Lecture 4 Notes: 06 / 30

Energy carried by a wave

We want to find the total energy (kinetic and potential) in a sine wave on a string. A small segment of a string at a fixed point x_0 behaves as a harmonic oscillator with amplitude A and angular frequency ω:

$$y(x_0, t) = Asin(kx_0 - \omega t) \equiv Asin(-\omega t + \phi)$$

The mass of this segment is equal to $m = \mu\, dx$, where μ is the mass per unit length of the string, and dx is the (small) length of the segment. What is the energy of this oscillator?

Recall that the total energy of a harmonic oscillator is

$$E = \frac{1}{2}KA^2$$

We are using capital K for the spring constant to avoid confusion with the wave number k. The spring constant is related to the frequency as follows:

$$\omega = \sqrt{\frac{K}{m}} \qquad K = m\omega^2$$

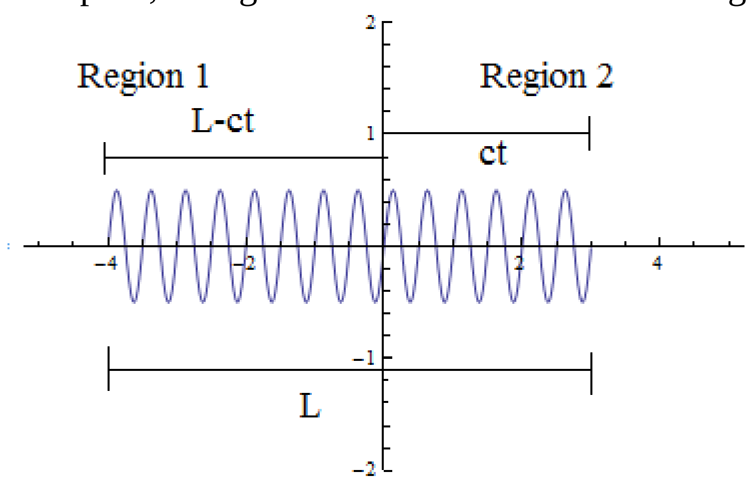
Thus, the energy of the segment of string is equal to

$$E = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}\mu\omega^2 A^2 dx$$

The length of the segment is dx, so the energy per unit length (energy density) is

$$\rho_E = \frac{E}{L} = \frac{1}{2}\mu\omega^2 A^2$$

At what rate is the energy carried by the traveling wave? Consider a traveling wave of finite length L. At time $t = 0$, this wave begins to cross from region 1 into region 2. After a time t has elapsed, a length of ct will have moved into region 2:



So, the energy in Region 2 is equal to $E_2 = \rho_E\, ct$ and the energy remaining in Region 1 is equal to $E_1 = \rho_E\, (L - ct)$. Region 2 therefore gains energy at the rate $I = \rho_E\, c$ joules per second, while Region 1 loses energy at the same rate. This rate of energy transport is known as the ***energy flux*** of the wave.

Plugging in our result for the energy density ρ_E, the energy flux is

$$I = \frac{1}{2}\mu\omega^2 A^2 c$$

We can also express this quantity in terms of the tension in the string, rather than the mass per unit length. Recall that

$$c = \sqrt{\frac{F_T}{\mu}} \quad \text{so that} \quad \mu = \frac{F_T}{c^2}$$

Plugging this into our equation for the energy flux, we obtain

$$I = \frac{1}{2}\frac{\omega^2 F_T}{c}A^2$$

Example: How much energy is transmitted per second by a wave given by the following function, assuming that the tension in the string is $F_T = 100N$?

$$y(x,t) = (0.018m)sin[(6.0m^{-1})x - (180s^{-1})t]$$

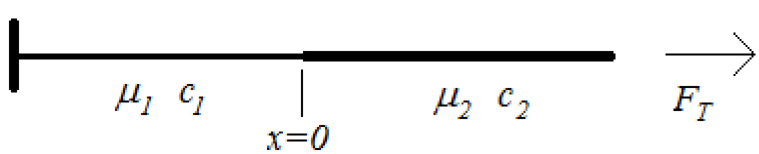
We can read off the amplitude and the angular frequency from the function, and are given the tension. We need to calculate the wave speed. It is $c = \omega / k$ = 30 m/s. Given all this, we can calculate the energy flux:

$$I = \frac{1}{2}\frac{\omega^2 F_T}{c}A^2 = \frac{1}{2}\frac{(180s^{-1})^2 \times 100N}{30m/s} \times (0.018m)^2 = 17.5W$$

The wave carries 17.5 watts of power.

