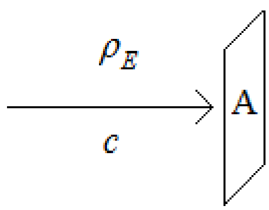
Lecture 5 Notes: 07 / 05

Energy and intensity of sound waves

Sound waves carry energy, just like waves on a string do. This energy comes in several types: potential energy due to the compression of the material, kinetic energy due to the movement of the material's particles, and also thermal energy, since when the material is compressed it heats up a bit. Just like for waves on a string, the energy carried by sound waves increases with amplitude and frequency; however, we won't derive the precise relationship here.

Let us say that a material with a sound wave traveling through it has an acoustic energy density ρ_E. On a string, this had units of energy per length, or J / m. However, a sound wave travels through a three-dimensional, rather than a one-dimensional, medium, so the units are now energy per volume, or J / m^3.

Consider an plane of area *A*, perpendicular to the direction of propagation of the wave:



Assume that *A* is sufficiently small that the amplitude and direction of travel of the wave is uniform throughout it. Then, the rate at which energy is transported across the surface *A* (called the ***energy flux across A***) is:

$$\Phi_E = \rho_E A c$$

The energy flux has units of (J / m^3) x (m^2) x (m / s) = J / s = W, as expected (since the flux is a rate at which energy is transported).

The ***intensity*** is defined as energy flux per unit area:

$$I = \frac{\Phi_E}{A} = \rho_E c$$

The intensity has units of W / m^2, and measures how loud (intense) the sound is.

Sound comes in a huge range of intensities. For example, the human ear can detect sounds with an intensity of about 10^{-12} W/m^2 (this is called the ***threshold of hearing***). There are, of course, inaudible sounds that have much lower intensities than this. The ear can handle, at least for a short while, sounds with an intensity up to about 1 W/m^2. Sounds with intensities much above 1 W/m^2 are very uncomfortable to be around without earplugs, and can quickly cause hearing damage.

Standing close to a jet engine or a firing artillery piece can expose one to sound intensities in excess of thousands of W / m^2. Such sound waves can instantly rupture the eardrum unless appropriate ear protection is used (opening the mouth also helps, as the sound wave can then reach both the inside and the outside of the eardrum at the same time, possibly preventing it from being ruptured.)

Since we want to be able to describe sounds which differ in intensity by more than 16 orders of magnitude, it is useful to introduce a logarithmic scale. The ***intensity level*** is given in decibels, and defined as follows:

$$\beta = (10db) log_{10} \left(\frac{I}{10^{-12} W/m^2} \right)$$

Thus, a sound on the threshold of hearing has an intensity level of about 0 decibels. Sounds too quiet to be audible have negative intensity levels. The loudest sounds that one can comfortably handle have intensity levels of around 120 decibels, while extremely loud sound sources, such as jet engines, rockets or explosions, are capable of producing intensity levels of 150 - 170 decibels or even more.

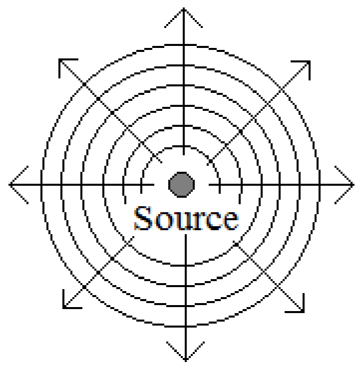
Note that when the intensity goes up by a factor of 10, the intensity level increases by 10 decibels. Thus, a 60-decibel sound is 10 times more intense than a 50-decibel sound, and 100 times more intense than a 40-decibel sound.

Example: 25 identical generators are located in one large space. With only one generator running, the intensity level of the noise within this space is 90 decibels. What is the intensity level when all generators are running?

The intensity becomes 25 times as large, so the intensity level increases by $10 \log_{10}(25) = 14$ decibels. Thus, the intensity level with all 25 generators running is 104 decibels.

Spherical waves

To an observer located sufficiently far away from a localized source of sound, such as a loudspeaker, the source looks approximately like a point source. If there is nothing that gets in the way or absorbs the sound, a point source emits sound in ***spherical waves***:



This diagram is a cross-section centered on the source. The circles show the locations of the wave peaks at a particular snapshot at time *t*; in fact, they are not circles but spherical surfaces, but they appear as circles in cross-section. The arrows show the direction of propagation of the wave at various points on the sphere.

