Lecture 14 Notes: 07 / 25

4-vectors
Interval between events

Last time, we implicitly made use of the concept of an ***event***, which is described by the coordinates (x, y, z) and the time t. The coordinates and the time vary depending on the observer, and can be translated from one observer's frame to another via the Lorentz transformation. The coordinates and the time together make up a ***4-vector*** $X = (ct, x, y, z)$. A 4-vector is defined as a 4-component object that takes on different values for different observers according to the Lorentz transformation.

Suppose we have two events at $X_1 = (ct_1, x_1, y_1, z_1)$ and $X_2 = (ct_2, x_2, y_2, z_2)$. The ***separation*** between these two events is the difference between their coordinates in time and space:

$$\Delta X = X_2 - X_1 = (ct_2 - ct_1, x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Since X_1 and X_2 are 4-vectors, ΔX is a 4-vector as well, and transforms according to the Lorentz transformation. Moreover, it has an invariant length, which is a ***4-scalar***, meaning that it is the same according to all observers. The invariant length of the interval is known as the ***invariant interval***.

$$\Delta s^2 = X^2 = (ct_2 - ct_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

Let us consider the meaning of this invariant interval. Suppose that there exists an observer B who is moving in such a way that he is at position (x_1, y_1, z_1) at t_1 and at position (x_2, y_2, z_2) at t_2. In this case, according to this observer, both events occur at his own position, that is, at the origin: $x_{1B} = x_{2B} = y_{1B} = y_{2B} = z_{1B} = z_{2B} = 0$. The observer measures the invariant interval to be

$$\Delta s^2 = (ct_{2B} - ct_{1B})^2 = c^2 \Delta t_0^2$$

Thus Δs is the ***proper time*** between the two events, multiplied by a factor of c. Recall from last lecture that the proper time for a process was the time elapsed for that process according to the observer who is at rest with respect to it. In this case, the positions of the two events are the same according to the observer, so if the events were caused by the same process, then the observer would be at rest with respect to it.

Thus, the invariant interval Δs is equal to Δt_0, the proper time that elapses between the two events. This is the time from one event to another according to an observer who is present at one event and moves uniformly in a straight line, in such a way as to be present at the next. Defined this way, this quantity is clearly a 4-scalar: different observers might measure different times between these events, but they all agree that an observer moving from one event to another would measure a length of time equal to Δs.

Note that Δs^2 can be negative, implying an imaginary "proper time" between the two events. This happens when

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 > (ct_2 - ct_1)^2$$

$$\Delta x^2 + \Delta y^2 + \Delta z^2 > c^2 \Delta t^2$$

$$\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \Delta r > c\Delta t$$

$$\frac{\Delta r}{\Delta t} = v > c$$

Thus if is Δs^2 negative, the observer would have to move faster than the speed of light to get from event 1 to event 2. This is not possible; thus there is no proper time between the two events, and Δs^2 does not have this physical interpretation. However, it can be shown that in this case, there exists an observer for which the two events are simultaneous. $-\Delta s^2$ then gives the square of the distance between the two events according to this observer, known as the ***proper distance***.

Note also that Δs^2 can be zero for a nonzero 4-vector. This is the case if getting from one event to another requires traveling exactly at the speed of light. This is unlike the length of a regular 3-vector, which if zero implies that the vector itself is zero.

Some terminology: If Δs^2 is positive, the separation between the two events is called ***timelike***. This is because there exists an observer (one traveling from one event to the next) for whom the events are separated in time, but not in space. If Δs^2 is negative, then the separation is ***spacelike***. This is because there exists an observer for whom the events are simultaneous, that is, separated in space but not in time. If Δs^2 is zero, the separation is ***lightlike***, because then the events can be connected by a ray of light moving at speed c.

Note that the timelike, spacelike or lightlike character of a separation between two events is independent of an observer, since Δs^2 itself is a 4-scalar.

Example: Suppose that one event occurs on Earth, while another one occurs 1.5 seconds later on the Moon, 384000 km away, according to an Earth-bound observer. Is the separation between these events timelike, spacelike or lightlike? If timelike, what is the proper time between the two events? If spacelike, what is the proper distance?

Let *x* be the direction from the event on Earth to that on the Moon. Then, the square of the invariant interval is equal to

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = \left(1.5s \times 3 \times 10^5 km/s\right)^2 - \left(3.84 \times 10^5 km\right)^2 =$$

$$5.50 \times 10^{10} km^2$$

This is positive, so the interval is timelike. The proper time is

$$\Delta s^2 = c^2 \Delta t_0^2$$

$$\Delta t_0 = \frac{\Delta s}{c} = \frac{\sqrt{5.50 \times 10^{10} km^2}}{3 \times 10^5 km/s} = 0.782s$$

The proper time is less than the time elapsed on Earth, as expected. This is because an observer who is present at both events would be moving at a sizable fraction of the speed of light, so the time measured by this observer would be dilated when viewed by an observer on Earth.

