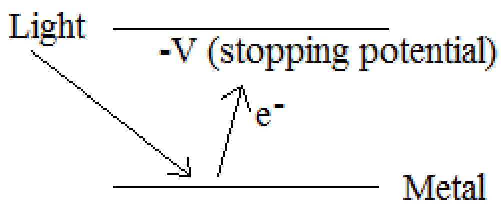
Lecture 15 Notes: 07 / 26

The photoelectric effect and the particle nature of light

When diffraction of light was discovered, it was assumed that light was purely a wave phenomenon, since waves, but not particles, diffract. However, the discovery of the ***photoelectric effect*** changed this.

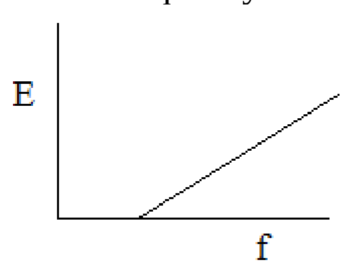
When light strikes the surface of a metal, it will, under certain conditions, eject electrons from the metal. The presence of ejected electrons can be detected when they reach another metal plate, creating a current; the energy of the electrons can be determined by applying a stopping voltage. Electrons ejected with an energy of, say, 1 eV, will be able to get through up to 1 volt of stopping voltage, but increasing the voltage higher than that will stop the photoelectric current.



From the wave picture of light, we would expect both the number of electrons ejected and the average energy of the electrons to be simply proportional to the intensity of the wave. However, that's not what we find. Up to a certain frequency of light, no electrons are ejected at all; above that frequency, the number of electrons ejected is in fact proportional to the intensity of light, but the energy of the electrons depends on the frequency of light, and is given by

$$E = hf - \phi$$

The graph of the electron energy vs. the frequency of incident light looks like this:



The slope of this graph is found to be the same in all photoelectric effect experiments, and is given by ***Planck's constant*** $h = 6.63 \times 10^{-34}$ m^2 kg/s = 4.14×10^{-15} eV s. The quantity ϕ is known as the ***work function***, and depends on the particular kind of metal used in the experiment.

The photoelectric effect is readily explained if light consists of particles called ***photons***. These particles are associated with electromagnetic waves, and have an energy equal to Planck's constant times the frequency of the wave:

$$E = hf$$

When a photon strikes an electron, it can be absorbed. The electron in the metal is bound to the metal by a binding potential ϕ; if the energy of the photon is sufficient, it can liberate the electron from the metal. The leftover kinetic energy of the electron is equal to $E = hf - \phi$, where hf is the energy of the photon (which has nothing to do with the kind of metal or experimental apparatus used, just the frequency of light) and ϕ is the binding energy of the electrons in the metal. Thus the constant h is expected to be independent of the metal used, while ϕ is expected to depend on the metal, just as observed.

The relationship between the energy and the frequency can be rewritten in terms of the angular frequency:

$$E = hf = \frac{h\omega}{2\pi} = \hbar\omega \qquad \hbar \equiv \frac{h}{2\pi} = 1.055\times10^{-34} m^2 kg/s = 6.59\times10^{-16} eV\,s$$

The momentum is similarly related to the wave number:

$$p = \hbar k$$

Photons have zero mass, and thus move at the speed of light, as discussed in the previous lecture.

Example: What photon energies (in eV) does the visible range correspond to?

Light in the visible range has wavelengths from about 390 to about 750*nm*. The frequency is related to the wavelength as:

$$f = \frac{c}{\lambda}$$

Therefore, $E = hc / \lambda$. For a wavelength of 390*nm*, this gives E = 3.18 eV. For a wavelength of 750*nm*, this gives E = 1.66eV. Thus the visible range corresponds to photon energies between 1.66 eV and 3.18 eV. The red light corresponds to lower-energy photons, while the blue and violet light corresponds to higher-energy photons.

Example: Ultraviolet light with a wavelength of 250*nm* is incident on a metal. The electrons ejected from the metal have a kinetic energy of 2.40*eV*. What is the work function of the metal?

The work function is

$$\phi = hf - E = \frac{hc}{\lambda} - E = 4.97eV - 2.40eV = 2.57eV$$

Example: A 100W light bulb emits light with an average wavelength of 600*nm*.

(a) How many photons per second does the bulb emit?

(b) If an astronaut looks at this light bulb from the International Space Station, approximately 300*km* away, how many photons from the bulb enter each eye per second? Assume a pupil diameter of 4*mm*.

