Lecture 13 Notes: 07 / 20

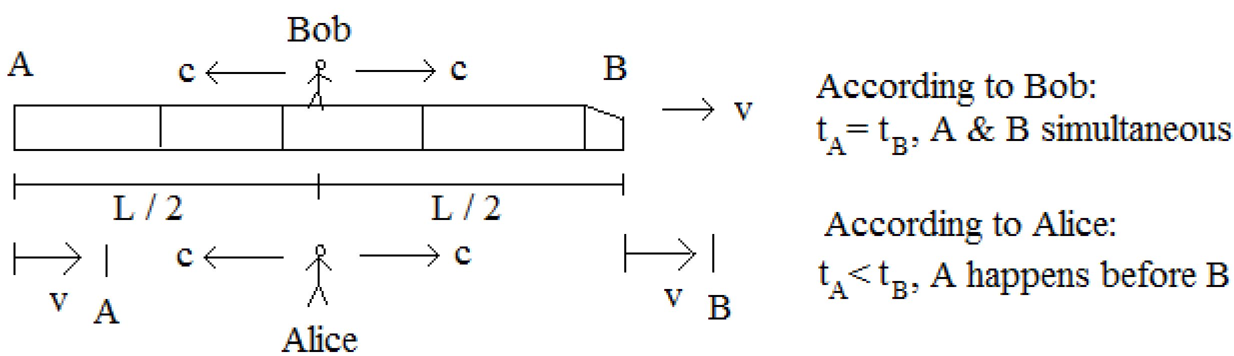
Invariance of the speed of light

The Michelson-Morley experiment, among other experiments, showed that the speed of light in vacuum is a universal constant, as predicted by Maxwell's equations. It is the same for all observers, and for light coming from any direction. This is at odds with our everyday experience with speeds, where the speed of any object relative to an observer depends on the motion of the observer.

Albert Einstein started with the assumption that Maxwell's equations, along with all other fundamental laws of physics, in fact hold for all observers, and so the speed of light really is independent from the observer. The assumption that the laws of physics are the same for all observers, together with a finite and observer-independent speed of light, is known as the ***principle of relativity.***

The principle of relativity requires us to abandon the notion of absolute time. In particular, with the principle of relativity, two events that are simultaneous according to one observer are not simultaneous according to another. To see this, consider the following example:

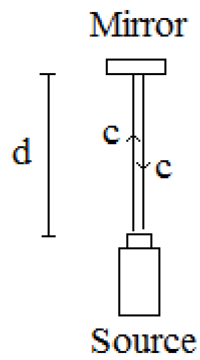
A train of length *L* is moving to the right with speed *v*. An observer, let's call him Bob, stands on top of the train, right in the middle, and sends a light pulse to both ends. Since the distance to both ends of the trains is the same, the pulses arrive at the same time. If we call the signal arriving at the back of the train event A, and the signal arriving at the front of the train event B, then A and B are simultaneous according to Bob.



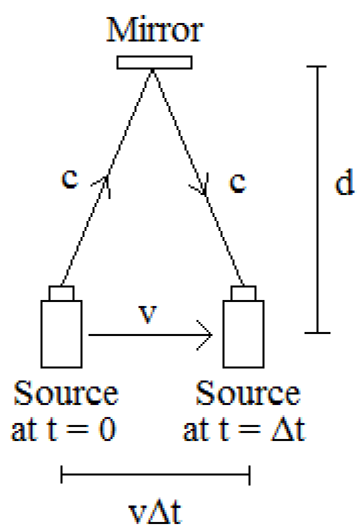
However, now consider that Alice is standing near the tracks, in the same spot as Bob when he sends the light pulses. By the principle of relativity, she also sees both light pulses moving at speed *c*. But now the back of the train moves towards the light pulse, while the front of the train moves away from it, so the light reaches the back of the train first. Thus A happens before B, according to Alice.

Proper time and time dilation

Consider the following thought experiment. Bob is on the train. He constructs a clock by bouncing a pulse of light off a mirror, and measuring the amount of time it takes the light to get back. If the mirror is a distance *d* from the light source, then the light moves a distance 2*d* at speed *c*, and Bob will see the light come back a time $\Delta t_0 = 2d / c$ later.



