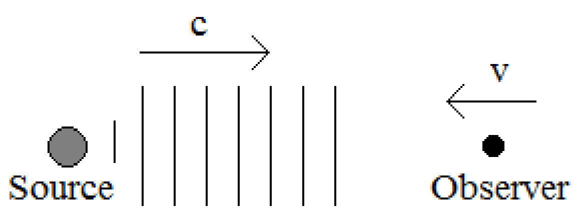
Lecture 6 Notes: 07 / 06

The Doppler effect

The Doppler effect is a shift in frequency that results from motion of the source or the observer (or both).

Consider first the case of a stationary source and a moving observer. Let f_S be the frequency of the source, and f_O be the frequency measured by the observer.



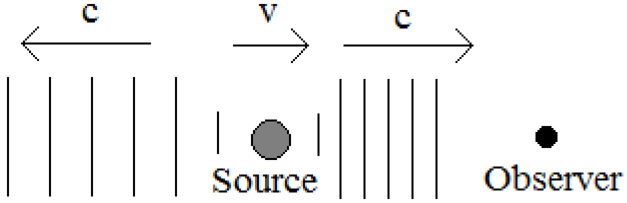
Observer approaching the source crosses more wavefronts per second

The source creates a bunch of sound waves with wave fronts separated by a wavelength λ. If the observer were stationary, the number of wave fronts passing him every second would be equal to the frequency of the source, $f_S = c / \lambda$. However, if the observer is moving towards the source with speed v, he will pass more wavefronts per second: $f_O = (c+v) / \lambda$. Dividing the frequency of the observer by that of the source, we get $f_O / f_S = (c+v) / c$, or

$$f_O = \frac{c+v}{c} f_S \qquad \text{Moving observer}$$

If the observer moves away from the source, rather than towards it, the sign of v is simply reversed.

Now consider a moving source and a stationary observer. Suppose the source is moving towards the observer with speed v:



Moving source creates shorter wavelengths in front of it, longer wavelengths behind.

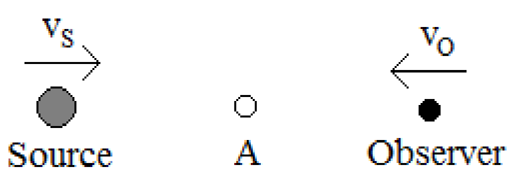
If the source was stationary, it would emit wave crests a distance λ apart, but now, over one period, the source moves a distance vT, so the next wave crest is emitted a distance of only $\lambda - vT = \lambda - v / f_S$ from the first. The observer therefore sees a wavelength of $l_O = \lambda - v / f_S$ and a frequency of
$f_O = c / \lambda_O = c / (\lambda - v / f_S)$. Multiplying both the numerator and the denominator by f_S and using $\lambda f_S = c$, we obtain

$$f_O = \frac{c}{c - v} f_S \qquad \text{Moving source}$$

Again, if the source is receding from the observer rather than approaching, the sign of v is reversed. The rule of thumb is that the frequency increases if the source and the observer are moving closer with time, and decreases if the source and the observer are getting farther and farther away from each other.

Example: Suppose the source is moving towards the observer with speed v_S, and the observer is simultaneously moving towards the source with speed v_O. What is the frequency measured by the observer, in terms of the source frequency?

We can derive the equation for both the source and the observer moving by breaking up the problem into two steps. First, introduce an additional observer A, located between the source S and the observer O, who is stationary with respect to the medium through which the sound is propagating:



The stationary observer A measures the frequency of the moving source to be

$$f_A = \frac{c}{c - v_S} f_S$$

The sound wave passes through A at this frequency, and continues on the observer O. Therefore, A can be treated as a stationary source with frequency f_A. The frequency measured by the moving observer O is then

$$f_O = \frac{c + v_O}{c} f_A = \frac{c + v_O}{c - v_S} f_S$$

Example: Suppose that an active sonar on a stationary ship is used to determine a receding ship's speed. A sonar ping is emitted at a frequency of 1400 *hz*, and the echo returns with a frequency of 1390 *hz*. The speed of sound in water is approximately 1500 *m/s*. How fast is the ship moving away?