Reflection and transmission from an interface

Now suppose that we join together two strings, each with a different mass per unit length. Tension is applied to the combined string, so that the tension force is the same everywhere, but because the mass per unit length is different, the wave speed will be different between the two segments. Let the interface be located at $x = 0$:



Let $y_1(x,t)$ be the displacement of the string to the left of the interface, and $y_2(x,t)$ be the displacement of the string to the right of the interface. We can conclude a couple of things about how these functions must match up at the interface:

1. y_1 and y_2 must be continuous at the interface, since the two strings are connected together. This means that $y_1(0, t) = y_2(0, t)$.

2. The derivatives of y_1 and y_2 with respect to x must also be continuous at the interface. The wave equation contains second derivatives with respect to x. If the derivative is discontinuous, then the second derivative is undefined, and the wave equation cannot be valid. Thus, the wave equation demands that $dy_1/dx(0, t) = dy_2/dx(0, t)$.

3. If the functions y_1 and y_2 are sine waves, they must oscillate with a common frequency. If they did not, then at the point $x = 0$, one side would move faster than the other, and even if we started with the two sides being continuous there, they would immediately become discontinuous. The wave numbers will then be different, since $k = \omega / c$, and c is different on the two sides. The wave number on the left is k_1 and the one on the right is k_2.

Suppose that a sine wave is incident from the left with amplitude A. Then, in region 1, there will generally be two sine waves: an incident wave going to the right, and wave reflected from the interface going back to the left. Thus,

$$y_1(x,t) = \underbrace{Asin(k_1 x - \omega t)}_{\text{Incident}} + \underbrace{Bsin(-k_1 x - \omega t)}_{\text{Reflected}}$$

Note that we chose a different sign convention, making all waves depend on $-\omega t$. This is completely equivalent to our previous convention, but will be slightly more convenient for this problem.

In region 2, there will only be a right-moving transmitted wave, since there is nothing farther to the right for the wave to reflect from, and no sources of waves from the right. Therefore, in region 2,

$$y_2(x,t) = \underbrace{Csin(k_2 x - \omega t)}_{\text{Transmitted}}$$

The continuity condition, $y_1(0, t) = y_2(0, t)$, means that

$$Asin(-\omega t) + Bsin(-\omega t) = Csin(-\omega t)$$
$$A + B = C$$

The continuity of the derivative with respect to *x*, $dy_1/dx(0, t) = dy_2/dx(0, t)$, gives us

$$k_1 Acos(-\omega t) - k_1 Bcos(-\omega t) = k_2 Ccos(-\omega t)$$
$$k_1(A - B) = k_2 C$$

Plugging in *C* from the first equation into the second gives the reflected amplitude *B*:

$$k_1(A - B) = k_2(A + B) \qquad (k_1 - k_2)A = (k_1 + k_2)B$$

$$B = \frac{k_1 - k_2}{k_1 + k_2} A$$

Using the first equation to determine the transmitted amplitude *C* gives

$$C = A + B = A + \frac{k_1 - k_2}{k_1 + k_2} A = \frac{k_1 + k_2 + k_1 - k_2}{k_1 + k_2} A$$
$$C = \frac{2k_1}{k_1 + k_2} A$$

We can put these expressions for the reflected and the transmitted amplitudes in terms of the mass per unit length of the strings, μ_1 and μ_2, using the relationship

$$k = \frac{\omega}{c} = \frac{\omega}{\sqrt{F_T/\mu}} = \frac{\omega\sqrt{\mu}}{\sqrt{F_T}}$$

