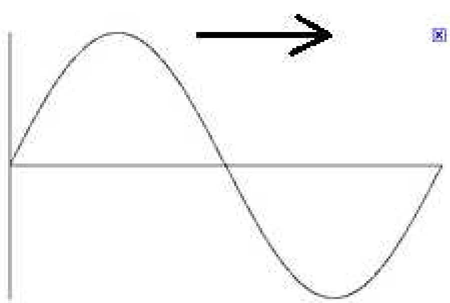
Lecture 3 Notes: 06 / 29

Intro to waves

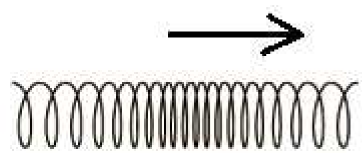
A wave is some kind of periodic disturbance that can move through a medium (waves on a string, sound waves in air, water waves in the ocean) or just through space (electromagnetic waves, matter waves in quantum mechanics).

Waves in a medium propagate by particles pulling and pushing on each other. These waves can carry energy and momentum over long distances, as each particle transmits energy to the next, but individual particles move only slightly from the original position.

Waves in a medium can be transverse (a string oscillating up and down, or an ocean wave on the surface of the water) or longitudinal (compression of a spring, or a sound wave in air):



Transverse wave



Longitudinal wave

Waves can have all kinds of different shapes, but a particularly simple form is

$$y(x,t) = Asin(kx \pm \omega t) \qquad \text{or} \qquad y(x,t) = Acos(kx \pm \omega t)$$

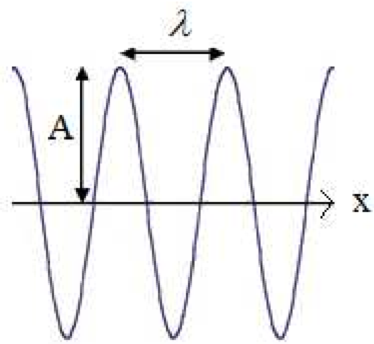
Note that the quantity k here has nothing to do with the spring constant (there aren't enough letters in all the alphabets of the world for physicists.) We will call these simple waves ***sine waves*** (whether they are a sine or a cosine). Note that a cosine is just a sine shifted over by an angle of $\pi/2$, so these two solutions are really the same thing. The plus or minus sign corresponds to left- or right-traveling waves, as we will see below.

Here, $y(x,t)$ is the value of the disturbance at position x and time t. In the case of a wave on a string oscillating up and down, it can be the height of a particular point on the string. In the case of a sound wave, it can be the change in the pressure of air at a particular place.

The quantity $\theta = kx \pm \omega t$ is called the ***phase*** of the wave. The quantity A is the maximum displacement attained by the wave, and is known as the ***amplitude***.

Motion of the wave

First, let's see what the wave looks like at a fixed time t. This is a snapshot of a wave at some particular time:



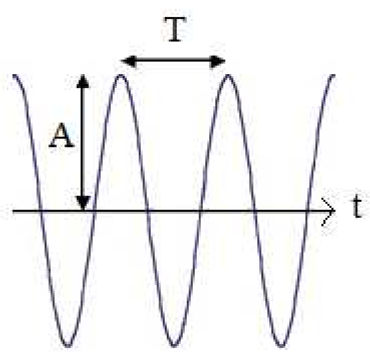
The horizontal axis is the distance (along a string, for example) and the vertical axis is the displacement of the wave at each point. The height of the wave crests is the amplitude A. The peak-to-peak distance is called the ***wavelength***, and is denoted by the Greek letter λ.

When one moves a distance λ, the wave goes through a full cycle. This means that the phase θ changes by 2π. Since we are holding the time fixed, this means that

$$k\lambda = 2\pi \qquad \lambda = \frac{2\pi}{k}$$

The quantity k (again, not to be confused with the spring constant) is known as the ***wave number***. It is equal to the number of radians by which the phase changes per meter, so its units are m^{-1}.