Now consider what would happen if the time separation was not 1.5, but 1.0 seconds. Then, we would have

$$\Delta s^2 = \Big(1.0s \times 3 \times 10^5 km/s)^2 - \left(3.84 \times 10^5 km/s\right)^2 = -5.75 \times 10^{10} km^2$$

This is negative, meaning that the interval is spacelike. The time is so short that an observer cannot get from one event to another in time, as that would require moving faster than light. The proper distance between the two events would be

$$\Delta x = \sqrt{-\Delta s^2} = \sqrt{5.75 \times 10^{10} km^2} = 2.40 \times 10^5 km$$

This is less than the distance from the Earth to the Moon. This can be understood by considering what kind of observer would see these two events as simultaneous. We can calculate the observer's speed by writing down the Lorentz transformation and solving for the speed, but clearly, in order to see these events as simultaneous, the observer would have to be moving at a significant fraction of the speed of light relative to the Earth and the Moon. Thus the distance from the Earth to the Moon would be contracted.

Note that the proper distance between two events is not the same as the proper distance between the two places where they occur! It is only the same if the events occur simultaneously according to an observer at rest with respect to their locations.

4-velocity

So far, we have seen two 4-vectors: the coordinate 4-vector and the separation 4-vector, and one 4-scalar: the invariant interval associated with a separation between two events, related to the proper time or the proper distance depending on the sign.. We can now form additional 4-vectors. As the first example, consider the motion of an object moving at speed *v* in the x-direction relative to an observer. We can take the derivative of the object's coordinates (a 4-vector) with respect to its proper time (a 4-scalar). Since the numerator of the derivative transforms under Lorentz transformation as a 4-vector, and the denominator is a 4-scalar and doesn't change, the result will be a 4-vector:

$$U = \frac{dX}{dt_0} = \frac{d}{dt_0}\left(ct, vt, y, z\right)$$

Now, $t = \gamma t_0$ so that $dt_0 = dt / \gamma$, and $x = vt$, since the object is moving at speed *v* in the *x*-direction. *y* and *z* are constant. Thus,

$$U = \gamma \frac{d}{dt}\left(ct, vt, y, z\right) = \left(\gamma c, \gamma v, 0, 0\right)$$

In general, for an object moving with any velocity in any direction,

$$U = \left(\gamma c, \gamma v_x, \gamma v_y, \gamma v_z\right) = \left(\gamma c, \gamma \vec{v}\right)$$

Recall that in the previous lecture we defined γ as

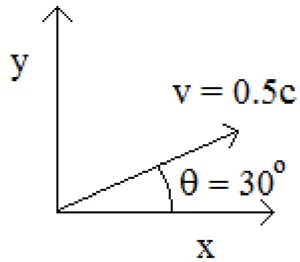
$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Let us compute the invariant length of the 4-velocity:

$$U^2 = \gamma^2 c^2 - \gamma^2 v^2 = \gamma^2 c^2\left(1 - v^2/c^2\right) = c^2 \frac{1 - v^2/c^2}{1 - v^2/c^2} = c^2$$

Thus the invariant length of the 4-velocity of any object is equal to *c*. To see this, consider the moving object in its own frame. It's obviously not moving relative to itself, so its speed in its own frame is zero, and so the spatial components of the 4-velocity in this frame are zero. The time elapsed in this frame is the object's proper time, and so *ct* changes by *c* whenever t_0 changes by 1 second. Since the change in elapsed time is the only piece of the 4-velocity that is not zero in this frame, the magnitude of the 4-velocity must be *c*. But as a 4-scalar, it is the same for all observers, so it must be *c* in any frame.

Example: An object is moving at a speed of 0.5*c* in at an angle of 30 degrees from the *x*-axis, relative to an observer on Earth:



(a) What is the 4-velocity of this object relative to the observer on Earth?

(b) What is the 4-velocity relative to an observer *B* moving with speed 0.7*c* in the *x* direction?

(c) What is the regular 3-velocity with respect to that observer? What is the speed of the object and the direction of its motion?