The result is

$$B = \frac{\omega\sqrt{\mu_1}/\sqrt{F_T} - \omega\sqrt{\mu_2}/\sqrt{F_T}}{\omega\sqrt{\mu_1}/\sqrt{F_T} + \omega\sqrt{\mu_2}/\sqrt{F_T}} A = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A$$
$$A = \frac{2\omega\sqrt{\mu_1}/\sqrt{F_T}}{\omega\sqrt{\mu_1}/\sqrt{F_T} + \omega\sqrt{\mu_2}/\sqrt{F_T}} A = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A$$

Consider a couple of limiting cases. First, consider the case that the two pieces of the string that are tied together are in fact the same, so that $\mu_1 = \mu_2 = \mu$ Then,

$$B = \frac{\sqrt{\mu} - \sqrt{\mu}}{\sqrt{\mu} + \sqrt{\mu}} A = 0$$
$$C = \frac{2\sqrt{\mu}}{\sqrt{\mu} + \sqrt{\mu}} A = \frac{2\sqrt{\mu}}{2\sqrt{\mu}} A = A$$

There is no reflection, and the entire wave is transmitted into region 2 with the same amplitude. This is exactly what we would expect, since the two regions in this case are indistinguishable, and there is really no interface for the waves to reflect from.

Now consider the case where the right-hand piece of string is infinitely heavy, $\mu_2 >> \mu_1$. This corresponds to the case of the string end being tied down to an immovable object, for example a wall. In that case,

$$B = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A = \frac{-\sqrt{\mu_2}}{\sqrt{\mu_2}} A = -A$$
$$C = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2}} A = 0$$

In this case, the entire wave is reflected and nothing is transmitted, as expected. The amplitude of the reflected wave is minus the amplitude of the incident wave. This means that the wave is inverted.

Finally, consider a case where the right-hand piece of string has zero mass. In effect, it doesn't really exist; this corresponds to the case where the right end of a string is left free to slide up and down rather than tied down. In this case, we have

$$B = \frac{\sqrt{\mu_1}}{\sqrt{\mu_1}} A = A$$

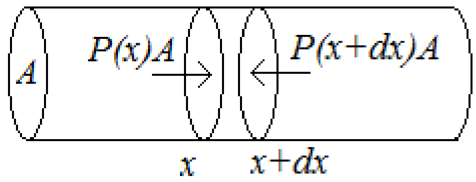
In this case, the wave is reflected with the same amplitude, upright. We also get $C = 2A$: this means that the end of the string moves up and down through twice the amplitude of the incident wave, but as there is no string beyond the end, in this case C is not the amplitude of any transmitted wave.

Sound Waves

Sound waves are longitudinal waves in a material. A part of the material is compressed, then recoils and expands from the increased pressure, pushing on the nearby pieces of the material, thus compressing them and propagating the wave.

First, we will establish the existence of such waves and determine their speed. Consider air, or some other medium, in a cylindrical pipe of cross-sectional area A. Suppose that the properties of this material depend only on the distance along the pipe x, not on the location relative to the central axis of the pipe.

Consider a short slab of material within the pipe, having a length *dx*. If a longitudinal wave travels through the pipe, we will expect this piece to move back and forth a little bit. First, let us find the net force on the piece:



The force is pressure times area, so there is a force *P(x)A* acting from the left, across the boundary at *x*, and a force *P(x+dx)A* acting from the right, across the boundary at *x+dx*. Newton's Second Law tells us that

$$F_{net} = -P(x+dx)A + P(x)A = ma$$

The mass of the piece is the material density times the piece's volume:

$$m = \rho V = \rho A dx$$

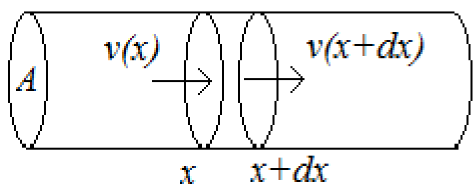
Plugging this into the equation above, we get

$$\rho A dx \; a = -A\left[P(x+dx) - P(x)\right]$$

$$a = -\frac{1}{\rho}\frac{P(x+dx) - P(x)}{dx} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

(This derivative is a partial derivative, because *P* actually depends on time as well as on position *x*, but we are holding time fixed when the derivative is calculated.)

