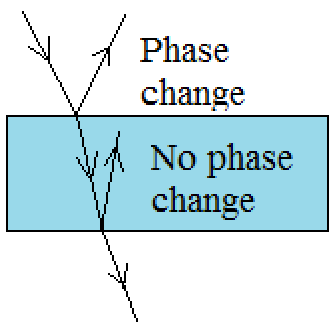
Lecture 10 Notes: 07 / 14

Reflections and thin-film interference

Instead of sending light through slits to interfere with itself, we can get interference between a source of light and its reflection, or two different reflections. First, we must understand how the phase of electromagnetic waves changes when they are reflected from an interface.

Recall that for waves on a string, when a wave was reflected from an interface with a string that had a slower wave speed, the amplitude of the reflected wave was inverted. This corresponds to a phase change of π. When the incident wave encountered an interface with a string that had a higher wave speed, the amplitude of the reflected wave had the same sign as that of the incident wave. This corresponds to no phase change.

Electromagnetic waves behave the same way. When they reflect from a material with a higher index of refraction (lower wave speed) than the medium the wave is traveling in, they change phase by π, but when they reflect from a material with a lower index of refraction, they do not change phase. The part of the wave that is transmitted into the new material never changes phase.

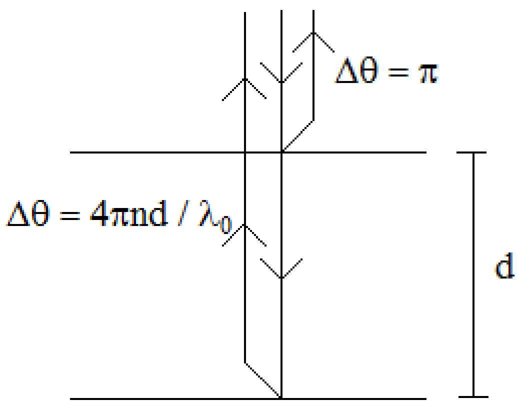


Example: A plane wave is incident on a thin film of material with n = 1.36, with air on both sides of the film. The film is 1.00μm thick. The direction of travel of the plane wave is normal to the film. For what wavelengths of the plane wave is the reflection enhanced by constructive interference? For what wavelengths is the reflection suppressed? Which of these wavelengths are visible light?

Let the thickness of the film be d = 1.00μm. Light that is reflected off the front surface of the film undergoes a phase change of π. Light that is reflected off the back surface of the film undergoes no phase change, but it has an additional distance of 2.00μm to travel. Over this distance, the light accumulates a phase at a rate of 2π per wavelength, except the wavelength in the material is λ_0 / n, where λ_0 is the wavelength of the incident wave in air. This because the wave number $k = \omega / v$ is inversely proportional to the wave speed, and thus proportional to the index of refraction (since ω is the same on both sides). $\lambda = 2\pi / k$ is inversely proportional to k, and therefore is inversely proportional to the index of refraction.

Thus the additional phase accumulated by the wave that reflects off the back surface, due to its greater traveling distance, is $4\pi d / \lambda = 4\pi n d / \lambda_0$.

This is a schematic diagram of the situation. The collinear rays have been separated a bit to show which one is which.



The difference in phase between the two rays is $4\pi n d / \lambda_0 - \pi$. For constructive interference, we must have the difference in phase equal to an even multiple of π, while for destructive interference, it must be an odd multiple of π:

$$\frac{4\pi n d}{\lambda_0} - \pi = 2j\pi \qquad \text{Constructive}$$
$$\frac{4\pi n d}{\lambda_0} - \pi = (2j-1)\pi \qquad \text{Destructive}$$

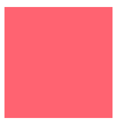
Simplifying and solving for λ_0, we obtain

$$\lambda_0 = \frac{4nd}{2j+1} \qquad \text{Constructive}$$
$$\lambda_0 = \frac{2nd}{j} \qquad \text{Destructive}$$