Again, this problem is derived by breaking it up into steps. Let f_S be the sonar frequency, f_E be the echo frequency, and f_O be the sonar frequency as measured by the ship. Then, we have

$$f_O = \frac{c-v}{c} f_S$$

Note that the minus sign appears because the ship is moving away from the source of the sound. With this notation, v is positive and will give the magnitude of the ship's speed. The ship receives the sonar ping at this frequency, and simply reflects it at the same frequency. Now the sonar can be treated as the observer, and the ship as a source with frequency f_O, moving away from the observer at speed v:

$$f_E = \frac{c}{c+v} f_O = \frac{c-v}{c+v} f_S$$

Now solve for v:

$$(c+v)f_E = (c-v)f_S \qquad v(f_S + f_E) = c(f_S - f_E)$$

$$v = \frac{f_S - f_E}{f_S + f_E} c = \frac{(1400-1390)hz}{(1400+1390)hz} \times 1500 m/s = 5.4 m/s$$

Interference of sound waves

Recall that the variation in pressure (or the displacement of air molecules from the equilibrium position) in a traveling sound wave is given by

$$y(x,t) = A sin(kx - \omega t + \phi) \equiv A sin(\theta)$$

If two sound waves meet, the values of y for each wave are simply added to determine the resultant wave:

$$y(x,t) = y_1(x,t) + y_2(x,t) = A_1 sin(\theta_1) + A_2 sin(\theta_2)$$

Suppose that the sound waves are perfectly in phase. This means that the phases θ_1 and θ_2 are different by an even multiple of π, and when $sin(\theta_1) = 1$, $sin(\theta_2) = 1$ as well. In this case, the maximum displacement of the combined wave is

$$y_{max} = A_1 + A_2$$

The amplitudes of the two waves add. This is known as ***constructive interference***.

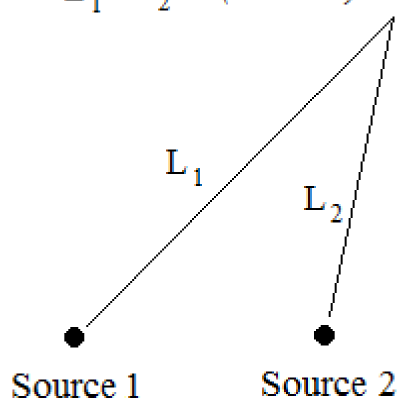
Now let the waves be perfectly out of phase, so that when one wave is at a maximum, the other one is at a minimum. This happens when θ_1 and θ_2 are different by an odd multiple of π. In this case, y attains a maximum when the wave with the larger amplitude is positive, and the one with the smaller amplitude is negative:

$$y_{max} = |A_1 - A_2|$$

The overall amplitude is reduced. This is known as ***destructive interference***. If $A1$ and $A2$ are equal, the waves will cancel perfectly.

Suppose we have two sources of sound waves which emit the exact same wave form (same frequency, and the sources are in phase with each other). In this case, the phase difference between the two waves at a point away from the speakers is determined by the path length difference $L_1 - L_2$. If the path lengths differ by a multiple of the wavelength λ, the waves are in phase and constructive interference occurs. The sound is louder at these points. If the path lengths differ by $L_1 - L_2 = (n + 1/2)\lambda$, the waves are out of phase and we get destructive interference, and the sound is quieter.

$L_1 - L_2 = n\lambda$: Constructive
$L_1 - L_2 = (n + 1/2)\lambda$: Destructive



Example:
An observer is located directly between two speakers, located 20 meters apart. The speakers are in phase with each other, and both are emitting a sound with a frequency of 60 *hz*. How far away from the center should the observer move to get destructive interference?

In the center, the observer is 10 meters away from each speaker, which gives constructive interference (L_1 - L_2 = 0 = $n\lambda$, n = 0). If the observer moves a distance x from the center towards speaker 2, the distance from one speaker is now $L_1 = 10 + x$ meters, and the distance from the other speaker is $L_2 = 10 - x$ meters.
Therefore, $L_1 - L_2 = 2x$. For destructive interference, we want

$$2x = \left(n + \frac{1}{2}\right)\lambda$$

The first point of destructive interference occurs at $n = 0$, or $x = 1/4\ \lambda$. Therefore,

$$x = \frac{1}{4}\lambda = \frac{1}{4}\frac{c}{f} = \frac{1}{4}\frac{331m/s}{60s^{-1}} = 1.4m$$

Thus if the observer moves 1.4 meters from the center towards one speaker or the other, this will be a point of destructive interference, where the sound waves from the two speakers will tend to cancel each other and amplitude of the sound will be at a minimum.

We will revisit interference in more detail, in the context of electromagnetic waves, later on in the course.