To find the 4-velocity, we'll need γ, v_x and v_y ($v_z = 0$). These are

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.5^2}} = 1.155$$

$$v_x = v\cos 30^\circ = 0.5c \cdot 0.866 = 0.433c$$

$$v_y = v\sin 30^\circ = 0.5c \cdot 0.5 = 0.250c$$

The 4-velocity is

$$U = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) = (1.155, 0.500, 0.289, 0)\, c$$

The observer *B* moves with speed 0.7*c* in the *x* direction. The 4-velocity according to observer B is given by the Lorentz transformation:

$$U_B^0 = \gamma\left(U^0 - \beta U^1\right) = \frac{1}{\sqrt{1 - 0.7^2}}(1.155c - 0.7 \times 0.500c) = 1.127c$$

$$U_B^1 = \gamma\left(U^1 - \beta U^0\right) = \frac{1}{\sqrt{1 - 0.7^2}}(0.500c - 0.7 \times 1.155c) = -0.432c$$

$$U_B^2 = U^2 = 0.289c \qquad U_B^3 = U^3 = 0$$

Thus in observer B's frame,

$$U_B = (1.127, -0.432, 0.289, 0)\, c$$

To determine the object's 3-velocity, notice that the time component of the 4-velocity is equal to γc, and so γ for the object's motion is 1.127. The other components of the 4-velocity are just γ times the components of the 3-velocity. Thus, the 3-velocity according to observer *B* is

$$\vec{v} = \frac{1}{1.127}(-0.432, 0.289, 0)\,c = (-0.383, 0.256, 0)\,c$$

The speed is

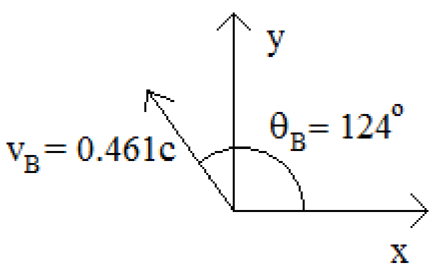
$$v = \sqrt{\vec{v}^2} = c\sqrt{0.383^2 + 0.256^2} = 0.461c$$

The direction of motion with respect to *B*'s *x*-axis is given by

$$\tan\theta_B = \frac{v_{yB}}{v_{xB}} = \frac{-0.383}{0.256} = -1.495$$

$$\theta_B = 124°$$

Observer B thus sees the object moving at 124 degrees from the *x*-axis, or 34 degrees from the *y* axis, relative to himself:



We could also use the velocity addition formula to get this result; however, since in this case the direction of motion of the object and the observer is not the same, we would need to derive the rule for velocity addition in a direction different than the direction of motion. The method using the 4-velocity can be summarized as follows, and is always valid provided the speed of the object is not equal to *c*:

(1) Determine the 4-velocity of the object in the initial reference frame
(2) Transform it to the desired reference frame using the Lorentz transformation
(3) Calculate the 3-velocity of the object in the new frame from the 4-velocity

4-momentum

In classical mechanics, the momentum was obtained by multiplying the velocity of an object by its mass. Similarly, in relativistic mechanics, there is a 4-vector quantity called the ***4-momentum***, obtained by multiplying the 4-velocity by the mass of the object:

$$P = mU = (\gamma mc, \gamma m\vec{v})$$

The mass of the object is defined as the mass measured in the frame where the object is at rest; by this definition, it is a 4-scalar, since even if different observers might measure a different "effective mass", they would all agree that if the mass was measured in the object's rest frame, it would be equal to *m*. The 4-momentum is thus a product of a 4-vector and a 4-scalar, and is therefore a 4-vector. This means it transforms between different observers according to the Lorentz transformation, just like the position 4-vector and the 4-velocity 4-vector.

Let us write the components of the 4-momentum as follows:

$$P = (\gamma mc, \gamma m\vec{v}) = \left(p^0, \vec{p}\right)$$

The time component of the 4-momentum is $p_0 = \gamma mc$. For speeds small compared to c, γ can be approximated by the binomial expansion:

$$p^0 = \gamma mc = \frac{mc}{\sqrt{1 - v^2/c^2}} \approx mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + ...\right)$$

The first term is just a constant, while the second term is equal to 1/2 mv^2 / c. This is just the kinetic energy in classical mechanics divided by c. We can therefore identify p^0 with the energy of the particle:

$$p^0 = \frac{E}{c}$$

$$E = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2 + ...$$

The first term in the approximate expansion is a constant, present even for a particle at rest. Thus for a particle at rest, the energy is proportional to its mass: $E = mc^2$. This is known as the particle's ***rest energy***. The second term gives the particle's kinetic energy, while the higher-order terms give relativistic corrections to the kinetic energy. The actual relativistic kinetic energy, without approximations, is given by

$$E_K = E - mc^2 = (\gamma - 1)\, mc^2$$

Note that for a nonzero mass, the energy approaches infinity as *v* approaches *c*. Thus it is impossible to accelerate a massive particle to exactly the speed of light, as that would require infinite energy. We will see below that massless particles always move at the speed of light.