Now, let's look at a particular point on the string, at position x, and follow its motion as a function of time. The graph of the displacement y at this point as a function of time looks something like this:



The picture looks much the same as the snapshot of the wave at a fixed time, but now the horizontal axis is time, not position. The point moves with simple harmonic motion. The amount of time that passes between one instance of the displacement reaching its maximum value and the next is known as the ***period***, denoted by T. The period is the amount of time it takes the phase θ to change by 2π, and since we are holding the position x fixed, this means that

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega}$$

Just like with the harmonic oscillator, the quantity ω is known as the ***angular frequency***. It is equal to the number of radians by which the phase changes per second. The angular frequency is related to the ***frequency***, which is the number of complete cycles per second, as follows:

$$f = \frac{\omega}{2\pi} = \frac{1}{T}$$

Now let's see how the wave moves. Suppose we have a wave crest for some particular values of x and t: $x = x_0$, $t = t_0$. Now, let us advance the time by a small amount Δt. Clearly, if we choose a new position where the phase remains the same, we will still be at the crest of the wave. Suppose the crest moves by some amount Δx. Then, we have

$$\theta = kx \pm \omega t = \text{constant}$$

$$kx_0 \pm \omega t_0 = k(x_0 + \Delta x) \pm \omega(t_0 + \Delta t)$$

$$k\Delta x \pm \omega \Delta t = 0$$

$$\frac{\Delta x}{\Delta t} = \mp\frac{\omega}{k} \equiv \mp c$$

Therefore, the crest moves with a speed $c = w / k$, either in the positive or the negative direction. This is known as the ***wave speed*** or the ***phase speed***. Note that if we have $y = A sin(kx + \omega t)$, then the wave is moving in the negative direction (to the left). If we have $y = A sin(kx - \omega t)$, then the wave is moving in the positive direction (to the right).

Spectral decomposition and group speed

It turns out that different waves can be added together, provided that their amplitudes are sufficiently small. The resultant combination is a physically possible displacement of the string (or another medium). This is known as the ***superposition principle***. We can build up arbitrarily complicated functions from simple waves. Symbolically,

$$y(x,t) = \sum_k A_k sin(kx - \omega_k t) + B_k cos(kx - \omega_k t)$$

$$\text{where } \omega_k = ck$$

Breaking down a function into simple sine and cosine waves is called ***spectral decomposition,*** and requires solving for the set of constants A_k and B_k. The technique for finding these constants is called ***Fourier analysis***. This is not difficult to learn, but is somewhat beyond the scope of this class; it is usually taught in second or third-year physics, engineering and math classes.

A ***wave packet*** is a bunch of sine waves added together to form a localized disturbance moving as one unit. The wave packet doesn't necessarily move at the phase speed. Instead, it moves at the ***group speed***, which is given by

$$v_g = \frac{d\omega}{dk}$$

I won't give the proof here. Note that if the speed c does not depend on k, then we have $\omega = ck$ and $v_g = c$. In other words, if all waves move at speed c regardless of wave number, the phase speed is the same as the group speed. This makes sense, since in that case all the waves making up the wave packet move together, carrying the wave packet along at the same speed. However, if c depends on k, then the group speed will be different from the phase speed.

<u>Example:</u>

Suppose that the displacement of a string is given by the equation

$$y(x,t) = (0.025m)cos[(50m^{-1})x - (100s^{-1})t]$$

What is the amplitude, wave number, angular frequency, wavelength, frequency, period and wave speed of this wave? Is the wave moving to the left or to the right?