Now Alice watches this experiment from the ground near the train tracks. According to her, the apparatus is moving to the right, so the light takes a path shown below:



The light has to travel a greater distance, but by the principle of relativity, it still moves at speed *c*. Therefore, the time it takes the pulse to go to the mirror and back is longer according to Alice than according to Bob. Since Bob could use this clock to time any kind of a physical process on the train (such as how fast a regular clock runs, or his heartbeat, or the rate at which radioactive particles decay), all processes on the train seem to run slower to Alice. This is known as ***time dilation***: clocks (and all other processes) that are moving relative to an observer appear to run slow to that observer.

We can calculate how much slower than normal Bob's time appears to run. The length of the path of light according to Alice is

$$l = 2\sqrt{d^2 + (v\Delta t/2)^2} = \sqrt{4d^2 + v^2\Delta t^2}$$

The amount of time it takes the light to cover this path is

$$\Delta t = \frac{l}{c} = \frac{\sqrt{4d^2 + v^2\Delta t^2}}{c}$$

Square both sides and solve for Δt:

$$\Delta t^2 = \frac{4d^2 + v^2\Delta t^2}{c^2}$$

$$(c^2 - v^2)\Delta t^2 = 4d^2$$

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d/c}{\sqrt{1 - v^2/c^2}}$$

Finally, note that $2d / c = \Delta t_0$, the time that elapses according to Bob. Therefore,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$$

Again, we can see that more time elapses for Alice than for Bob for the same process occurring on the train, and therefore processes on the train appear to run slowly.

The following functions of velocity appear often in relativity, and therefore get their own symbols:

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

We can write our equation for time dilation as

$$\Delta t = \gamma \Delta t_0$$

The time Δt_0, which is the amount of time that passes according to the clock's own reference frame, is known as the ***proper time***.

Example: An unstable elementary particle called the muon has an average lifetime of $2.2\mu s$ before decaying into other particles. This lifetime is in a frame in which the muon is at rest. What is the lifetime of a muon moving with a speed of $0.5c$? $0.99c$? If muons with these speeds are produced in a nuclear reaction, how far, on average, will they travel before decaying?

At 0.5c, we have, for the muon lifetime,

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{2.2\mu s}{\sqrt{1 - 0.5^2}} = 2.54\mu s$$

At 0.99c, the muon lifetime becomes

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{2.2\mu s}{\sqrt{1 - 0.99^2}} = 15.6\mu s$$

You can see that the time dilation is fairly modest even for a particle moving at half the speed of light; the lifetime of such a particle is only slightly longer than that of a particle at rest. However, for a particle moving at 99% the speed of light, the lifetime is considerably larger than for a resting particle.

The average range prior to decay of a muon moving at 0.5c is

$$r = \tau v = (2.54 \times 10^{-6} s)(0.5 \times 3 \times 10^8 m/s) = 380m$$

The average range of a muon moving at 0.99c is

$$r = \tau v = (15.6 \times 10^{-6} s)(0.99 \times 3 \times 10^8 m/s) = 4600m$$

This is more than 10 times the range of the slower muon, even though the speed is only twice as much. This is because the faster-moving muon has a much longer lifetime, due to time dilation.

Muons are produced in the upper atmosphere by an interaction of solar radiation and cosmic rays with air molecules. We can see that without time dilation, muons from the upper atmosphere would not be able to reach the surface. However, fast-moving muons are detected on the surface, thus providing evidence for time dilation and for the theory of relativity.

Length Contraction

Suppose now that Bob is in his high-speed train, moving at speed v, traveling in a straight line from San Diego to San Francisco. His clock measures that an amount of time t_0 has passed, and Bob therefore concludes that the distance between San Diego and San Francisco is $x = vt_0$. Note that t_0 is the proper time, but x is not the proper distance, since this is a distance measured according to Bob, in a frame where San Francisco and San Diego are not at rest. This is why x does not get a 0 subscript.