Beats

Now suppose that two waves with equal amplitude A have slightly different angular frequencies ω_1 and ω_2. Let us specialize to the point $x = 0$ and assume that the two waves are in phase at $t = 0$. Then, the value of the wave function at $x = 0$ at time t is

$$y = A\left[sin(\omega_1 t) + sin(\omega_2 t)\right]$$

We have arbitrarily chosen the sign of the argument of the sine to be positive, to avoid writing all the minus signs. Now let us change variables to the average frequency, and the difference in frequency between the two waves:

$$\frac{1}{2}(\omega_1 + \omega_2) \equiv \omega_0 \qquad \omega_2 - \omega_1 = \Delta\omega$$

Solving for ω_1 and ω_2 and plugging this into the equation for y gives

$$\omega_1 = \omega_0 - \frac{1}{2}\Delta\omega \qquad \omega_2 = \omega_0 + \frac{1}{2}\Delta\omega$$

$$y = A\left\{sin\left[\left(\omega_0 - \frac{1}{2}\Delta\omega\right)t\right] + sin\left[\left(\omega_0 + \frac{1}{2}\Delta\omega\right)t\right]\right\}$$

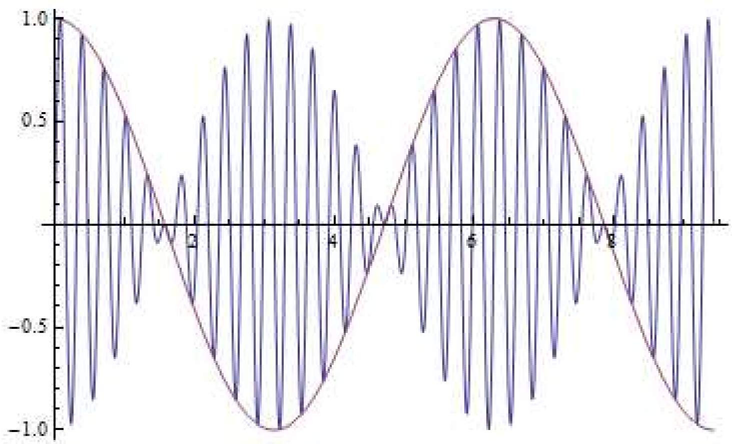
Finally, use the trigonometric identity *sin(A+B) = sin A cos B + cos A sin B*:

$$y = A\left\{sin\left[\left(\omega_0 - \frac{1}{2}\Delta\omega\right)t\right] + sin\left[\left(\omega_0 + \frac{1}{2}\Delta\omega\right)t\right]\right\}$$

$$y = A\left[sin(\omega_0 t)\cos\left(\frac{\Delta\omega}{2}t\right) - cos(\omega_0 t)\sin\left(\frac{\Delta\omega}{2}t\right)\right.$$
$$\left. + sin(\omega_0 t)\cos\left(\frac{\Delta\omega}{2}t\right) + cos(\omega_0 t)\sin\left(\frac{\Delta\omega}{2}t\right)\right]$$

$$y = 2Asin(\omega_0 t)cos\left(\frac{\Delta\omega}{2}t\right)$$

Thus there is a fast oscillation, $sin(\omega_0 t)$, with an amplitude that varies slowly between zero and $2A$ in time (again, we are assuming that the two frequencies are nearly the same, so that $\Delta\omega$ is much smaller than ω_0, and the cosine piece oscillates slowly). The graph of y as a function of time looks something like this:



The blue line indicates $y(t)$, while the red line indicates the amplitude of the oscillations $A\cos(\Delta\omega t/2)$. You can see that the sound wave alternates between high intensity (the large bumps in the graph, called ***beats***) and low intensity (the nodes in between them). The angular frequency of the modulating cosine function is $\Delta\omega/2$, so its frequency is

$$\frac{1}{2\pi}\frac{1}{2}|\omega_2 - \omega_1| = \frac{1}{2}|f_2 - f_1|$$

However, note from the graph that there are in fact two beats for every period of the cosine function. Thus, the ***beat frequency*** (defined as the number of beats per second) is twice the frequency of the cosine:

$$f_{beat} = |f_2 - f_1|$$

If the two waves have different amplitudes, the amplitude at the nodes won't be zero, but there will still be high and low-intensity periods alternating at the beat frequency.

Example: An observer is riding in a car that has just emerged from a tunnel and is moving directly away from the cliff at a speed of 6.5 *m/s*. The car is sounding its horn at a frequency of 350 *hz*. The observer hears rapid beats due to interference of the sound of the horn with the echo from the cliff. What is the beat frequency?