The number of photons per second is the power divided by the energy of each photon:

$$\Phi_N = \frac{P}{E} = \frac{P\lambda}{hc} = \frac{(100W)(6 \times 10^{-7}m)}{(6.63 \times 10^{-34}m^2kg/s)(3 \times 10^8 m/s)} = 3.02 \times 10^{20}$$

Thus the light bulb emits about 3 x 10^{20} photons per second. At a distance of 300*km* away, the photon flux per area is

$$\frac{\Phi_N}{A} = \frac{\Phi_N}{4\pi R^2} = \frac{3 \times 10^{20} s^{-1}}{4\pi(3 \times 10^5 m)^2} = 2.65 \times 10^8 m^{-2} s^{-1}$$

The number of photons entering a 4*mm* pupil is the area of the pupil times the flux per area:

$$\frac{\Phi_N}{A}(\pi r^2) = (2.65 \times 10^8 m^{-2} s^{-1})(\pi \times (2 \times 10^{-3} m)^2) = 3330 s^{-1}$$

Thus about 3000 photons from the light bulb enter each of the astronaut's eyes per second. If there were no competing light sources, this would be enough to be able to see the light bulb.

Wave nature of matter

If we direct a beam of electrons onto a crystal, we see a diffraction pattern similar to that when we shine a light on a diffraction grating. The presence of diffraction indicates that electrons are not just particles, but are associated with waves, just as photons are associated with electromagnetic waves. These waves are known as ***de Broglie waves*** or ***matter waves***, and have a frequency and wave number that are related to the energy and momentum of the electrons just like they are for photons:

$$E = \hbar\omega \qquad p = \hbar k$$

It turns out that all kinds of particles are associated with these de Broglie waves. The relationship between the energy and momentum on one hand and the frequency and wave number on the other is always the same.

Example:

Find the de Broglie wavelength for an electron with a kinetic energy of 1.0 eV.

The electron's kinetic energy is much smaller than its rest energy of 0.511MeV, so we can use nonrelativistic expressions for the energy to a very good approximation. The kinetic energy in the nonrelativistic limit is given by

$$E_K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

We now calculate the wave number and finally the wavelength:

$$k = \frac{p}{\hbar} = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2(0.511 \times 10^6 eV/c^2)(1.0eV)}}{\hbar} = \frac{1011eV}{\hbar c} =$$

$$= \frac{1010eV}{(6.59 \times 10^{-16} eVs)(3 \times 10^8 m/s)} = 5.11 \times 10^9 m^{-1}$$

$$\lambda = \frac{2\pi}{k} = 1.229 \times 10^{-9} m = 1.229nm$$

You can see that the de Broglie wavelengths of electrons are quite small compared to wavelengths of visible light. Electron beams can be focused by electromagnetic fields just as light can be focused by lenses, and so electron beams can be used in ***electron microscopes***. An electron microscope illuminates the object with a beam of electrons, rather than a beam of light; the electrons are then focused to form a magnified image, just as light is in an optical microscope. Since the wavelengths are so short, and resolution of a microscope is limited by the wavelength, electron microscopes have much higher resolution than optical ones.

Example:
Electrons with a kinetic energy of 1 keV are scattered off a crystal with an interatomic separation of 0.2 *nm*. What is the angle between two nearby diffraction peaks?

First we find the de Broglie wavelength of the electrons:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mE}} = 3.89 \times 10^{-11} m$$

Second, for diffraction, we have $d \sin \theta = n \lambda$. The first peak is separated from the second by an angle given by $\sin \theta = \lambda / d = 3.89 \times 10^{-11}\text{m} / 2 \times 10^{-10}\text{m} = 0.1945$. Therefore, $\theta = 11.2^{o}$.

The frequency-wave number 4-vector

The relationship

$$E = \hbar\omega \qquad p = \hbar k$$

can be written in terms of the energy-momentum 4-vector as follows:

$$P = \hbar K$$

$$P = \left(\frac{E}{c}, \vec{p}\right) \qquad K = \left(\frac{\omega}{c}, \vec{k}\right)$$

Since *K* is proportional to the 4-vector *P*, it is also a 4-vector (known as the frequency-wave number 4-vector). A sine wave can be written as follows:

$$A \sin\left(\omega t - \vec{k} \cdot \vec{x}\right) = A \sin\left(K \cdot X\right)$$

$$K = \left(\frac{\omega}{c}, \vec{k}\right) \qquad X = (ct, \vec{x})$$

The phase of the wave is

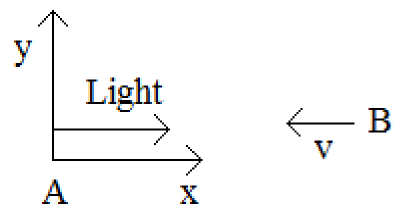
$$\theta = K \cdot X$$

This is a 4-scalar, as it should be, since if two different observers look at the same part of the wave, they should agree on whether it is at a minimum or a maximum or somewhere in between.