The total acoustic energy flux emitted by the speaker is the power of the speaker (this is, of course, less than the amount of power the speaker draws from the power source, since the speaker is not 100% efficient at converting the energy input into sound). By spherical symmetry, the acoustic energy must flow uniformly through any spherical surface centered on the speaker.

Also, by conservation of energy, if there is no absorption of sound, the flux through any closed surface that contains the speaker must be equal to the power of the speaker (since all the energy emitted by the speaker must leave across this surface). Choosing a spherical surface of radius *r*, so that the intensity is uniform everywhere on the surface, we obtain the intensity at radius *r*:

$$I(r) = \frac{\Phi_E}{A(r)} = \frac{\Phi_E}{4\pi r^2}$$

Thus, the intensity is inversely proportional to the distance squared. If the intensity at a distance r_0 is known to be I_0, then the intensity at a different distance r_1 is given by

$$I_1 = \left(\frac{r_0}{r_1}\right)^2 I_0$$

Example:
A sound source has a power of 10 milliwatts. What is the intensity level of the sound 10 meters away? 1 kilometer away?

10 meters away, the intensity level is

$$I(10m) = \frac{0.010W}{4\pi \times (10m)^2} = 7.96 \times 10^{-6} W/m^2$$

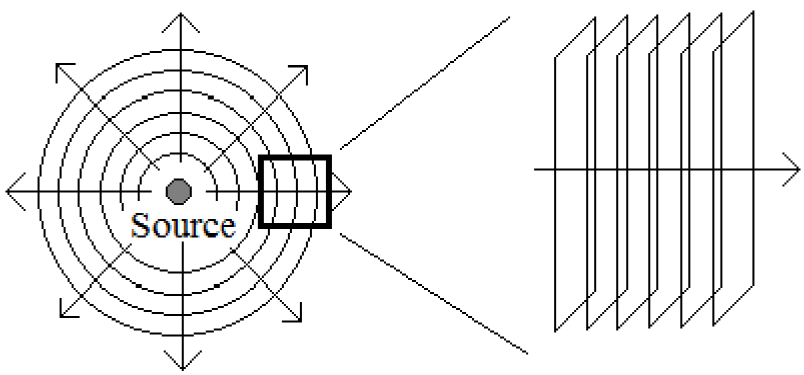
The intensity is

$$\beta = (10db) log_{10} \left(\frac{7.96 \times 10^{-6} W/m^2}{10^{-12} W/m^2} \right) = 69\ db$$

One kilometer away is 100 times as far. Therefore, the intensity drops by a factor of $100^2 = 10^4$. This means that intensity level decreases by 40 decibels (remember that it decreases by 10 each time the intensity decreases by a factor of 10). The intensity at a distance of 1 km is therefore 29 db.

Plane waves

Far away from the source and in a small enough region, the wave fronts in a spherical wave are approximately planar and parallel to each other:



The planes on the right and the circles on the left once again represent the locations of maximum pressure, and the arrows show the direction of propagation of the wave.

If we choose *x* to be the direction of propagation of the wave in this region, the displacement of the pressure is independent of *y* and *z*, but depends on *x* and *t*:

$$\Delta P = y(x, t) = A sin(kx - \omega t)$$

This is exactly the form of the wave we found for sound traveling in one dimension (for example, down a pipe).

Standing waves in a pipe

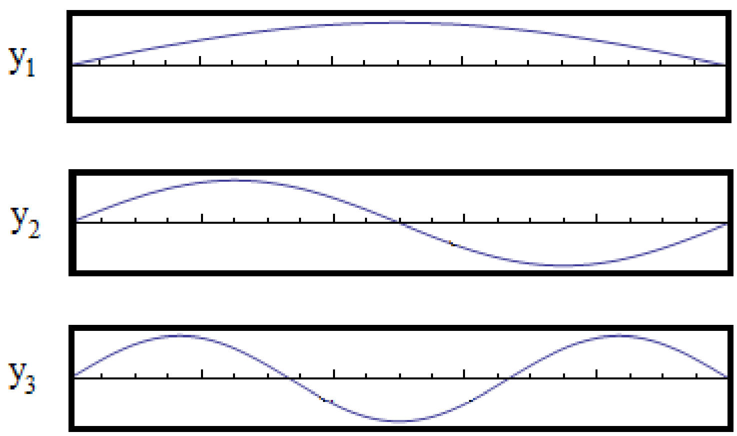
Suppose we have a pipe of finite length, and want to know what kind of sound waves can exist inside it.

