Homework Set 1

Due Thursday, 06/30

<u>Problem 1</u>: A spring has a relaxed length of 15.0cm. When a 100g mass is hung from the spring, the spring is stretched to 17.5cm. When an unknown mass M is hung from the spring, the spring stretches to 18.0cm.

(a) What is the mass of M?

(b) What is the frequency of oscillations with the 100g mass and with mass M?

(c) If a 250g mass is attached to the spring while the spring is relaxed, and then dropped from that position, what will be the amplitude of the oscillations?

Solution:

(a) First, calculate the spring constant. From $mg = kz_0$, where z_0 is the stretching of the spring beyond the relaxed point, we get

$$k = \frac{mg}{z_0} = \frac{0.100kg \times 9.8m/s^2}{0.175m - 0.150m} = 39.2kg/s^2$$

Now we can use the spring constant to calculate the second object's mass:

$$m = \frac{kz_0}{g} = \frac{39.2kg/s^2 \times (0.180m - 0.150m)}{9.8m/s^2} = 0.12kg = 120g$$

(b) The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{39.2kg/s^2}{0.100kg}} = 3.15s^{-1} \quad \text{for the 100g mass}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{39.2kg/s^2}{0.120kg}} = 2.88s^{-1} \quad \text{for the 120g mass}$$

(c) When a 250g mass is added to the relaxed spring, the equilibrium position shifts down by

$$z_0 = \frac{mg}{k} = \frac{0.250kg \times 9.8m/s^2}{39.2kg/s^2} = 0.0625m = 6.25cm$$

Since the mass is released from rest 6.25cm from the new equilibrium, that distance will be the amplitude of the oscillations. So, the amplitude is 6.25cm.

<u>Problem 2</u>: The position of an oscillator is given by

$$x(t) = 0.0135m \times cos\left(6.58s^{-1} \times t + 0.14\right)$$

(a) What is the amplitude, angular frequency, frequency and period of the oscillations?

(b) What is the speed of the oscillator at $t = 0$?

(c) If the oscillator has a mass of 0.05kg, what is the spring constant?

Solution:

(a) The amplitude is the maximum displacement reached by the oscillator. It is 0.0135m, or 1.35 cm. The angular frequency is the factor multiplying the time in the cosine: it is equal to 6.58 s^{-1}. The frequency is $f = \omega / 2\pi = 6.58s^{-1} / 2\pi = 1.05s^{-1}$. The period is $T = 1/f = 0.95s$.

(b) Differentiating the position with respect to time, or recalling the kinematic equations of a harmonic oscillator, the speed is given by

$$v(t) = -A\omega sin(\omega t + \phi) = -0.0888m/s \times sin(6.58s^{-1} \times t + 0.14)$$

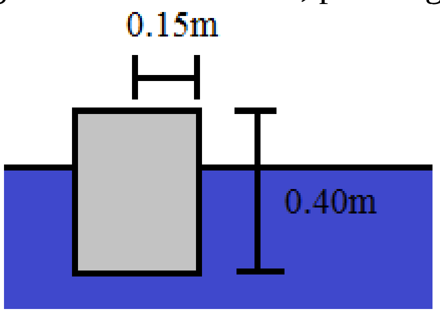
At $t = 0$, the speed is

$$v(0) = -0.0888m/s \times sin(0.14) = 0.0124m/s = 1.24cm/s$$

(c) The spring constant is related to the angular frequency by

$$\omega = \sqrt{\frac{k}{m}} \qquad k = m\omega^2 = 0.05kg \times (6.58s^{-1})^2 = 2.16kg/s^2$$

<u>Problem 3:</u> A cylindrical barrel with length L = 0.40m, radius R = 0.15m and density ρ = 600 kg/m^3 floats in water (density 1000 kg/m^3). The forces acting on the barrel are gravity, pointing down, and the buoyant force, equal to the weight of the displaced water according to Archimedes' law, pointing up.



(a) If the barrel is in equilibrium (no net force), how high is the top of the barrel above the surface of the water?