The ***proper distance*** between the two cities is the distance according to Alice, who is having a beer at Porters Pub, at rest with respect to both San Diego and San Francisco. She sees Bob ride the train at speed v, but due to time dilation, according to Alice, he takes a longer time $t = \gamma t_0$ to get from one place to the other. Alice therefore concludes that the distance between San Diego and San Francisco is $x_0 = vt = \gamma v t_0 = \gamma x$. Again, this is the proper distance, because Alice is at rest with respect to the points between which the distance is measured. Thus, if the proper distance is x_0, the distance according to an observer moving relative to the object at speed v is

$$x = \frac{x_0}{\gamma} = x_0\sqrt{1 - v^2/c^2}$$

Unlike time, which increases compared to the proper time with motion, the distance decreases. It can be shown that only the component of the distance parallel to the motion is decreased; perpendicular distances are unaffected. This is known as ***length contraction***.

Example: A spaceship is flying past an asteroid at a speed of 0.8c. An observer on the asteroid marks the time that elapses between when the nose of the ship passes the asteroid, and when the tail of the ship passes, and finds that this time is 1.6 μs. What is the length of the spaceship according to the crew on board?

The length of the spaceship according to the observer on the asteroid is

$$l = vt = (0.8 \times 3 \times 10^8 m/s)(1.6 \times 10^{-6} s) = 380m$$

This length is contracted compared to the proper length, so the length according to the spaceship's crew is longer:

$$l = l_0\sqrt{1 - v^2/c^2}$$

$$l_0 = \frac{380m}{\sqrt{1 - 0.8^2}} = 630m$$

The spaceship is thus 630m long, not just 380m as the observer might think if he didn't know about relativity.

If the ship was moving past the asteroid at 0.9c, what length would the observer measure?

At 0.9c, the length measured by the observer on the asteroid would be

$$l = l_0\sqrt{1 - v^2/c^2} = 630m\sqrt{1 - 0.9^2} = 275m$$

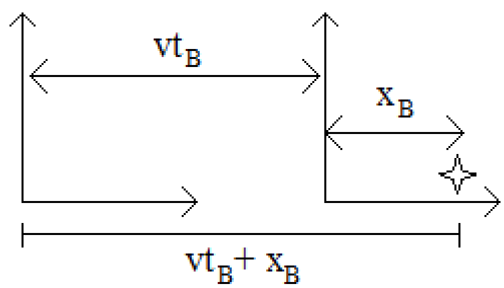
The observer would see the ship even more contracted than at 0.8c.

Lorentz Transformations

Suppose that Bob passes Alice at a relative speed v, and the moment they pass each other, they decide to measure the distance x along Bob's direction of travel from the point where they met, and the time t from the moment they met. In other words, the origins of their x-axes and t-axes coincide at the meeting point. Of course, later on, the origins of their x-axes and t-axes are no longer the same.

Suppose Bob observes an event (say, an explosion) at position x_B relative to himself, after an amount of time t_B passed according to his clock. At what position and time does Alice observe this event?

In Bob's frame, Alice is now a distance vt_B to the left of Bob, and the event occurs a distance x_B to the right of Bob. Thus, Bob sees the distance between the event and Alice as $vt_B + x_B$:



However, Bob is moving relative to Alice, so this distance is not the proper distance. According to Alice, the distance is larger by a factor of γ:

$$x_A = \gamma\left(x_B + vt_B\right)$$

If Bob is moving with velocity v relative to Alice, then Alice is moving with velocity $-v$ relative to Bob. Therefore, we can write the same equation for translating positions of events from Alice's frame to Bob's frame, by replacing v with $-v$. Note that this doesn't change the value of γ, since γ depends on v^2:

$$x_B = \gamma\left(x_A - vt_A\right)$$

Now, if the event occurs at time t_B in Bob's frame, at what time does it occur in Alice's frame? We can plug the expression for x_A into the equation above and solve for t_A:

$$x_B = \gamma\left(x_A - vt_A\right) = \gamma\left(\gamma(x_B + vt_B) - vt_A\right)$$