In general, sound waves inside a pipe can be broken up into longitudinal modes (waves that depend on the coordinate *x* along the length of the pipe), radial modes (those that depend on the distance *r* from the pipe's axis) and azimuthal modes (those that depend on the angle ϕ along the pipe's circumference). For the purposes of this class, we will only deal with longitudinal modes. For a sufficiently narrow pipe, these will give the dominant contribution to the sound inside the pipe.

Consider first a pipe of length *L* that is closed on both ends. This means that the air can't move back and forth right at the ends, since there is a solid surface there. This situation is completely analogous to a string that is tied down at both ends. By analogy with the string, the displacement of air molecules along the pipe as a function of the position *x* is given by a standing wave that is zero on both ends:

$$y_n(x,t) = Asin\left(\frac{n\pi x}{L}\right)cos\left(\frac{n\omega t}{L}\right) = Asin\left(\frac{n\pi x}{L}\right)cos\left(\frac{n\pi ct}{L}\right)$$

The first three modes at maximum displacement ($t = 0$) look like this:



The frequency associated with the *n*th mode is

$$f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L}$$

The fundamental frequency occurs for $n = 1$, since the $n = 0$ mode doesn't exist (for $n = 0$, displacement is zero everywhere):

$$f_F = f_1 = \frac{c}{2L}$$

Now consider a pipe that is open on one end and closed on the other. It can be shown that the displacement of the air molecules is maximum at the open end (we won't go into the proof here, due to lack of time). Let $x = 0$ be the closed end: using the standing wave $y(x,t) = A\sin(kx)\cos(\omega t)$ satisfies the boundary condition $y(x=0) = 0$, since $\sin(0) = 0$. The constraint on k comes from the condition that the displacement is maximized at $x = L$. This is true if

$$\frac{\partial y}{\partial x}(L) = 0 \qquad Ak\cos(kL)\cos(\omega t) = 0 \qquad \cos(kL) = 0$$

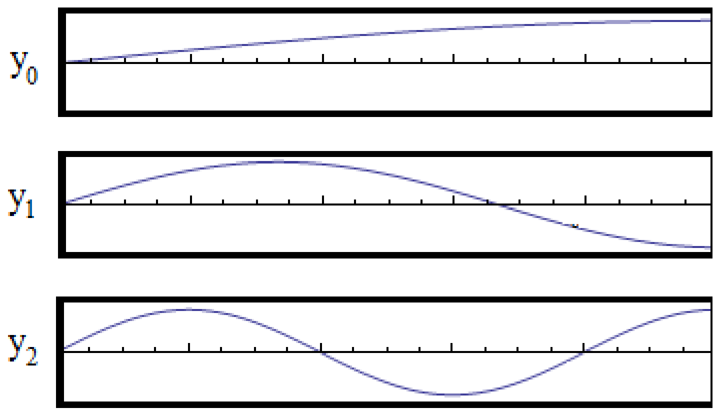
$$kL = \left(n + \frac{1}{2}\right)\pi, \quad n = (0, 1, 2...)$$

$$k = \left(n + \frac{1}{2}\right)\frac{\pi}{L}$$

The standing wave solutions for a pipe open at one end are thus given by

$$y(x,t) = A\sin\left[\left(n + \frac{1}{2}\right)\frac{\pi x}{L}\right]\cos\left[\left(n + \frac{1}{2}\right)\frac{\pi c t}{L}\right]$$

The first three modes at $t = 0$ look like this:



The frequency associated with the nth mode is

$$f_n = \frac{\omega_n}{2\pi} = \left(n + \frac{1}{2}\right)\frac{c}{2L}$$

In this case, the $n = 0$ mode exists, and gives the following fundamental frequency:

$$f_F = f_0 = \frac{c}{4L}$$

Finally, consider a pipe that is open both at $x = 0$ and $x = L$. In this case, the displacement is maximized on both sides. We should use a different standing wave to describe this situation: $y(x,t) = A\cos(kx)\cos(\omega t)$ is a standing-wave solution of the wave equation that always has a maximum at $x = 0$, and thus satisfies the boundary condition on the left. All that remains is the boundary condition on the right, which is