(b) Show that the barrel will obey Hooke's law if displaced slightly up or down from equilibrium. What is the effective "spring constant" and the period of oscillations?

Solution:

(a) Let z be the height of the top of the barrel above the water. The net force on the barrel (up is positive) is equal to

$$F = m_w g - m_b g = (V_w \rho_w - V_b \rho_b) g = \pi R^2 (L - z) \rho_w g - \pi R^2 L \rho_b g$$

Here, the subscript w denotes the mass, volume and density of the displaced water, while the subscript b indicates the same quantities pertaining to the barrel. At the equilibrium position z_0, the net force is zero:

$$F = F(z_0) = \pi R^2 g[(L - z_0)\rho_w - L\rho_b] = 0 \qquad (L - z_0)\rho_w - L\rho_b = 0$$

$$z_0 = \frac{\rho_w - \rho_b}{\rho_w} L = \frac{400 kg/m^3}{1000 kg/m^3} \times 0.40m = 0.16m = 16cm$$

(b) If the top of the barrel is z meters above the water, the net force is given by the first equation in part (a). Now we define x as the displacement from equilibrium: $x = z - z_0$, or $z = x + z_0$. Plugging this into the force equation, using the result for z_0 and simplifying, we obtain Hooke's Law:

$$F = -\pi R^2 \rho_w g x \qquad F = -kx \qquad k = \pi R^2 \rho_w g$$

The "spring constant" and the period of oscillations are:

$$k = \pi R^2 \rho_w g = \pi (0.15m)^2 \times 1000 kg/m^3 \times 9.8 m/s^2 = 693 kg/s^2$$

$$T = 2\pi \sqrt{\frac{m_b}{k}} = 2\pi \sqrt{\frac{\pi (0.15m)^2 \times 0.40m \times 600 kg/m^3}{693 kg/s^2}} = 0.98s$$

Problem 4: A 6.0 kg block is attached to a spring with k = 120kg/s, and slides on a floor with a coefficient of kinetic friction $\mu_k = 0.25$. The block begins at the equilibrium position. It is struck by a bullet with a mass of 15 grams, moving at 650 m/s and coming from the right. The bullet becomes embedded in the block.

How far to the left does the block move?

Solution: First, use conservation of momentum to determine the speed of the block after it is struck by the bullet. (Energy is clearly not conserved when the bullet strikes the block, since the collision is inelastic). The bullet has momentum p = 0.015kg x 650 m/s = 9.75 kg m/s. The combined block-bullet system has the same momentum. The mass of the bullet is insignificant compared to the mass of the block; it's OK to include it, but we will take the combined mass of the block and bullet to be approximately 6.0 kg. Its speed is therefore

$$v = \frac{p}{m} = \frac{9.75kg \cdot m/s}{6.0kg} = 1.63m/s$$

Next, we use conservation of energy to see how far the block goes. The amount of energy lost to friction is equal to the frictional force times distance traveled. If we call the distance traveled x (we will consider it to be positive) we have the following expression for "conservation" of energy:

$$E = E_0 - F_f x = \frac{1}{2}mv_0^2 - \mu_k mgx = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

We are looking for the point where the block comes to a stop, so $v = 0$ there. The equation for conservation of energy thus becomes

$$\frac{1}{2}mv_0^2 - \mu_k mgx = \frac{1}{2}kx^2$$

$$\frac{1}{2}kx^2 + \mu_k mgx - \frac{1}{2}mv_0^2 = 0$$

$$(0.5\times 120kg/s^2)x^2 + (0.25\times 6.0kg\times 9.8m/s^2)x - (0.5\times 6.0kg\times(1.63m/s)^2) = 0$$

$$(60kg/s^2)x^2 + (14.7kg\cdot m/s^2)x - (7.97kg\cdot m^2/s^2) = 0$$

$$x = \frac{\left(-14.7 + \sqrt{14.7^2 + 4\cdot 60\cdot 7.97}\right)kg\cdot m/s^2}{120kg/s^2} = 0.26m = 26cm$$

Note that we have taken the positive root. That is because we have defined the displacement of the mass as positive when we calculated energy lost to friction.

