Reflection and refraction

When an electromagnetic wave, such as light, encounters the surface of a medium, some of it is reflected off the surface, while some crosses the boundary and enters the material. The direction of travel of the wave in the material will typically change; this is called refraction.

The arrows in the diagrams are known as rays, and represent the direction of travel of the wave. The rays are always perpendicular to the wave fronts. For large-scale phenomena (large-scale compared to the wavelength) involving plane waves, the electromagnetic waves all travel in the same direction and in straight lines, and rays are a useful way to represent their path. The behavior of rays under these circumstances is described by geometric optics. For phenomena that occur on a scale comparable to the wavelength, the ray approximation is not adequate, and we must keep detailed track of the actual electromagnetic waves; this is described by wave optics.

The law of reflection

The law of reflection states that the angle of reflection of the wave is the same as the angle of incidence:

\[ \theta_I = \theta_R \]

This is due to a pair of symmetries the problem possesses. Time reversal invariance is a property of electromagnetism, states that if we reverse the direction of an EM wave, it will retrace its path. The setup in the diagram is also symmetric if we flip the diagram.
Reflection and images

Reflection of the rays off a smooth surface results in the creation of an *image*. An image is an apparent source of light rays. This is a ray diagram for an arbitrarily selected pair of rays from an object that reflect off a mirror, or another smooth reflective surface:

The rays are reflected off the surface of the mirror and travel to the observer, as shown on the left side of the diagram. The observer detects these rays and traces them back to their apparent source on the right, behind the mirror. These rays appear to come from an image. Of course, there are really no rays behind the mirror, but from the observer's point of view, the situation is indistinguishable from the one where the image is the real source of the rays. Thus, the observer sees the image as if it was the actual object.

An image like this, which is not a source of actual rays but rather the result of the observer's eyes tracing the rays back to their apparent origin, is called a *virtual image*. In other situations, we will see *real images*, which are formed at intersections of actual rays of light.

Tracing rays from other points on the object establishes that the reflected image the same size as the object, and is the same distance from the mirror as the object but on the opposite side of the mirror. The image is flipped front to back; this flipping results in an apparent change of handedness.
Example: Two mirrors are placed at a right angle to each other. A ray of light is incident on one of the mirrors at angle \( \theta \), as shown. What is the direction of the outgoing ray?

The reflection will look something like this, with the relevant angles labeled:

By the law of reflection, angle \( a = \theta \). Since angles \( a \) and \( b \) are two angles of a right triangle, and the third angle is 90 degrees, \( b = 180^\circ - 90^\circ - a = 90^\circ - a \). Again by the law of reflection, \( b = c \), and again from simple geometry, \( d = 90^\circ - c \). Thus, \( d = 90^\circ - (90^\circ - \theta) = \theta \). The final outgoing ray is therefore parallel to the incident ray, but going in the opposite direction. This is a commonly known property of reflection from the inside of a right angle.

Refraction

If the material is transparent, some of the light will enter the material rather than being reflected. In a material, light is no longer a pure electromagnetic wave, but involves oscillations in the polarization of the material as well; this is a periodic movement of the electrons. The presence of polarization changes the wave equation, altering the speed at which electromagnetic waves propagate. We define the index of refraction, which tells us how much the wave is slowed down in a particular material:

\[
n \equiv \frac{c}{v}
\]

Here, \( v \) is the speed of light in the material, and \( c \) is the speed of light in vacuum. Since light travels slower in matter than in vacuum, the index of refraction is always greater than or equal to 1. Most common transparent materials, such as glass, have an index of refraction between 1.3 and 2.0. A few materials, such as diamond, have a higher index of refraction. Water has an index of refraction that depends on the temperature; for water near room temperature, it is about 1.33.
Snell's law

When a wave front crosses a boundary, for example from a material with a faster wave speed to one with a slower wave speed, it will change direction a bit, since the part of the wave front that hits the slower part of the material first will slow down first:

We can see that crossing into a material with a slower wave speed makes the waves turn closer to the direction normal to the surface. Let us label the angles as follows:

Over time $T$, the wave fronts in material 1 move a distance $v_1 T$ while the wave fronts in material 2 move a distance of $v_2 T$. Let us call this distance $d$. The intersection of the wave front with the interface moves a distance $\Delta x$, related to $d$ by $d = \Delta x \sin \theta$. But the wave crests on the two sides must remain joined together across the surface. This means that the distance $\Delta x$ is the same on both sides. This gives

$$d_1 = v_1 T = \Delta x \sin \theta_1$$

$$d_2 = v_2 T = \Delta x \sin \theta_2$$

Dividing one equation by the other gives

$$\frac{v_1}{v_2} = \frac{\sin \theta_1}{\sin \theta_2}$$

Since the speed is inversely proportional to the index of refraction, this equation can be rewritten as follows:

$$\frac{n_2}{n_1} = \frac{\sin \theta_1}{\sin \theta_2}$$

or

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

This is Snell's law of refraction.
**Example:** A beam of light is incident on a pane of glass 1.0 centimeters thick, with an index of refraction $n = 1.40$, at an angle of 60 degrees to the normal. Show that the beam that emerges on the other side moves in the same direction as the incident beam, but is shifted a distance $d$. What is the distance $d$?