The amplitude is maximum displacement of the string: A = 0.025m. The wave number is the factor multiplying x in the argument: k = 50m^{-1}. Similarly, the angular frequency is ω = 100s^{-1}. We can now calculate the wavelength, frequency, period and wave speed:

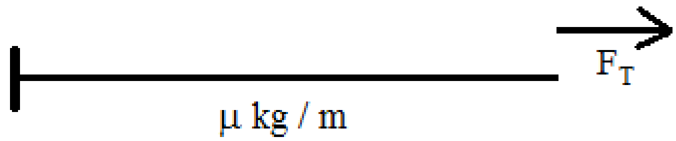
$$\lambda = \frac{2\pi}{k} = 0.126m \qquad f = \frac{\omega}{2\pi} = 15.9s^{-1}$$

$$T = \frac{1}{f} = 0.0628s \qquad c = \frac{\omega}{k} = 2.0m/s$$

If the time t increases, the position x must increase as well to keep the phase constant. Therefore, the wave is moving to the right.

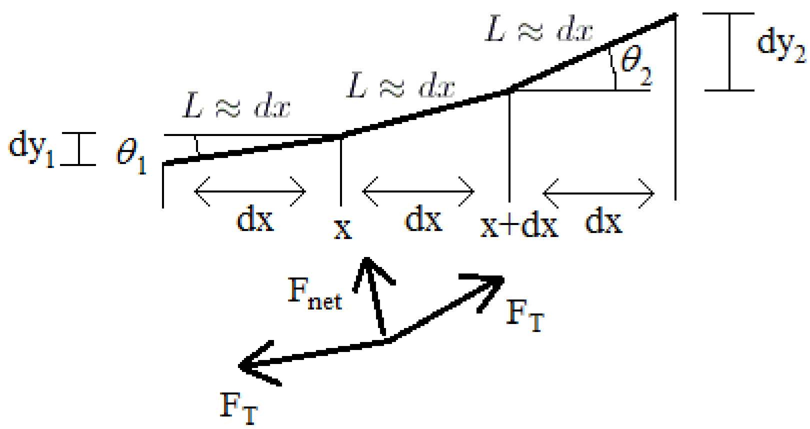
Mechanics of a string

Consider a string held under tension F_T. The string has mass per unit length μ. If the force due to gravity is negligible, then at equilibrium, the string is stretched out in a straight line:



Now let the string be deformed slightly from equilibrium. We will require that the string is fairly horizontal at every point; that is, the maximum angle the string makes with the horizontal is always small. The deformed string will look something like this:

To see how this deformation changes with time, let us zoom in on the small segment of string enclosed by the box. Let us also approximate the string in the box by a set of arbitrarily short line segments. In this approximation, the small piece of string looks something like this:



The force diagram indicates the forces acting on the middle segment. Since the angles are small (remember that in the diagram, they are exaggerated for clarity), the length of each segment L is approximately equal to dx, and the net force is almost vertical. If the angles get too big, then the net force will not be vertical, and the string will oscillate back and forth, not just up and down. We are assuming this doesn't happen. The net force is thus given by the sum of the vertical components of the tension forces:

$$F_{net} \approx F_T sin(\theta_2) - F_T sin(\theta_1) = F_T \frac{dy_2}{L} - F_T \frac{dy_1}{L}$$
$$\approx F_T \left(\frac{dy_2}{dx} - \frac{dy_1}{dx} \right) = F_T \left(\frac{dy}{dx}(x+dx) - \frac{dy}{dx}(x) \right)$$

Now we use Newton's second law:

$$ma = (\mu\ dx)a = F_T\left(\frac{dy}{dx}(x+dx) - \frac{dy}{dx}(x)\right)$$

$$a = \frac{F_T}{\mu}\frac{\frac{dy}{dx}(x+dx) - \frac{dy}{dx}(x)}{dx} = \frac{F_T}{\mu}\frac{d}{dx}\frac{dy}{dx} = \frac{F_T}{\mu}\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dt^2} = \frac{F_T}{\mu}\frac{d^2y}{dx^2}$$