$$\gamma vt_A = \gamma^2 x_B + \gamma^2 vt_B - x_B = \left(\gamma^2 - 1\right)x_B + \gamma^2 vt_B$$

$$\left(\gamma^2 - 1\right) = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2\frac{v^2}{c^2}$$

$$\gamma vt_A = \gamma^2\frac{v^2}{c^2}x_B + \gamma^2 vt_B$$

$$t_A = \gamma\left(t_B + \frac{v}{c^2}x_B\right)$$

The coordinates perpendicular to the direction of motion are the same for Bob and for Alice. Thus we have the following “dictionary” for translating coordinates and times of events from Bob's frame to Alice's frame and vice versa:

$$x_A = \gamma\left(x_B + vt_B\right) \qquad x_B = \gamma\left(x_A - vt_A\right)$$

$$t_A = \gamma\left(t_B + \frac{v}{c^2}x_B\right) \qquad t_B = \gamma\left(t_A - \frac{v}{c^2}x_A\right)$$

$$. \qquad y_A = y_B \qquad z_A = z_B$$

These equations, relating the position and time of an event as measured by one observer to that as measured by a different observer, are known as the ***Lorentz transformation.***

Velocity Addition

Suppose there is an object that is moving with speed v_B relative to Bob, while Bob is moving with speed v relative to Alice. What is the speed of the object, v_A, relative to Alice?

Without relativity, this would be simply $v_A = v + v_B$. However, this cannot be true, as this kind of equation does not keep the speed of light constant between observers. We now derive the relativistically correct way of adding velocities.

Suppose as the time according to Bob, t_B, changes by an amount dt_B, the position of the object according to Bob, x_B, changes by an amount dx_B. The speed according to Bob is then simply $v_B = dx_B / dt_B$. The speed according to Alice is

$$v_A = \frac{dx_A}{dt_A} = \frac{\gamma(dx_B + vdt_B)}{\gamma(dt_B + (v/c^2)dx_B)} = \frac{dx_B/dt_B + v}{1 + (v/c^2)dx_B/dt_B} = \frac{v_B + v}{1 + v_B v/c^2}$$

Notice that for v_B and v small, the denominator is almost 1, and the familiar non-relativistic relationship, $v_A = v_B + v$, is recovered. What if Bob is observing a ray of light, which he sees moving at speed $v_B = c$? Then,

$$v_A = \frac{c+v}{1+v/c} = c\frac{c+v}{c+v} = c$$

As required by relativity, if Bob sees the light moving at speed c, so does Alice, even though she is moving relative to Bob.

Example: Alice sees a muon fly by at a speed of 0.98c, but Bob, who is chasing after the muon in his high-speed train, sees it moving overtaking him at a speed of only 0.7c. What is Bob's speed relative to Alice?

By the velocity addition rule,

$$v_A = \frac{v_B + v}{1 + v_B v/c^2} \qquad 0.98c = \frac{0.70c + v}{1 + 0.70v/c}$$

$$0.98c + 0.70 \cdot 0.98v = 0.70c + v \qquad (1 - 0.70 \cdot 0.98)v = (0.98 - 0.70)c$$

$$v = \frac{0.28}{0.314}c = 0.89c$$

Bob is thus moving at a speed of 0.89c.

Generalization of Lorentz transformations, 4-vectors

The Lorentz transformations relate the coordinates and time of an event according to one observer to the coordinates and times of the same event according to another observer. We can write the time and coordinates of an event as a 4-component object, known as a ***4-vector***:

$$(ct, x, y, z) \equiv \left(x^0, x^1, x^2, x^3\right)$$

x^0 is known as the time component of the 4-vector, and x^i $(i = 1, 2, 3)$ are known as the space components. Note that the superscripts simply label which component of the 4-vector we are talking about; we are not raising anything to any power. In this notation, the components of the 4-vector, which we will denote by the capital letter X, transform according to the Lorentz transformation as follows:

$$x_A^0 = \gamma\left(x_B^0 + \beta x_B^1\right) \qquad x_B^0 = \gamma\left(x_A^0 - \beta x_A^1\right)$$

$$x_A^1 = \gamma\left(x_B^1 + \beta x_B^0\right) \qquad x_B^1 = \gamma\left(x_A^1 - \beta x_A^0\right)$$

$$x_A^2 = x_B^2 \qquad x_A^3 = x_B^3$$

We used $\beta = v / c$ for simplicity of notation. We will see later on that there are other quantities (such as the 4-velocity, the frequency plus wave number, and the energy plus momentum) that transform from one observer to another in exactly the same way.