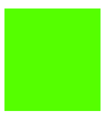
Running through a few values of j, we get constructive interference for wavelengths of 5440*nm*, 1813*nm*, 1088*nm*, 777*nm*, 605*nm*, 494*nm*, 418*nm*, 362*nm*, 320*nm*, etc. The first four values are near infrared, while the last two and the following ones are ultraviolet; the visible wavelengths for which we see constructive interference are 605, 494 and 418 *nm*. We see destructive interference for 2720, 1360, 907, 680, 554, 453, 389 and 340 *nm*; of those, 680, 554, 453 and possibly 389*nm* are visible.

If white light, which is a mixture of all wavelengths, is incident on this film, the reflection will be bright in the 605, 494 and 418*nm* visible bands, and dark in the intermediate parts of the spectrum. The combination of light with these three wavelengths will produce a specific color. The wavelengths for which there is constructive interference will depend on the thickness *d* of the film, and so will the color of the reflected light. This explains the multicolored appearance of soap bubbles and oil slicks on the surface of the water: the thickness of the film making up the wall of the soap bubble or the oil floating on the water varies from point to point, and so does the color of the reflected light.

In case you want to know, the color reflected from this particular thickness of film would look approximately like this. This was done by generating the spectrum of the reflection (intensity as a function of wavelength) and then using a spectrum to RGB converter.

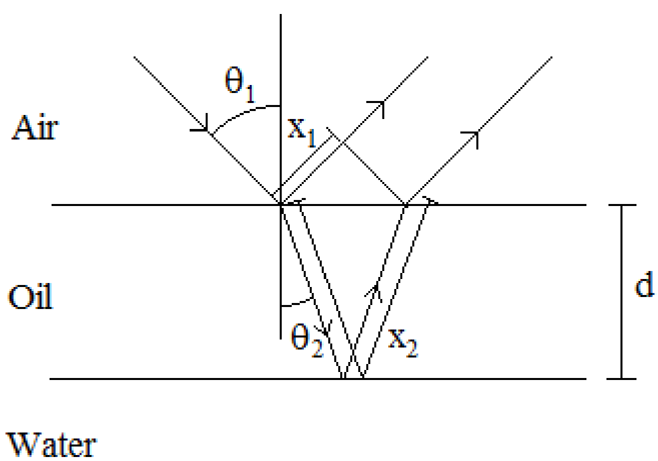


If the thickness of the film was instead 500*nm*, the color would look like this:

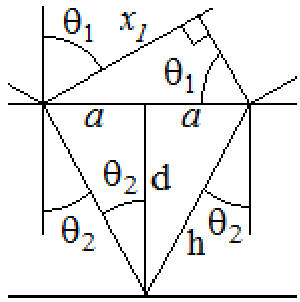


Example: Light of wavelength λ_0 = 500*nm* is reflected off a film of oil on water. The oil has an index of refraction of 1.20, while the water has an index of refraction of 1.33. If the thickness of the oil layer is 600 *nm*, at what angles of reflection is there constructive interference?

Now there is a phase change of π when the light reflects off both interfaces, so that phase change will be the same for both rays and will cancel. However, there will be additional path differences to take into account due to the geometry:



The beam reflected off the top of the oil film travels an additional distance x_1 through the air, compared to the other beam. However, the beam reflected off the oil-water interface travels an additional distance x_2 through the oil. To determine x_1, look at a close-up of a part of the diagram and draw some additional lines:



From this diagram, $x_1 = 2a \sin \theta_1$. a is given by $a / d = \tan \theta_2$, or $a = d \tan \theta_2$. Therefore, $x_1 = 2d \sin \theta_1 \tan \theta_2$.

Next, we determine x_2. This is just the total path length of the ray that enters the oil and is reflected by the surface of the water. If we label one side of the path as h (see diagram above), then $x_2 = 2h$. h is related to d by $d = h \cos \theta_2$, so $h = d / \cos \theta_2$. Therefore, $x_2 = 2d / \cos \theta_2$.