The Doppler effect

The angular frequency of a photon, or any other particle, is a component of a 4-vector. Thus it will take on different values relative to different observers, according to the Lorentz transformation. The dependence on the frequency of light on the motion of the observer is the relativistic analog of the Doppler effect with sound waves. We will derive an expression for the relativistic Doppler effect.

Suppose a source *A* produces light with frequency f_A. The light goes in the positive *x* direction. An observer *B* is approaching *A* with velocity *v* from the *x* direction; thus *B* is moving with speed *v* in the -*x* direction:



First, we will write down an expression for the frequency - wave number 4-vector of the light in *A*'s frame. Find the angular frequency and the wave number, and plug into the expression for *K*:

$$\omega_A = 2\pi f_A \qquad k_A = \frac{\omega_A}{c}\hat{x} = \frac{2\pi f_A}{c}\hat{x}$$

$$K_A = \left(\frac{2\pi f_A}{c}, \frac{2\pi f_A}{c}, 0, 0\right)$$

Now Lorentz transform this to B's frame. Notice that B moves in the negative *x* direction, so we should use the transformation with the positive sign in front of *v*:

$$K_B^0 = \gamma\left(K_A^0 + \beta K_A^1\right) = \gamma\left(1+\beta\right)\frac{2\pi f_A}{c} = \frac{2\pi f_B}{c}$$

$$K_B^1 = \gamma\left(K_A^1 + \beta K_A^0\right) = \gamma(1+\beta)\frac{2\pi f_A}{c} = \frac{2\pi f_B}{c}$$

Both components of the 4-vector are the same, just as in *A*'s frame. Solving for f_B yields

$$f_B = \gamma\left(1+\beta\right) f_A = \frac{1+\beta}{\sqrt{1-\beta^2}} f_A = \frac{1+\beta}{\sqrt{(1+\beta)(1-\beta)}} f_A$$

$$f_B = f_A\sqrt{\frac{1+\beta}{1-\beta}}$$

The frequency for *B* is increased if *B* moves towards the source of light, as expected. If *B* was moving away instead, the sign of β would be reversed, which would cause the numerator and the denominator under the square root to be exchanged, and the frequency would decrease instead. Note that unlike in the case of sound, the equation doesn't change depending on whether the source or the observer moves, because the situation is symmetric: each moves relative to the other. In the case of air, either the source or the observer could be stationary relative to the air, but with light, there is no air to serve as a background reference frame.

Example: Consider a slightly more complicated case, where the light is moving in the positive *y* direction in A's frame, and *B* is approaching from the right with a speed of 0.5*c*. The light has a wavelength of 600*nm* in *A*'s frame. What wavelength does *B* see, and what direction is the light traveling relative to *B*?

First, we construct the frequency-wave number 4-vector in *A*'s frame. The angular frequency is $\omega_A = 2\pi c / \lambda = 3.142 \times 10^{15} s^{-1}$. The wave number is $k_A = \omega_A / c = 1.047 \times 10^7 m^{-1}$. This points in the *y* direction, so *K* is equal to

$$K_A = \left(\frac{\omega_A}{c}, \vec{k_A}\right) = (1.047, 0, 1.047, 0) \times 10^7 m^{-1}$$

Now we transform this to *B*'s frame. The Lorentz transformation gives

$$K_B^0 = \gamma\left(K_A^0 + \beta K_A^1\right) = \frac{1}{\sqrt{1-0.5^2}}\left(1.047 + \beta \times 0\right) \times 10^7 m^{-1}$$