Consider the following quantity composed from components of a 4-vector:

$$s^2 = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2$$

Note the similarity to the length of a regular 3-vector from the Pythagorean theorem, except that there are 4 components rather than 3, and there is an additional minus sign between the time component and the space components.

According to observer *A*, this quantity is equal to

$$s_A^2 = \left(x_A^0\right)^2 - \left(x_A^1\right)^2 - \left(x_A^2\right)^2 - \left(x_A^3\right)^2$$

According to observer *B*, it is equal to

$$s_B^2 = \left(x_B^0\right)^2 - \left(x_B^1\right)^2 - \left(x_B^2\right)^2 - \left(x_B^3\right)^2$$

We will show that these two are in fact equal, so that s^2 is an absolute quantity, the same for all observers. Start with the expression according to *A*, and change all the coordinates to those in *B*'s frame by using the Lorentz transformation:

$$s_A^2 = \gamma^2 \left(x_B^0 + \beta x_B^1\right)^2 - \gamma^2 \left(x_B^1 + \beta x_B^0\right)^2 - x_B^2 - x_B^3 =$$

$$= \gamma^2 \left[\left(x_B^0\right)^2 + 2\beta x_B^0 x_B^1 + \beta^2 \left(x_B^1\right)^2 - \left(x_B^1\right)^2 - 2\beta x_B^0 x_B^1 - \beta^2 \left(x_B^0\right)^2\right] - x_B^2 - x_B^3 =$$

$$= \gamma^2 \left(1 - \beta^2\right) \left(x_B^0\right)^2 - \gamma^2 \left(1 - \beta^2\right) \left(x_B^1\right)^2 - \left(x_B^2\right)^2 - \left(x_B^3\right)^2$$

$$\gamma^2 \left(1 - \beta^2\right) = \frac{1 - \beta^2}{1 - \beta^2} = 1$$

$$s_A^2 = \left(x_B^0\right)^2 - \left(x_B^1\right)^2 - \left(x_B^2\right)^2 - \left(x_B^3\right)^2 = s_B^2$$

Thus the value of s^2 is independent of the observer. Quantities such as this, which are the same when measured by any observer, are known as ***4-scalars***. Other examples of 4-scalars include the total number of particles inside an object, proper length and proper time (note that while the length and time depend on the observer measuring them, the *proper* length and time are intrinsic properties of the object or process itself, and are thus independent of any observer.)

Generalizing to any kind of 4-vector (not just coordinates and time of an event), we define the ***invariant length*** of the 4-vector in the same way:

$$V = \left(v^0, v^1, v^2, v^3\right) \equiv \left(v^0, \vec{v}\right)$$

$$V^2 = \left(v^0\right)^2 - \left(v^1\right)^2 - \left(v^2\right)^2 - \left(v^3\right)^2 = \left(v^0\right)^2 - \vec{v}\cdot\vec{v}$$

If V is a 4-vector, then V^2 is a 4-scalar (the same for all observers). This is exactly analogous to the behavior of vectors in 3D with respect to rotation of the coordinate system: the components of the vector change depending on the coordinates chosen, but the length of the vector remains the same.

Similarly, we define an ***invariant dot product*** between two 4-vectors U and V:

$$U \cdot V \equiv u^0 v^0 - \vec{u}\cdot\vec{v}$$

Since U and V are both 4-vectors, they transform the same way under Lorentz transformation. Therefore the dot product above is independent of any observer and is a 4-scalar, just like the invariant length of a 4-vector.

We will make use of 4-vectors when we discuss the relativistic description of waves, and the relationship between energy, momentum and mass.