There is a small clarification that must be made regarding the kind of derivatives we are taking: note that when we compared nearby points on the string, we were looking at a snapshot, so in the equation we should make clear that we are holding the time fixed when evaluating the *x* derivatives. Similarly, the acceleration refers to a single small piece of the string, so when we are evaluating the time derivatives, we are holding *x* fixed. Derivatives with respect to only one coordinate, holding the others constant, are partial derivatives, so we should use the partial derivative notation. Also, we will put everything on the left side of the equation, obtaining the wave equation for the string:

$$\frac{\partial^2 y}{\partial t^2} - \frac{F_T}{\mu}\frac{\partial^2 y}{\partial x^2} = 0$$

Sine wave solutions

Now let's see if our sine wave solution, $y(x,t) = A\sin(kx \pm \omega t)$, satisfies the wave equation. First, take the derivatives:

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A sin(kx \pm \omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A sin(kx \pm \omega t)$$

Plug this into the wave equation:

$$-\omega^2 A sin(kx \pm \omega t) + \frac{F_T}{\mu}k^2 A sin(kx \pm \omega t) = 0$$

$$-\omega^2 + \frac{F_T}{\mu}k^2 = 0 \qquad \omega = k\sqrt{\frac{F_T}{\mu}}$$

The phase speed and the group speed are

$$c = \frac{\omega}{k} = \sqrt{\frac{F_T}{\mu}} \qquad v_g = \frac{d\omega}{dk} = \sqrt{\frac{F_T}{\mu}} = c$$

Thus, on a string, the group speed is the same as the phase speed. Wave packets move at the same speed as individual sine waves.

Since $F_T/\mu = c^2$, we can rewrite the wave equation as follows:

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$$

This is how the wave equation is usually written. In this form, it is true for a number of different systems, not just for waves on a string.

Example:

In an example above, we had a sine wave on a string described by the function

$$y(x,t) = (0.025m)cos[(50m^{-1})x - (100s^{-1})t]$$

We can now ask an additional question about this system. Suppose that the string is held under tension of 2.0*N*. What is the mass per unit length of this string?

First, we calculate the wave speed:

$$c = \frac{\omega}{k} = \frac{100s^{-1}}{50m^{-1}} = 2.0m/s$$

Now, we use the relationship between the wave speed, the tension and the mass per unit length:

$$c = \sqrt{\frac{F_T}{\mu}} \qquad \mu = \frac{F_T}{c^2} = \frac{2.0N}{(2.0m/s)^2} = 0.50kg/m$$

The wave equation and superposition principle

We can now prove that the superposition principle holds for waves on a string. A mathematical way to state the superposition principle is that if two functions, *g* and *h*, are solutions of the wave equation, then any combination of these functions, $y = Ag + Bh$, where *A* and *B* are constants, is also a solution. This allows us to add multiple sine waves together to form more complicated solutions.

To prove the superposition principle, assume that *g* and *h* are solutions, and write down the wave equation for the combined wave, $y = Ag + Bh$:

$$\frac{\partial^2}{\partial t^2}(Ag+Bh) - c^2\frac{\partial^2}{\partial x^2}(Ag+Bh) = A\frac{\partial^2 g}{\partial t^2} + B\frac{\partial^2 h}{\partial t^2} - Ac^2\frac{\partial^2 g}{\partial x^2} - Bc^2\frac{\partial^2 h}{\partial x^2} =$$

$$= A\left(\frac{\partial^2 g}{\partial t^2} - c^2\frac{\partial^2 g}{\partial x^2}\right) + B\left(\frac{\partial^2 h}{\partial t^2} - c^2\frac{\partial^2 h}{\partial x^2}\right) = A\cdot 0 + B\cdot 0 = 0$$

In the last step, we have used the given fact that *g* and *h* each satisfy the wave equation. Therefore, $y = Ag + Bh$ satisfies the wave equation, and the superposition principle holds.

Standing Waves

So far, we have considered traveling waves, which moved either to the left or to the right with a wave speed *c*. There is another kind of waves, which do not move, but oscillate in place. These are called ***standing waves***.