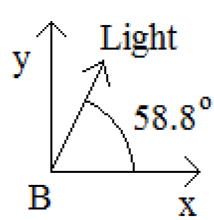
$$= 1.209 \times 10^7 m^{-1}$$

$$K_B^1 = \gamma\left(K_A^1 + \beta K_A^0\right) = \gamma\left(0 + \beta K_A^0\right) = 6.329 \times 10^6 m^{-1}$$

$$K_B^2 = K_A^2 = 1.047 \times 10^7 m^{-1} \qquad K_B^3 = K_A^3 = 0$$

$$K_B = (1.209, 0.6329, 1.047, 0) \times 10^7 m^{-1}$$

The direction of the light according to *B* is the direction of the wave number 3-vector. Its *x* component is 0.6329, while the *y* component is 1.047 (times a common factor of $10^7 m^{-1}$). The tangent of the angle from the axis is thus 1.047 / 0.6329, and the angle is equal to atan(1.047 / 0.6329) = 58.8°. Observer B thus sees the rays of light point at an angle of 58.8° above the *x* - axis:



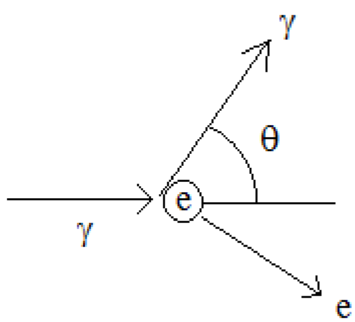
The wavelength is 2π divided by the magnitude of the wave number 3-vector:

$$\lambda_B = \frac{2\pi}{k_B} = \frac{2\pi}{\sqrt{1.047^2 + 0.6329^2} \times 10^7 m^{-1}} = 514nm$$

Observer B thus sees light with a wavelength of 514*nm*.

Compton scattering

Now that we know that a photon is a particle of zero mass, we can consider the problem of a photon scattering of an electron. It was observed that when photons scatter off electrons, they experience a frequency shift that depends on the scattering angle. This can only be explained if light consists of particles that follow the energy-frequency relationship. The dependence on the angle is only correctly predicted if relativity is used to derive it. Thus, Compton scattering verifies both quantum mechanics and relativity. Consider a photon (traditionally labeled by the letter γ) moving in the positive x direction. The photon strikes a stationary electron, and is deflected by an angle θ. The electron is similarly deflected.



We can write the expression for conservation of energy and momentum in this process. The initial 4-momentum of the photon is

$$P_{I,\gamma} = \left(\frac{E_{I,\gamma}}{c}, p_\gamma\right) = \left(\frac{E_{I,\gamma}}{c}, \frac{E_{I,\gamma}}{c}, 0, 0\right) \quad \text{since } e = pc \text{ for } m = 0$$

The initial 4-momentum of the electron is

$$P_{I,e} = \left(\frac{E_{I,e}}{c}, 0, 0, 0\right) = (mc\ , 0, 0, 0) \quad \text{since } \vec{v}_{I,e} = 0 \text{ and } E_{I,e} = mc^2$$

The final 4-momentum of the photon and the electron are

$$P_{F,\gamma} = \left(\frac{E_{F,\gamma}}{c}, \frac{E_{F,\gamma}}{c}\cos\theta, \frac{E_{F,\gamma}}{c}\sin\theta, 0\right)$$

$$P_{e,\gamma} = \left(\frac{E_{F,e}}{c}, p^x_{F,e}, p^y_{F,e}, 0\right)$$

We have chosen our coordinates so that the particles scatter in the *xy* plane, and therefore we don't have to worry about the *z* axis.

Conservation of 4-momentum yields three equations:

$$E_{I,\gamma} + mc^2 = E_{F,\gamma} + E_{F,e}$$

$$E_{I,\gamma} = E_{F,\gamma}\cos\theta + p^x_{F,e}c$$

$$0 = E_{F,\gamma}\sin\theta + p^y_{F,e}c$$

We are going to solve for the final energy of the photon as a function of the initial energy and the scattering angle. If the initial energy and the scattering angle are given, there remain four unknowns: the final energy of the photon, the final energy of the electron, and the two components of the electron's momentum after the collision. We thus need another equation. This is provided by the mass-energy-momentum relationship for the electron:

$$E^2_{F,e} = \vec{p}^{\,2}_{F,e}c^2 + m^2c^4$$

Solving these equations is a straightforward but tedious algebra exercise, which we won't repeat here. The solution of a particularly simple case (where the photon bounces straight back, so everything always moves in the *x* direction) is left as homework problem #17. The final result is usually given in terms of the change in the photon wavelength (which is of course related to the energy):

$$\lambda_F - \lambda_I = \frac{h}{mc}(1 - \cos\theta)$$

This result, which follows from both the energy-frequency relationship and from relativity, describes the actual experimental findings, and was historically one of the key pieces of evidence for relativity and for quantum mechanics.