<u>Problem 5:</u> A 70kg bungee jumper jumps off a very tall bridge using a bungee cord with a relaxed length of 30 meters.

Assume the bungee cord exerts no force until it reaches its relaxed length, and obeys Hooke's Law when stretched beyond the relaxed length. To avoid dislocating the jumper's leg, the deceleration of the bungee jumper should not exceed 30 m/s^2. What is the maximum allowable spring constant of the bungee cord?

Solution: This problem combines kinematics of constant acceleration with conservation of energy. Our strategy will be to calculate the bungee jumper's energy when the bungee cord begins to stretch, determine the resulting amplitude of the oscillation, and from that, determine the maximum acceleration as a function of *k*. We will then solve for the maximum possible *k*.

When the bungee cord first begins to stretch, the speed at which the jumper is falling is

$$v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8m/s^2 \times 30m} = 24.2m/s$$

The jumper is also well above the equilibrium point of the oscillator. Equilibrium is reached when the bungee cord is sufficiently stretched that the elastic force is the same as the force of gravity, $kz_0 = mg$. Therefore, the equilibrium is $z_0 = mg / k$ meters below the point where the cord begins to stretch. In other words, when the bungee cord first begins to stretch, the jumper is still mg / k meters above the equilibrium point. The oscillator energy is therefore

$$E = \frac{1}{2}mv_0^2 + \frac{1}{2}kz_0^2 = \frac{1}{2}kA^2$$

Solve for the amplitude of the oscillation:

$$A = \sqrt{z_0^2 + \frac{mv_0^2}{k}} = \sqrt{\frac{m^2g^2}{k^2} + \frac{mv_0^2}{k}}$$

The acceleration is greatest when the displacement is equal to the amplitude. At that point, the magnitude of the force is *kA*, and so the magnitude of the acceleration is kA / m. Therefore, the maximum acceleration is

$$a_{max} = \frac{k}{m}\sqrt{\frac{m^2g^2}{k^2} + \frac{mv_0^2}{k}} = \sqrt{g^2 + \frac{kv_0^2}{m}} < 30m/s^2$$

Next, we square both sides and obtain an inequality for *k*:

$$g^2 + \frac{kv_0^2}{m} < 900m^2/s^4 \qquad k < \frac{70kg}{(24.2m/s)^2}\left(900m^2/s^4 - (9.8m/s^2)^2\right)$$

$$k < 96kg/s^2$$

Thus, the maximum safe spring constant of this bungee cord is 96 kg/s^2.

<u>Problem 6:</u> A simple pendulum is in an elevator that is accelerating upward at 1.5 m/s^2. The pendulum is swinging with a period of T = 0.8s.

(a) What is the length of the pendulum?

(b) When the elevator begins to decelerate at 1.5 m/s^2, what will be the period of this pendulum?

Solution:

(a) When the elevator is accelerating upward at 1.5m/s^2, the effective "acceleration due to gravity" felt within the elevator is 9.8m/s^2 + 1.5m/s^2 = 11.3m/s^2. The period of the pendulum is

$$T = 2\pi\sqrt{\frac{L}{g_{eff}}}$$

The length of the pendulum is thus

$$L = \frac{g_{eff}T^2}{4\pi^2} = \frac{11.3m/s^2 \times (0.8s)^2}{4\pi^2} = 0.183m = 18.3cm$$

(b) When the elevator is decelerating, the effective gravity is 9.8m/s^2 + 1.5m/s^2 = 8.3m/s^2. The period is now

$$T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{0.183m}{8.3m/s^2}} = 0.93s$$