First, draw a diagram tracing the path of the light rays:

![Diagram of light ray through glass pane](image)

In air, the index of refraction is approximately 1.00. Snell's law gives $\sin \theta_1 = n \sin \theta_2$ for light entering the pane of glass from the left and $n \sin \theta_2 = \sin \theta_3$ for light leaving the glass on the right. Thus, $\theta_1 = \theta_3$; the outgoing ray points in the same direction as the incident ray. The beam is shifted a distance $d$, given by the following geometric construction:

![Geometric construction](image)

Since $d = L \sin(\theta_1 - \theta_2)$ and $w = L \cos \theta_2$, this gives $d = w \sin(\theta_1 - \theta_2) / \cos \theta_2$. Plugging in the numbers and using Snell's law to calculate $\theta_2$, we obtain

$$d = 1.0\text{cm} \times \frac{\sin(60^\circ - 38^\circ)}{\cos(38^\circ)} = 0.475\text{cm}$$

You can see that these kinds of refraction problems rely heavily on basic geometry. Hence the term geometric optics.
It can be shown that light traveling from point A to point B minimizes the time it takes to get from one point to another. Suppose that we have two materials, 1 and 2, with different light speeds. We wish to send a beam of light from point A in material 1, a distance \( x_A \) from the interface, to point B in material 2, a distance \( x_B \) from the interface and shifted a distance \( y_B \) along the interface from point A. Let \( y \) be the point where where the beam of light crosses the interface:

The amount of time it takes the light to travel from A to B is the sum of the times it takes for each segment of the path:

\[
t = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x_A^2 + y^2}}{v_1} + \frac{\sqrt{x_B^2 + (y_B - y)^2}}{v_2}
\]

When the time is minimized, the derivative of the time with respect to \( y \) is zero. This gives the following expression:

\[
\frac{y}{v_1 \sqrt{x_A^2 + y^2}} - \frac{y_B - y}{v_2 \sqrt{x_B^2 - (y_B - y)^2}} = 0
\]

By labeling the angles \( \theta_1 \) and \( \theta_2 \) as before and looking at the geometry of the problem, we realize that the following fractions are just a side of a right triangle over the hypotenuse, and thus

\[
\frac{y}{\sqrt{x_A^2 + y^2}} = \sin(\theta_1) \quad \frac{y_B - y}{\sqrt{x_B^2 - (y_B - y)^2}} = \sin(\theta_2)
\]

Thus we obtain Snell's law:

\[
\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}
\]

Snell's law is therefore equivalent to the statement that light always takes the path between two points that minimizes the amount of time for light to travel that path.
Snell's law involves sines of angles. As we know, sines of real numbers are limited to values less than or equal to one. Thus it follows that there are some angles at which refraction into certain different materials is not possible. If this is the case, all light is reflected. This is known as total internal reflection; internal because, as we will see, it can only occur when refracting from a medium with a higher index of refraction into one with a lower index, for example from the inside of a piece of glass into air.

Snell's law is
\[ \frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \quad \text{or} \quad \sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1 \]

Clearly, if \( n_1 > n_2 \), there will be some values of \( \theta_1 \) for which Snell's law will give us \( \sin \theta_1 > 1 \). This means that there is no angle at which refraction can occur, and all light is reflected. The critical angle is the value of \( \theta_1 \) at which \( \sin \theta_2 \) becomes equal to 1:
\[ \frac{n_1}{n_2} \sin \theta_C = 1 \quad \sin \theta_C = \frac{n_2}{n_1} \]

Since the sine of an angle must be less than or equal to one, the critical angle only exists if the index of refraction \( n_1 \) is greater than \( n_2 \). Thus, total internal reflection only occurs when light strikes the interface from the side with the higher index of refraction. In that case, if the incident angle is less than the critical angle, some of the light is refracted out of the medium; if the incident angle is greater than the critical angle, then all light is reflected.

Total internal reflection is used in fiber-optic cables: if light is sent down the length of a transparent cable, it will reflect perfectly off the walls as long as it strikes the walls at a large enough angle from the normal. This will be true if the cable is not bent too much.