The phase accumulated by the first ray is $\Delta\theta_1 = 2\pi x_1 / \lambda_0$. The phase accumulated by the second ray is $\Delta\theta_1 = 2\pi x_2 / \lambda = 2\pi n x_2 / \lambda_0$, where n is the index of refraction of the oil (the factor of n appears because the wavelength is decreased by a factor of the index of refraction). Therefore the difference in phase is

$$\Delta\theta_2 - \Delta\theta_1 = \frac{2\pi(nx_2 - x_1)}{\lambda_0} = \frac{2\pi}{\lambda_0}\left(\frac{2nd}{\cos\theta_2} - 2d\sin\theta_1\tan\theta_2\right) =$$

$$= \frac{4\pi d}{\lambda_0}\frac{1}{\cos\theta_2}\left(n - \sin\theta_1\sin\theta_2\right)$$

Now we use Snell's law. This tells us that $\sin\theta_2 = \sin \theta_1 / n$. Writing the cosine in terms of sine as well, we obtain

$$\Delta\theta_2 - \Delta\theta_1 = \frac{4\pi d}{\lambda_0}\frac{1}{\sqrt{1-\sin^2\theta_1/n^2}}\left(n - \sin^2\theta_1/n\right) = \frac{4\pi n d}{\lambda_0}\frac{1-\sin^2\theta_1/n^2}{\sqrt{1-\sin^2\theta_1/n^2}} =$$

$$= \frac{4\pi d}{\lambda_0}\sqrt{n^2 - \sin^2\theta_1}$$

To get constructive interference, the phase difference must be an even multiple of π:

$$\frac{4\pi d}{\lambda_0}\sqrt{n^2 - \sin^2\theta_1} = 2j\pi$$

Simplifying and solving for the angle gives

$$\frac{2d}{\lambda_0}\sqrt{n^2 - \sin^2\theta_1} = j$$

$$n^2 - \sin^2\theta_1 = \frac{j^2\lambda_0^2}{4d^2}$$

$$\sin\theta_1 = \sqrt{n^2 - \frac{j^2\lambda_0^2}{4d^2}} = \sqrt{1.440 - 0.174j^2}$$

Choosing *j* = 0 or *j* = 1 gives the sine as greater than one, while choosing *j* = 3 or higher gives a negative number under the square root. Only *j* = 2 works. Thus, the only angle for which we get constructive interference is

$$\sin\theta_1 = \sqrt{1.440 - 0.174 \times 4} = 0.863$$

$$\theta_1 = 59.7^\circ$$

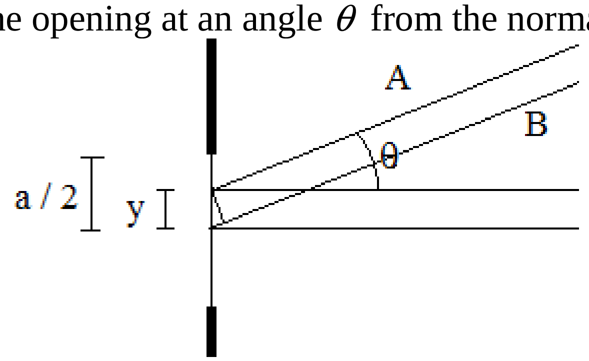
Thus, with this thickness of film and wavelength of light, we only get total constructive interference if the incident angle is approximately 60 degrees to the surface normal.

Single-slit diffraction

Diffraction is the spreading of light as it passes through a small opening or around a small obstacle. It is not possible for a beam of electromagnetic waves to have a very sharp edge (sharp compared to the scale of the wavelength) as such a wave configuration would not be a solution to the wave equation.

We will first consider the spreading of light emerging through an opening. For the double-slit experiment, we assumed that the light simply went from the opening in all directions; this is true if the opening is very small. For larger openings, the spreading will not be so drastic.