The motion of the system is driven by variations in pressure. Pressure changes from point to point because the volume elements are alternatively compressed and stretched out. Let us now look how this volume element's volume changes with time:



If the speed of the element on the left of our piece is not the same as of the element on the right, the length and therefore volume of our piece will change. We will, however assume that the length stays close to *dx* (this is the case if the amplitude of the oscillations is small). We will let the actual length of the segment be $L = (1+\phi)dx$, where ϕ is much less than 1. The rate of change of *L* is clearly:

$$\frac{\partial L}{\partial t} = dx\frac{\partial \phi}{\partial t} = v(x+dx) - v(x)$$

We already have an equation for the acceleration of each piece, so to connect the equations, differentiate this with respect to *t*:

$$dx\frac{\partial^2 \phi}{\partial t^2} = a(x+dx) - a(x)$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{a(x+dx) - a(x)}{dx} = \frac{\partial a}{\partial x}$$

Finally, we plug the expression we derived for the acceleration earlier:

$$a = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2}$$

This is beginning to look like the wave equation. All that remains is to relate ϕ (which is a fractional change in length, and therefore in volume, of our element) to the pressure *P*. If you recall the chapter on the elastic properties of materials, the fractional change in volume is related to the change in pressure through a quantity known as the ***bulk modulus,*** defined as minus the change in pressure over the fractional change in volume:

$$B = -\frac{\Delta P}{\Delta V/V}$$

In terms of our variables, the denominator is

$$\frac{\Delta V}{V} = \frac{A(1+\phi)dx - Adx}{Adx} = \phi$$

As expected, ϕ is equal to the fractional change in volume. Now let $P = P_0 + \Delta P$, where P_0 is the ambient pressure (a constant) and ΔP is the change in pressure due to the compression (which varies in space and time). Then, the bulk modulus is

$$B = -\frac{\Delta P}{\phi} \qquad \phi = -\frac{\Delta P}{B}$$

Now we substitute this into our equation

$$\frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\rho}\frac{\partial^2 P}{\partial x^2}$$

Also, use the following (since P_0 is constant):

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2}{\partial x^2}(P_0 + \Delta P) = \frac{\partial^2(\Delta P)}{\partial x^2}$$

The result is the wave equation for sound waves traveling in one dimension:

$$-\frac{1}{B}\frac{\partial^2(\Delta P)}{\partial t^2} = -\frac{1}{\rho}\frac{\partial^2(\Delta P)}{\partial x^2}$$

$$\frac{\partial^2(\Delta P)}{\partial t^2} - \frac{B}{\rho}\frac{\partial^2(\Delta P)}{\partial x^2} = 0$$

To clean up the notation, let $\Delta P = y$ and identify B/ρ as c^2, the speed of wave propagation squared (since this equation has exactly the same form as that for waves on a string, and we have already established that the factor multiplying the second term is equal to c^2):

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0 \qquad c = \sqrt{\frac{B}{\rho}}$$

We can now calculate the speed of sound in a material, if we know its density and bulk modulus.

Examples:

1. The bulk modulus of water is 2.1×10^9 Pa. Its density is 10^3 kg/m^3. What is the speed of sound in water?

$$c = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.1 \times 10^9 kg\ m^{-1}s^{-2}}{10^3 kg\ m^{-3}}} = 1450 m/s$$

2. The speed of sound in air at 0°C and 1 atm is 331 m/s, and the density of air under these conditions is 1.29kg/m^3. What is the bulk modulus of air?

$$c = \sqrt{\frac{B}{\rho}} \qquad B = \rho c^2 = (1.29 kg/m^3)(331 m/s)^2 = 8.49 \times 10^4 Pa$$

Note that this is fairly close to the atmospheric pressure. In general, for gases, *B* is approximately equal to the pressure, but slightly different by a numerical factor that varies depending on if it is a monoatomic, a diatomic, or some other type of gas.

3. A sound wave in a material with bulk modulus of 2.0×10^{10} Pa and density of 5.5×10^3 kg/m^3 has a frequency of 300 hz. What is its wavelength?

$$\lambda = \frac{c}{f} = \frac{\sqrt{B/\rho}}{f} = \frac{1910 m/s}{300 s^{-1}} = 6.36 m$$