To construct a standing wave, add a left-moving wave to a right-moving wave. The two component waves have identical amplitudes $A / 2$ as well as identical wave numbers and frequencies:

$$y(x,t) = \frac{1}{2}Asin(kx - \omega t) + \frac{1}{2}Asin(kx + \omega t)$$

Now use the trigonometric identities:

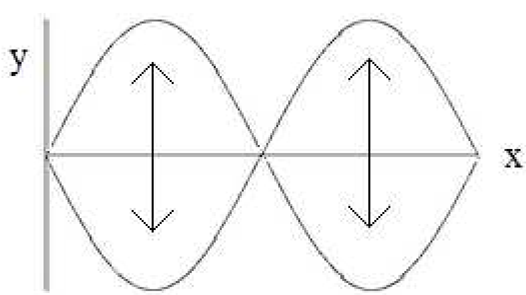
$$sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$$
$$sin(a-b) = sin(a)cos(b) - cos(a)sin(b)$$

This gives the following expression for the wave function:

$$y(x,t) = \frac{1}{2}A\left[sin(kx)cos(\omega t) - sin(\omega t)cos(kx) + sin(kx)cos(\omega t) + sin(\omega t)cos(kx)\right]$$

$$y(x,t) = Asin(kx)cos(\omega t) = Asin(kx)cos(kct)$$

Consider the wave function at $t = 0$. In that case, $cos(\omega t) = 1$, and the wave function is simply a sine curve with amplitude *A*, $y(x,t) = A\,sin(kx)$. As the time advances, the cosine becomes smaller than 1, then becomes negative, and finally reaches a value of -1 before turning back. This causes the amplitude of the sine curve to oscillate, but the wave doesn't move left or right. The wave therefore oscillates as follows:



The points at the edges and in the center, where the string does not move, are called ***nodes***. The peaks of the wave, where the string moves through the greatest distance, are called the ***antinodes***.

Standing waves on a string with fixed ends

Suppose we have a string of length *L*, which has its ends tied down so that they cannot move. Since standing waves possess nodes that do not move, it stands to reason that they can describe the oscillations on such a string, provided that a node is located at each end of the string.

Let us place one end of the string at $x = 0$ and the other end at $x = L$. Then, our wave must satisfy the conditions $y(0, t) = 0$ and $y(L, t) = 0$. These are called ***boundary conditions***, since they proscribe the behavior of the string at its boundaries. Using the standing wave solution above:

$$y(0,t) = Asin(k \cdot 0)cos(kct) = 0$$
$$y(L,t) = Asin(kL)cos(kct) = 0$$

Since $sin\ 0 = 0$, the first equation is automatically satisfied for our choice of standing wave solution. The second equation gives

$$sin(kL) = 0 \qquad kL = n\pi \qquad k = \frac{n\pi}{L}$$

Here, *n* is any positive integer (negative numbers just change the sign of the solution, so they do not really yield any independent solutions, and if $n = 0$, then the wave function is simply zero everywhere.) Our set of solutions, satisfying the wave equation as well as the boundary conditions, is therefore given by the following, for any choice of a positive integer *n*:

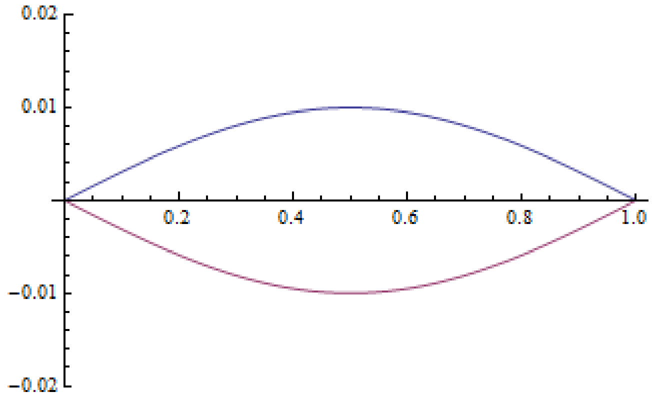
$$y_n(x,t) = Asin\left(\frac{n\pi x}{L}\right) cos\left(\frac{n\pi ct}{L}\right)$$

These solutions are called the ***normal modes*** of the string.