Consider an opening of size *a*, running from $y = -a / 2$ to $y = a / 2$. Consider the light coming from point *y* in the opening at an angle θ from the normal to the screen:



The path length difference between this beam, A and the beam B, which emerges from the center of the opening, is $y \sin \theta$. The phase difference compared to the central beam is thus $2\pi y \sin \theta / \lambda$. The total value the wave on the screen is proportional to the sum of the contributions for all different values of *y*, from *-a / 2* to *a / 2*:

$$E(\theta, t) \propto \int_{-a/2}^{a/2} dy \cos\left(\omega t + \frac{2\pi \sin\theta}{\lambda} y\right) =$$
$$\frac{\lambda}{\pi \sin\theta} \sin\left(\frac{\pi a \sin\theta}{\lambda}\right) \cos(\omega t)$$

The electric field at angle θ thus oscillates with an amplitude proportional to

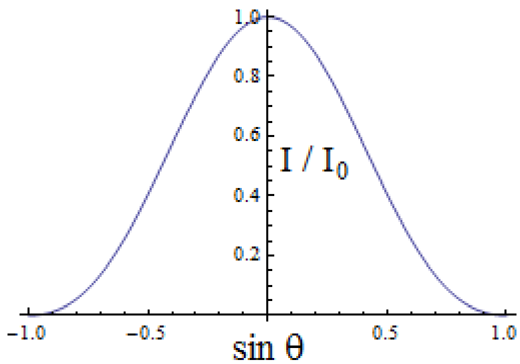
$$A(\theta) = \frac{\lambda}{\pi \sin\theta} \sin\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

The intensity of light is proportional to the square of the amplitude:

$$I(\theta) \propto \frac{1}{\sin^2\theta} \sin^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$$

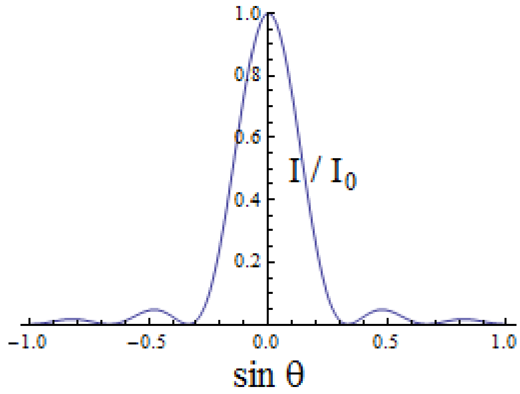
Since this is a proportionality relationship, let us define the intensity such that the intensity at the center is zero. The limit of this function at $\sin \theta = 0$ is $\pi^2 a^2 / \lambda^2$, so we will simply divide the function by this quantity. We'll call this the relative intensity (intensity relative to the intensity directly opposite the center of the hole).

For a hole the size of the wavelength, $a = \lambda$, the intensity as a function of $\sin \theta$ looks like this:

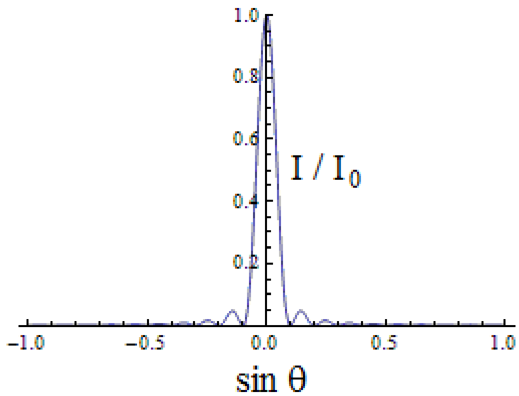


The light is peaked to the front of the hole, but a lot of it goes out sideways, as we expect, since the size of the opening is comparable to the wavelength.