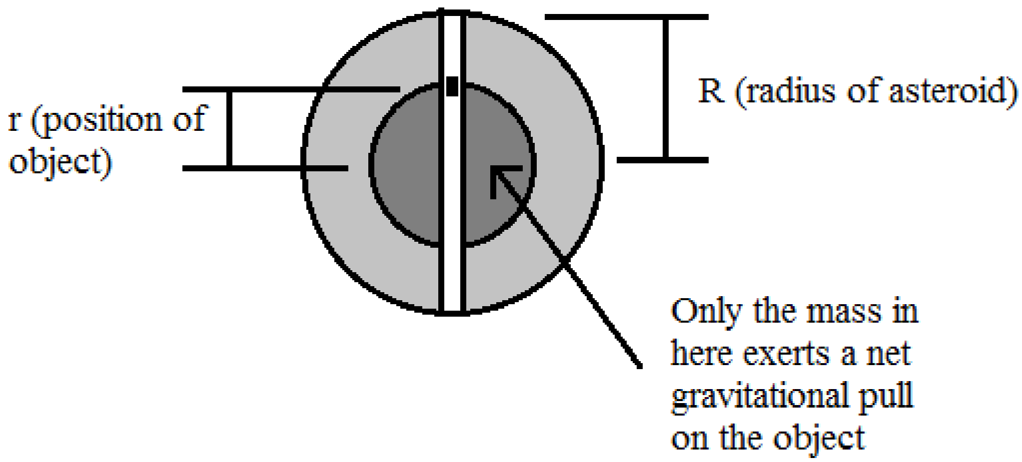
<u>Problem 6</u>: Darth Vader lands on an unknown planet. For some reason, he has a pendulum clock with him. On his spaceship, where the artificial gravity is kept at 8.0m/s^2, the clock keeps perfect time, but on the planet's surface, the clock's hour hand takes only 40 minutes to move through 1 hour. What is the acceleration due to gravity on the planet?

Solution: The amount of time it takes the clock to go through 1 second is proportional to its pendulum's period, which is in turn inversely proportional to the square root of the acceleration due to gravity. Therefore,

$$T_0 \propto 60min \propto \frac{1}{\sqrt{8.0m/s^2}}$$
$$T_1 \propto 40min \propto \frac{1}{\sqrt{g_P}}$$
$$\sqrt{\frac{g_P}{8.0m/s^2}} = \frac{60min}{40min} = \frac{3}{2}$$
$$g_P = \frac{9}{4} \times 8.0m/s^2 = 18.0m/s^2$$

The acceleration due to gravity on the planet is therefore 18.0 m/s^2.

Problem 7: Suppose we have a spherical asteroid with radius R = 50km and density ρ = 3000 kg/m^3. We drill a straight tunnel from the surface of the asteroid to the center and out to the other side, and drop an object into the tunnel. There is a theorem, by Isaac Newton, that an object within a spherically symmetric distribution of mass (such as our asteroid) is attracted only by the gravity of the stuff that is closer to the center of the distribution than the object itself. So, if our object is 5km from the asteroid's center, it will only be attracted by the mass that is within 5km of the center of the asteroid. You will need to use this theorem to solve this problem.



(a) Show that if the object doesn't collide with the walls or otherwise experience friction, it will move back and forth through the tunnel with simple harmonic motion.

(b) How long will it take for the object to come back to you, after you throw it in?

(c) What is the object's maximum speed?

Solution:

(a) Let r be the distance of the object from the center of the asteroid. The mass of the part of the asteroid that is enclosed within a sphere of radius r is

$$M(r) = V(r)\rho = \frac{4}{3}\pi r^3 \rho$$

The force due to gravity felt by the falling object is thus

$$F = -\frac{GmM(r)}{r^2} = -\frac{4}{3}\pi m G\rho r = -kr \qquad k = \frac{4}{3}\pi m G\rho$$

The object thus obeys Hooke's law, and undergoes simple harmonic back and forth in the tunnel.

(b) When the object comes back, that means one period of the oscillation has passed. The period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3}{4\pi G\rho}} = 2\pi\sqrt{\frac{3}{4\pi \times 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} \times 3000 kg/m^3}} =$$
$$= 6860s = 1.91h$$

The object will come back up the tunnel 1.91 hours after it is dropped in.