The angular frequency, frequency and period of a normal mode are given by the following:

$$\omega_n = \frac{n\pi c}{L} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L} \qquad T_n = \frac{1}{f_n} = \frac{2L}{nc}$$

Taking the amplitude to be 1 cm and the length of the string to be 1m, the first normal mode has *n = 1* and looks like this (the two curves show the extremes of the string's oscillations):



The scale on both axes is in meters, and the vertical scale is exaggerated.

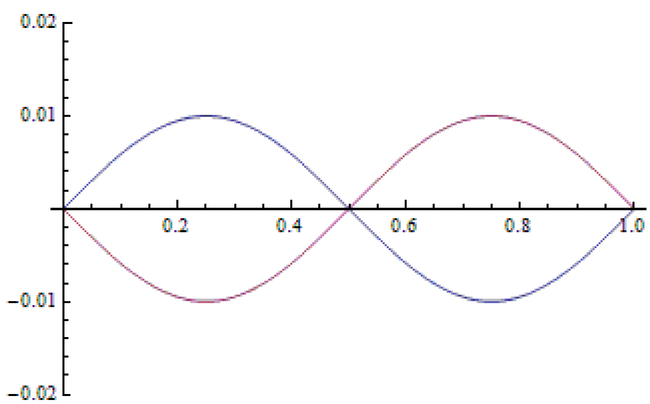
Suppose the string is under tension of 50N and has a mass per unit length of 25g/m. The wave speed is then given by

$$c = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{50N}{0.025kg/m}} = 44.7m/s$$

The frequency of oscillations of the first normal mode (called the ***fundamental frequency*** or the ***first harmonic***) is then equal to

$$f = \frac{c}{2L} = \frac{44.7m/s}{2.0m} = 22.4s^{-1}$$

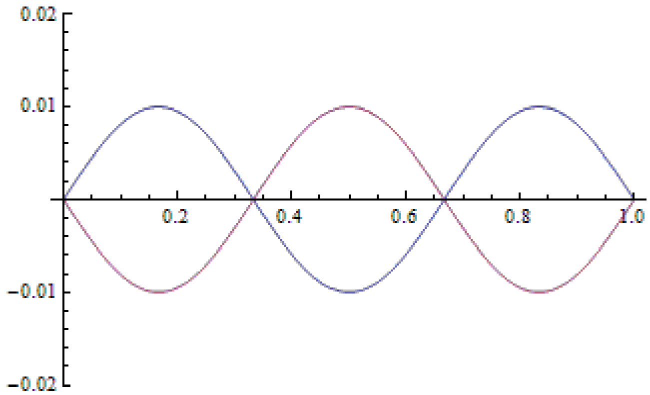
The second normal mode has *n* = 2, and looks like this:



The frequency of the second normal mode is called the ***second harmonic*** and is equal to

$$f_2 = \frac{2c}{2L} = \frac{2 \times 44.7m/s}{2.0m} = 44.7s^{-1}$$

The third normal mode has *n = 3* and looks like this:



Its frequency is

$$f_3 = \frac{3c}{2L} = \frac{3 \times 44.7m/s}{2.0m} = 67.1s^{-1}$$

Spectral decomposition on a string with fixed ends

Any possible displacement of a string with fixed ends can be expressed in terms of the normal modes. The only caveat is that the factor that oscillates in time can be proportional to a sine as well as a cosine. Any possible configuration of an oscillating string with ends tied down at $x=0$ and $x=L$ can be written as follows:

$$y(x,t) = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left[A_n sin\left(\frac{n\pi ct}{L}\right) + B_n cos\left(\frac{n\pi ct}{L}\right)\right]$$

As with the case of traveling waves, the set of coefficients A_n and B_n are determined from the initial displacement and speed of every point on the string through the techniques of Fourier analysis.