First determine the frequency of the echo. The car is a moving source, moving away from the cliff, so the frequency of sound as it arrives at the cliff f_C is

$$f_C = \frac{c}{c+v} f_S = \frac{331}{331+6.5} f_S = 343hz$$

The echo is reflected at the same frequency. Now the cliff is acting as a stationary source, and the car is moving away. The frequency of the echo as measured by the observer in the car is

$$f_O = \frac{c-v}{c} f_C = \frac{331-6.5}{331} \times 343hz = 337hz$$

Finally, the beat frequency is

$$f_{beat} = (350 - 337)hz = 13hz$$

The observer thus hears 13 beats per second - very fast, but distinguishable by the human ear as individual pulses of sound.

Note: The material past this point will be on Midterm 2, not Midterm 1

Section II: Electromagnetic Waves and Optics

Wave solutions to Maxwell's equations

The dynamics of electric and magnetic fields are governed by Maxwell's equations. You don't need to know these equations in detail, but they look something like this:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\mu_0} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

These are partial differential equations, the meaning of which will become clear if one takes a course in vector calculus. Here, *E* and *B* are the electric and magnetic fields, ρ is the charge density, and *J* is the current density.

It can be shown (in a different class) that in a vacuum ($\rho = J = 0$) the Maxwell equations are equivalent to the wave equation. What happens is that, even though there are no charges or currents, oscillations in the electric field induce oscillations of the magnetic field, and vice versa. Plane wave solutions to this wave equation satisfy the following:

$$\vec{E} = \vec{E}_0 sin\left(\omega t - \vec{k}\cdot\vec{r}\right)$$

$$\vec{E}_0 \cdot \vec{k} = 0 \qquad \omega = \frac{|k|}{\sqrt{\epsilon_0 \mu_0}}$$

$$\vec{B} = \sqrt{\epsilon_0 \mu_0}\left(\hat{k} \times \vec{E}_0\right)$$

Let us consider what this solution means. The first line tells us that the electric field oscillates like a simple sine wave. The electric field a vector, so its amplitude is a vector as well. The wave “number” is actually a vector, but that's not a problem: let us choose our coordinates so that the *z* axis lies along the direction of *k*. Then,

$$\vec{k} = k\hat{z}$$

$$\vec{E} = \vec{E}_0 sin(\omega t - kz)$$

This is simply an equation for a plane wave propagating in the positive *z* direction with wave number *k*. Therefore, the direction of the *k* vector is the direction of propagation of the wave, and is magnitude is equal to the wave number.

The second line states that the electric field is always perpendicular to the direction of propagation (so the dot product is zero) and gives the relationship between the angular frequency and the wave number. Since $\omega = kc$ for all waves, this means that the speed of propagation of electromagnetic waves is given by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 m/s$$

This speed is the same for all electromagnetic waves, regardless of frequency or amplitude. It also seems to be independent of an observer, since we haven't specified what the speed is relative to! For a string and for sound, that's obvious; the speed of waves on a string is given relative to the string itself, and the speed of sound waves is given relative to the air. But for EM waves, there is no string or air, so if we take Maxwell's equations seriously, the speed of light is the same relative to all observers.

This seems strange: you'd think if you were moving at speed *v* towards the source of electromagnetic waves, you should see them coming towards you at speed *v* + *c*. But this doesn't happen; electromagnetic waves always move at *c*. We will resolve this apparent paradox when we briefly talk about special relativity later in the course.

Finally, the last line states that the magnetic field *B* is perpendicular both to the electric field *E* and to the direction of propagation of the wave. Its amplitude is proportional to that of the electric field, and it oscillates in phase with the electric field at exactly the same frequency.

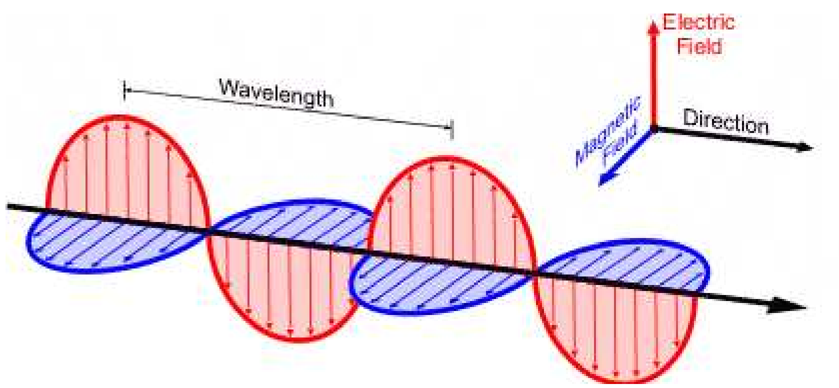
The direction of the electric field *E* is called the ***polarization vector***. Let us choose our coordinates so that the polarization vector lies in the positive *x* direction. Then, the electric field is given by