**Example:**
Reflection and refraction works the same way for sound as it does for light, except that with sound, the speed in denser materials is usually higher than in gases such as air, not lower. What is the critical angle for an interface between a piece of metal with a speed of sound of 3500 m/s, and air, with a speed of sound of 331 m/s?

Since the index of refraction is inversely proportional to wave speed, \( \sin \theta_C = \frac{n_2}{n_1} = \frac{v_2}{v_1} \). Therefore, \( \sin \theta_C = \frac{331}{3500}, \) or \( \theta_C = 5.4^\circ \). Thus, the sound will be reflected off the metal surface unless it comes in almost perpendicular to it, at 5.4 degrees or less from the normal. Since the speed of sound in solid materials is so much higher than in air, walls are very efficient at reflecting sound.
Dispersion

In general, the index of refraction of a material depends on wavelength of light, being higher for longer wavelengths than for shorter ones. If white light (which is a mixture of all the different wavelengths) is refracted through a material, blue light will be refracted through a greater angle than red light, and the wavelengths will separate. This is the principle used in prisms.

Example: Suppose we have the same pane of glass as in the refraction problem above, but now the index of refraction is 1.46 for violet light and 1.40 for red light. By what distance are the red and violet sides of the emerging beam separated?

We already did the calculation for the displacement of a beam with \( n = 1.40 \), and found that the beam was displaced by a distance of 0.475 centimeters. Repeating the calculation with \( n = 1.46 \), we obtain

\[
\begin{align*}
    d &= 1.0 \text{cm} \times \frac{\sin(60^\circ - 33^\circ)}{\cos(33^\circ)} \\
    &= 0.541 \text{cm}
\end{align*}
\]

Thus the violet light is displaced by 0.541 centimeters. The difference is 0.066 cm, or 0.66 mm. The red edge of the visible spectrum is separated from the violet edge by a distance of 0.66 mm. This is a rather small distance; for this reason, light is usually separated by prisms, which have sides that are not parallel and cause the angle at which the different wavelengths of light to emerge to be different. The separation between the different wavelengths can then be increased by projecting the image of the spectrum onto a screen far away.
Absorption

Any material, such as air, water or glass, is not perfectly transparent. As light travels through such a material, it is gradually absorbed and converted into heat. Some wavelengths are absorbed much more strongly than others. For example, the Earth's atmosphere is nearly transparent to visible light, but the ozone in the atmosphere absorbs strongly in the ultraviolet range, and carbon dioxide, water vapor, methane and other greenhouse gases absorb strongly in the infrared.

Due to absorption, the intensity of light as it travels through matter decays exponentially. If the initial intensity at is $I_0$, and the light has traveled a distance $x$, its intensity is

$$I = I_0 e^{-\alpha x}$$

This is known as the **Beer-Lambert law**. The constant $\alpha$ is known as the absorption coefficient, and has units of inverse length. It is equal to the average distance that light travels through the material before being absorbed.

**Example:**

A certain person can sit in the sun for 20 minutes without being sunburned, when the sun is 30 degrees from the vertical. When the sun is 45 degrees from the vertical, this person can last 60 minutes without sunburn. How quickly would she develop a sunburn if the sun is directly overhead?

The distance that sunlight travels through the atmosphere is $d_0 / \cos \theta$, where $\theta$ is the angle of the sun from the vertical, and $d_0$ is the distance when the sun is directly overhead. Thus, when the sun is 30 degrees from the vertical, the distance is $1.15d_0$, and when the sun is 45 degrees from the vertical, it is $1.41d_0$. The amount of time it takes to develop a sunburn is inversely proportional to the intensity of the ultraviolet light, which falls off with distance according to the Beer-Lambert law. Therefore,

$$\frac{1}{20\text{min}} \propto e^{-1.15\alpha d_0} \quad \frac{1}{60\text{min}} \propto e^{-1.41\alpha d_0} \quad \frac{1}{T} \propto e^{-\alpha d_0}$$

Dividing the first equation by the second yields

$$3 = \frac{e^{-1.15\alpha d_0}}{e^{-1.41\alpha d_0}} = e^{0.26\alpha d_0} \quad \alpha d_0 = \frac{\ln 3}{0.26} = 4.23$$

Dividing the first equation by the third yields

$$\frac{T}{20\text{min}} = \frac{e^{-1.15\alpha d_0}}{e^{-\alpha d_0}} = e^{-0.15\alpha d_0} \quad T = 20\text{min} \times e^{-0.15 \times 4.23} = 10.6\text{min}$$

This person can thus tolerate slightly over 10 minutes of sun directly overhead without developing a sunburn - about half the time with the sun at 30 degrees from the vertical.