Now we will try a hole 3 wavelengths across. The graph now looks like this:



There is now a sharp forward peak, and additional ***diffraction peaks*** have developed to the sides of the forward peak. Finally, try a hole that is 10 wavelengths across:



The central peak is now even sharper. The diffraction peaks are smaller and closer together. In the limit that the hole becomes much larger than the wavelength, the peak becomes infinitely sharp, and all the light goes precisely forward through the hole. This is why we can't see around the corner of a door; the door is much, much larger than the wavelength of light, so all light that enters the door goes straight through, without diffracting off to the side by any measurable amount. Sound, however, has a wavelength comparable to the size of the door, so it quite readily diffracts off to the side. We can thus hear sounds behind an open door even if we cannot see the source of the sound.

The intensity is exactly zero when

$$\frac{\pi a \sin\theta}{\lambda} = n\pi$$

Note that $n = 0$ doesn't count since that's the location of the central peak, and the limit of the intensity there is finite. Simplifying this expression a bit,

$$a \sin\theta = n\lambda$$

Example: A laser with $\lambda = 620nm$ produces a beam 1*mm* across. The beam is projected on a wall 20 meters away. The beam spreads due to diffraction. What is the diameter of the beam's central peak when it hits the wall?

Treat the laser beam as light going through a hole 1*mm* across. The beam will thus have a central peak, and some diffraction rings around it (they are rings because the beam is symmetric about the central axis). The central peak is delineated by the first diffraction minimum, which occurs when $a \sin \theta = \lambda$. Let D be the diameter of the beam at the wall. Then,

$$\sin\theta = \frac{\lambda}{a} = \frac{6.20 \times 10^{-7}m}{10^{-3}m} = 6.20 \times 10^{-4} \approx \theta$$

$$D \approx R\theta = 20m \times 6.20 \times 10^{-4} = 0.0124m = 12.4mm$$

In just 20 meters, the laser beam has spread to more than 12 times its original diameter. This illustrates the impossibility of sending a narrow beam over a long distance without the beam spreading out due to diffraction.

It turns out that coherent light encountering a small obstacle (such as a hair) will diffract around it, producing much the same diffraction pattern as with light going through a small opening.

Example: Suppose we illuminate a hair with laser light having a wavelength of 620 *nm*. The diffraction pattern is projected onto a screen 2 meters away. The difference between the first diffraction minimum on the left of the central peak and the first diffraction minimum on the right is measured to be 0.5*cm*. What is the diameter of the hair?

The distance from the center to the nearest diffraction minimum is 0.5 / 2 = 0.25*cm*. This corresponds to $n = 1$. The angle is

$$\sin\theta \approx \theta \approx \frac{0.25cm}{200cm} = 0.00125$$

(since the angle is very small, it is almost exactly equal to the distance between the two points divided by the distance from the screen to the source, and the sine of the angle is almost exactly equal to the angle itself.)

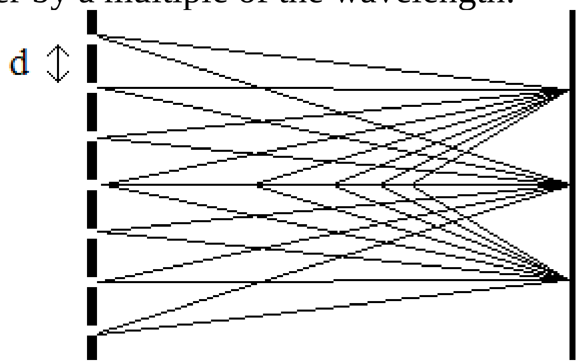
Now we solve for a:

$$a = \frac{n\lambda}{\sin\theta} = \frac{1 \times 6.2 \times 10^{-7}}{0.00125} = 4.96 \times 10^{-4}m \approx 0.50mm$$

The hair is thus half a millimeter across.