$$\vec{E} = \hat{x} E_0 sin(\omega t - kz)$$

The magnetic field is

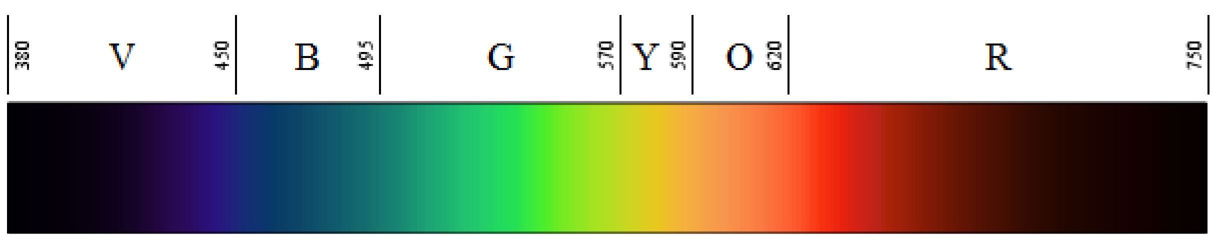
$$\vec{B} = \sqrt{\epsilon_0 \mu_0} \left(\hat{z} \times \vec{E} \right) = \hat{y} \frac{E_0}{c} sin(\omega t - kz)$$

At time as specific time *t*, a snapshot of the solution looks something like this:



The electromagnetic spectrum

Our eyes can perceive electromagnetic waves with wavelengths between about 390 and 750 *nm* as light. The color of the light depends on the wavelength: red light has long wavelengths, while violet light has short wavelengths:



EM waves beyond the red edge of the visible spectrum, from 790 *nm* to 300 μm (by convention) are known as ***infrared radiation***. Beyond infrared radiation is ***microwave radiation*** (wavelength of up to 1 meter), and beyond that is ***radio***.

Going to the other side of the spectrum, waves beyond the violet edge of the visible spectrum are known as ***ultraviolet radiation***, spanning wavelengths from 10 to 390 *nm*. Going to even shorter wavelengths we have ***X-ray radiation*** (0.10 to 10 *nm*). Electromagnetic waves with wavelengths shorter than 0.10*nm* are known as ***γ radiation (gamma radiation, or gamma rays)***.

Example: What is the visible range in terms of frequency?

Frequency is related to λ by $f = c / \lambda$. For the violet edge of the visible spectrum

$$f = \frac{3 \times 10^8 m/s}{390 \times 10^{-9} m} = 7.7 \times 10^{14} Hz = 770 \times 10^{12} Hz = 770 THz$$

For the red edge,

$$f = \frac{3 \times 10^8 m/s}{750 \times 10^{-9} m} = 4.0 \times 10^{14} Hz = 400 THz$$

The visible spectrum therefore includes frequencies between 400 and 770 terahertz.

Types of electromagnetic radiation

In sunlight, and in most other natural sources of EM radiation, there are waves of all kinds of polarizations, phases, wavelengths, etc., superimposed together (remember that we can add all these waves due to the superposition principle). There are various types of radiation where these parameters are restricted:

Monochromatic radiation contains only a narrow range of frequencies. The light of a laser is monochromatic; the light passed through a narrow color filter is also approximately monochromatic.

Coherent radiation consists of electromagnetic waves that are all in phase with each other. Note that this is only possible if the waves are also monochromatic (if they weren't, they would quickly develop different phases). Lasers produce coherent light; color filters do not.

Polarized radiation consists of EM waves that all have the same direction of the polarization vector. So we could have radiation that is polarized in the *x* direction, or in the *y* direction, or in any other direction perpendicular to the direction of travel. ***Circularly polarized radiation*** refers to a very specific superposition of *x* and *y* -polarized waves that are out of phase with each other. Polarized radiation is produced by passing light through a ***polarization filter***, which is essentially a material that only allows electric fields pointing in a particular direction to pass through.