In terms of the energy, the 4-momentum vector can be written as

$$P = \left(\frac{E}{c}, \vec{p}\right)$$

For this reason, it is also known as the ***energy-momentum 4-vector***.

Note that the velocity of the particle can be calculated from the components of the 4-momentum as follows:

$$\frac{\vec{v}}{c} = \frac{\gamma m \vec{v}}{\gamma m c} = \frac{\vec{p}}{p^0} = \frac{c\vec{p}}{E}$$

Example: An electron has rest energy of $mc^2 = 0.511\ MeV$. What is the speed of an electron that is accelerated through a potential of $10^6\ V$?

An electron accelerated through a million volts gains 1 MeV in kinetic energy. Its total energy (kinetic plus rest energy) thus becomes 1.511 MeV. Plug this into the expression for the energy and solve for the electron's speed:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = 1.511 MeV$$

$$1 - \frac{v^2}{c^2} = \left(\frac{mc^2}{E}\right)^2 \qquad \frac{v}{c} = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = \sqrt{1 - \left(\frac{0.511 MeV}{1.511 MeV}\right)^2} = 0.941$$

Thus the speed of the electron is 0.941*c*.

Mass-energy-momentum relationship

Note that since $P = mU$, $P^2 = m^2U^2 = m^2c^2$. Identifying the time component of the 4-momentum as E / c and the space component as the 3-momentum, this gives

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2c^2$$

$$E^2 = \vec{p}^2c^2 + m^2c^4$$

Massless particles

The energy-momentum relationship can be applied to particles with zero mass. In this case, $m = 0$, and the energy-momentum relationship gives

$$E^2 = \vec{p}^2 c^2$$

$$E = pc$$

In terms of the energy and the momentum, the speed of any particle is (see above):

$$\frac{v}{c} = \frac{pc}{E} = 1 \qquad v = c$$

Therefore, massless particles move at the speed of light.

Example: What is the speed of a massive particle with mass *m* and 3-momentum of magnitude *p*?

The equation relating the momentum, energy and speed still holds, but the energy can now be determined from the energy-momentum relationship for a massive particle:

$$\frac{v}{c} = \frac{pc}{E} \qquad E = \sqrt{p^2c^2 + m^2c^4}$$

$$v = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{c}{\sqrt{1 + m^2c^2/p^2}}$$

You can see that the denominator is greater than 1, so the speed is always smaller than *c* for any magnitude of the 3-momentum. As the 3-momentum approaches infinity, the denominator approaches 1, and the speed approaches *c*.

Conservation of energy and momentum

Just like in nonrelativistic mechanics, the energy and momentum are conserved. In relativistic mechanics, this can be expressed as follows:

$$\sum P_I = \sum P_F$$

The total 4-momentum of the system in the initial state is equal to the total 4-momentum in the final state. Let us apply this to a simple problem.

Example: Suppose that a single particle of mass *M*, initially at rest, decays into two particles, each of mass *m*. What is the speed of each final particle?

If the initial particle is at rest, it has a 4-momentum of $P_0 = (E_0/c, 0) = (Mc, 0)$. The final particles have 4-momenta given by

$$P_1 = (E_1/c, \vec{p}_1) \qquad P_2 = (E_2/c, \vec{p}_2)$$

The sum of the initial 4-momenta (only P_0) must be the same as the sum of the final 4-momenta ($P_1 + P_2$). This gives us

$$(Mc, 0) = \left(\frac{E_1 + E_2}{c}, \vec{p_1} + \vec{p_2}\right)$$

$$E_1 + E_2 = Mc^2 \qquad \vec{p_1} + \vec{p_2} = 0$$

The first equation on the bottom line gives conservation of energy (including mass) while the second equation is conservation of momentum. We can see that for an initial particle at rest, the momenta of the two outgoing particles must be in opposite directions. Now the energy of the outgoing particles is

$$E_1 = \sqrt{\vec{p}_1^2 c^2 + m^2 c^4} \qquad E_2 = \sqrt{\vec{p}_2^2 c^2 + m^2 c^4}$$

$$\vec{p}_1 = -\vec{p}_2 \qquad E_1 = E_2 = E$$

The energies of the particles are equal because their masses are equal, and the magnitudes of their 3-momenta are equal. Thus we get

$$2E = Mc^2 = 2\gamma mc^2$$

$$\gamma = \frac{M}{2m} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\frac{v}{c} = \sqrt{1 - \left(\frac{2m}{M}\right)^2}$$

Note that this only gives a valid result if $M > 2m$. Thus a particle cannot decay into two particles if the total mass of the two is more than the mass of the initial particle. If the mass of the products is less than the mass of the initial particle, however, the extra mass can be converted into kinetic energy.

We will give a more complicated example of how to use the conservation of energy-momentum when we discuss Compton scattering in the next lecture.