(c) The object's maximum speed occurs in the center, where the potential energy is zero, and can be determined from conservation of energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$
$$v_{max} = A\sqrt{\frac{k}{m}} = A\sqrt{\frac{4}{3}\pi G\rho} =$$
$$= 5 \times 10^4 m \times \sqrt{\frac{4}{3}\pi \times 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2} \times 3000 kg/m^3} =$$
$$= 45.8 m/s$$

<u>Problem 9 (Extra Credit):</u> A 0.10 kg mass is oscillating on a weakly damped spring with a period of 0.5 seconds. It begins with an amplitude of A_0 = 16 cm, but after 30 seconds, the amplitude is reduced to 10 cm due to the damping force.

(a) What will the amplitude of the oscillations be after 1 minute?

(b) What is the magnitude of the frictional force when the mass passes through the equilibrium position for the first time, before the amplitude has decreased?

Solution:
(a) Since this mass is oscillating, the oscillator is underdamped. From the second lecture, we know that the amplitude of an underdamped oscillator decreases exponentially with time as

$$A(t) = A_0 e^{-\gamma t}$$

We know that A_0 = 16cm, and $A(t{=}30s)$=10cm. We can now solve for the rate γ.

$$\gamma = \frac{1}{t} ln\left(\frac{A_0}{A(t)}\right) = \frac{1}{30s} ln\left(\frac{16cm}{10cm}\right) = 0.0157s^{-1}$$

The amplitude after 1 minute will be

$$A(60s) = 16cm \times e^{-0.0157s^{-1}\times 60s} = 6.2cm$$

(b) For an underdamped oscillator, the damping coefficient is related to γ as follows:

$$\gamma = \frac{b}{2m}$$

Therefore, $b = 2m\gamma$ = 0.00314kg s^{-1}. To get the damping force, we also need to know the speed of the mass as it passes through the equilibrium point. This can be determined from the period and the amplitude. Since the oscillator is only weakly damped, all the relations for an undamped oscillator are approximately true, and we have

$$\frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2 \qquad v_{max} = A\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

$$\text{therefore, } v_{max} = \frac{2\pi A}{T} = \frac{2\pi \times 0.16m}{0.5s} = 2.01m/s$$

The magnitude of the damping force at the equilibrium point is therefore

$$|F_d| = b v_{max} = 0.00314kg/s \times 2.01m/s = 6.31 \times 10^{-3} N$$

Problem 10 (Extra Credit): Suppose that a pendulum clock keeps perfect time on the equator. If this clock is moved to the South Pole, will it be too fast or too slow? How much time will it gain / lose over the course of 1 year? (Hint: assume that the difference in local gravity on Earth is primarily due to centrifugal acceleration due to the Earth's rotation.)

Solution: At the equator, the local gravity appears to be a bit lower, since there is a centrifugal "force" from the Earth's rotation pointing straight up. The difference is

$$a_c = \omega_E^2 R_E = \left(\frac{2\pi}{86400s}\right)^2 \times 6.38 \times 10^6 m = 0.0337 m/s^2$$

Therefore, gravity at the pole is 1.00343 that at the equator. The period of the pendulum is inversely proportional to the square root of the gravity, so at the pole, the pendulum will count 1 second in

$$\sqrt{\frac{1}{1.00343}} = 0.99829s$$

The clock will run too fast. In one year, it will count the following number of seconds:

$$N = \frac{365.24d/yr \times 86400s/d}{0.99829counts/s} = 3.1611 \times 10^7 s$$

$$\text{instead of } 365.24d/yr \times 86400s/d = 3.1556 \times 10^7 s$$

$$\text{Extra time: } (3.1611 - 3.1556) \times 10^7 s = 55000s \approx 15hrs!$$

So, the clock will be ahead by 15 hours if it runs for 1 year at its new location.