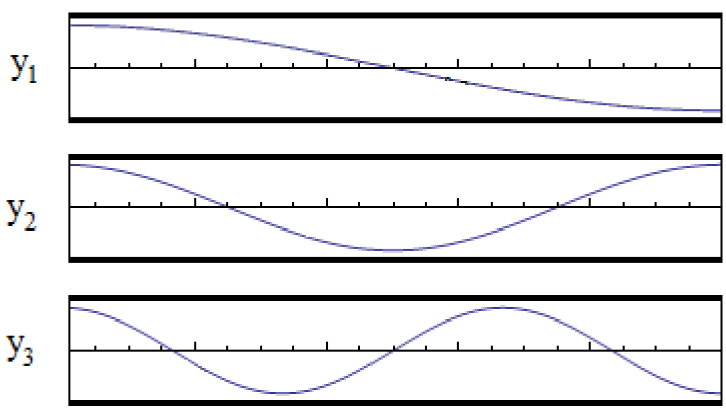
$$\frac{\partial y}{\partial x}(L) = 0 \qquad -Aksin(kL)sin(\omega t) = 0 \qquad sin(kL) = 0$$

$$kL = n\pi \qquad k = \frac{n\pi}{L}$$

The constraint on k is thus the same as for a pipe closed on both ends, or a string that is tied down on both ends. The $n = 0$ mode does not exist, since that would correspond to a constant displacement throughout the pipe, which just describes a movement of air in one side and out the other, not an oscillating wave; therefore, $n = 1, 2, 3...$ The standing-wave solutions for a pipe open on both ends are

$$y_n(x,t) = Acos\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi ct}{L}\right)$$

The first three modes at $t = 0$ look like this:



The frequency of the nth mode and the fundamental frequency are the same as for a pipe closed on both ends:

$$f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L}$$

$$f_F = f_1 = \frac{c}{2L}$$

Example: A bottle has a length of 20 centimeters. What is the lowest frequency produced when air blows across the top of the bottle? Estimate the fundamental frequency if this bottle is placed in an 1 atm of xenon (molecular weight 131 amu).

The bottle can be approximated as a pipe open on one end and closed on another. The fundamental frequency in air under standard temperature and pressure is therefore

$$f_F = \frac{c}{4L} = \frac{331m/s}{4 \times 0.20m} = 414\ hz$$

In xenon, the speed of sound is different. This is primarily because the density of xenon is higher; the bulk modulus of xenon at a given pressure is in fact also slightly different from that of air, since xenon is a monoatomic gas while air is primarily diatomic, but this effect is relatively small. We know from basic thermodynamics of gases that at a given temperature and pressure, the density is proportional to the molecular weight of the gas. Air is primarily diatomic nitrogen, so we will take its molecular weight to be approximately 2 x 14 = 28. Therefore,

$$\rho_{Xe} \approx \frac{m_{Xe}}{m_{N_2}}\rho_{Air} = \frac{131}{28}\rho_{Air} = 4.7\rho_{Air}$$

Recall that the speed of sound in a gas is inversely proportional to the square root of the density, provided that the bulk modulus is about the same (we will assume that it is). Therefore, the speed of sound in xenon is lower:

$$c_{Xe} \approx \frac{1}{\sqrt{4.7}}c_{Air} \approx 0.46c_{Air}$$

Since the fundamental frequency is proportional to the speed of sound, it will also be lower by the same factor:

$$f_{F,Xe} \approx 0.46 f_{F,Air} \approx 190hz$$

The frequency is therefore much lower in this heavier gas. If a gas lighter than air, such as helium, was used, the frequency would instead be higher. This is why someone's voice sounds high-pitched if one first takes a breath of helium.

Example:
A whistle can be approximately modeled as a pipe open on both ends. It has a length of 6 centimeters. One of Prof. Brian Keating's graduate students has this whistle at the South Pole during the Antarctic winter. The temperature is -70ºC, and the air pressure is 1.0 atm. What is the whistle's fundamental frequency?

The density of air, and therefore the speed of sound, is different at such a low temperature. Recall from thermodynamics that the density of a gas is inversely proportional to the absolute temperature. Therefore, the density at -70ºC is

$$\rho_{-70^\circ C} = \left(\frac{273K}{273K - 70K}\right)\rho_{0^\circ C} = 1.34\rho_{0^\circ C}$$

The speed of sound is

$$c_{-70^\circ C} = \frac{c_{0^\circ C}}{\sqrt{1.34}} = \frac{331m/s}{\sqrt{1.34}} = 285m/s$$

Finally, assuming that the whistle behaves as an pipe open on both ends, the fundamental frequency of the whistle is

$$f_F = \frac{c_{-70^\circ C}}{2L} = \frac{285m/s}{2 \times 0.06m} = 2380\ hz$$