Diffraction grating

A diffraction grating is a commonly used device consisting of a large number of evenly-spaced interference slits. The individual slits are narrow compared to the separation. The diffraction grating acts much like the double-slit interference apparatus, with constructive interference arising when the path length distances from all the slits are different from each other by a multiple of the wavelength:



The rays shown illustrate points of constructive interference. It can be demonstrated that for small diffraction angles and a screen sufficiently far away, the conditions for constructive and destructive interference are precisely the same as for the double-slit experiment:

$$d \sin\theta = n\lambda \qquad \text{Constructive}$$

$$d \sin\theta = \left(n + \frac{1}{2}\right)\lambda \qquad \text{Destructive}$$

Here, *d* is the slit separation.

Example: 450*nm* light is sent through a diffraction grating to a screen 5 meters away. The separation between the diffraction peaks on the screen is 6.5*cm*. How many slits per centimeter does the diffraction grating have?

$$\sin\theta \approx \theta \approx \frac{6.5cm}{500cm} = 0.013$$

$$d = \frac{\lambda}{\sin\theta} = \frac{4.20 \times 10^{-7}m}{0.013} = 3.23 \times 10^{-5}m = 3.23 \times 10^{-3}cm$$

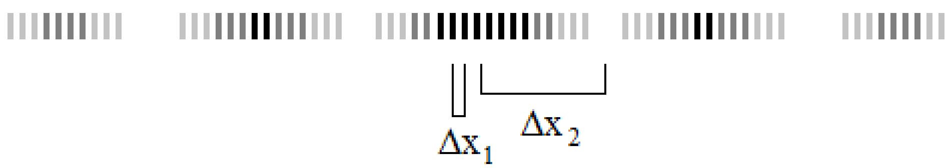
The grating thus has 1*cm* / *d* = 310 slits per centimeter.

In reality, the slits in the diffraction grating have a finite size, so we get both diffraction between the slits and single-slit diffraction from each individual slit. The resulting pattern looks something like this:

This is a negative, with black representing bright light and white representing darkness. There is a bright, wide central cluster of diffraction peaks. The individual diffraction peaks within the clusters are caused by diffraction between the different slits, while the clusters themselves are peaks resulting from single-slit diffraction. Because the separation of diffraction peaks is inversely proportional to the size of the feature on the screen they correspond to, and the slits are farther from each other than they are wide, the multiple-slit pattern is closer spaced than the single-slit pattern.

Example: 450*nm* light is sent through a diffraction grating to a screen 50 *cm* away. The separation between the diffraction fringes on the screen is 0.40*cm*. The separation between the middle of the large central cluster and the first spot where the diffraction peaks vanish is 5.0*cm*. How far apart are the slits separated? How many slits per centimeter is that? What is the width of each individual slit?

Let Δx_1 be the separation between the diffraction fringes on the screen and Δx_2 be the separation between the center of the pattern and the first spot where the fringes disappear:



We use many-slit diffraction to determine the separation between the slits:

$$d = \frac{\lambda}{\sin\theta} \approx \frac{\lambda R}{\Delta x_1} = \frac{4.50 \times 10^{-5}cm \times 50cm}{0.40cm} = 0.00563cm = 56.3\mu m$$

The slits are separated by 56.3 microns. That's about 178 slits per centimeter.

Use single-slit diffraction to determine the width of each slit:

$$a = \frac{\lambda}{\sin\theta_2} \approx \frac{\lambda R}{\Delta x_2} = \frac{4.50 \times 10^{-5}cm \times 50cm}{5.0cm} = 4.5 \times 10^{-4}cm = 4.5\mu m$$

The slits are 4.5 microns across.

We have seen that diffraction is quite useful for measuring the size of small objects. The smaller the wavelength of electromagnetic radiation used, the smaller the size of objects that can be usefully measured. Since x-rays have wavelengths suitable to the size of molecules, ***x-ray diffraction*** is often used to determine the structure of crystals.

If the crystal is made of complicated things such as protein molecules, the resulting pattern will be complicated as well, but the equations describing the interference that produces the pattern can be reversed, and the detailed structure of the molecule can be calculated from the pattern. This process is actually not that different from what we did above to calculate the slit separation and the width of each slit.