approaches infinity. Wave packets travel slower than c.

Problem 10 (Extra Credit)

A whale is swimming 300 meters under the surface of the ocean. The whale emits a call, using a power output of 120 Watts.

A swimmer is in the water directly above the whale. If the swimmer's head is underwater, what is the intensity level of the sound he hears (in decibels)? If his head is above the water, what is the intensity level then? Hint: the transmission and reflection of sound waves works the same way as for waves on a string. Assume that the speed of sound in air is 331 *m/s*, and the speed of sound in water is 1550 *m/s*.

Solution:

Below the surface of the water, the intensity of the sound is the total flux (equal to the whale's output) divided by the surface area at 300 meters:

$$I = \frac{\Phi}{4\pi r^2} = \frac{120W}{4\pi \times (300m)^2} = 1.06 \times 10^{-4} W/m^2$$

The intensity level, in decibels, is

$$\beta = 10 log_{10}\left(\frac{I}{10^{-12} W/m^2}\right) = 80.3db$$

To get the intensity level above the surface, we must know how much of the intensity is transmitted into the air (rather than reflected off the surface back into the water). The result we obtained for an interface between two types of string, in terms of the wave speed *c*, is still valid:

$$T = \frac{4c_1c_2}{(c_1 + c_2)^2} = \frac{4 \times 331m/s \times 1550m/s}{(331m/s + 1550m/s)^2} = 0.580$$

The intensity just above the surface is thus 0.58 times that below the surface. This means that the intensity level in decibels changes by 10 $log_{10}(0.58)$ = -2.4db. The intensity level of the sound just above the surface is therefore 80.3 - 2.4 = 77.9 db.