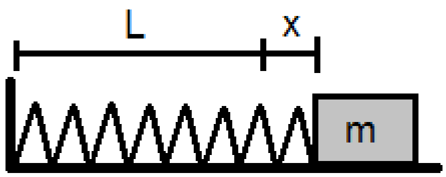
Lecture 1 Notes: 06 / 27

The first part of this class will primarily cover oscillating systems (harmonic oscillators and waves).

These systems are very common in nature - a system displaced from equilibrium by a small amount will tend to oscillate harmonically, unless the friction is too high for oscillations to take place.

Mass on a spring and Hooke's Law

Simple system: Mass on a spring. Consider a spring that has a relaxed length L, attached to a mass on a frictionless table:



Let x be the distance by which the spring is stretched or compressed from its relaxed length. It's positive if the spring is stretched, and negative if it is compressed. If x is sufficiently small (compared to L), we find that the force is proportional to the displacement:

$$F = -k\,x$$

This is Hooke's law. The sign is negative because if the mass is displaced to the right, the force tries to return it back to the left, towards the equilibrium position, and vice versa. For this reason, the force is called the ***restoring force***. The proportionality constant is called the ***spring constant***, and measures the stiffness of the spring.

Potential energy of the spring: Since $F = -dU / dx$, the potential energy is

$$U = \frac{1}{2}kx^2$$

The total energy is the kinetic energy plus the potential energy:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Systems that obey Hooke's Law are called ***harmonic oscillators*****.**

Why Hooke's Law?

Why does the spring (and many other systems in nature) follow such a simple force law? There is a mathematical reason.

In general, the force F is a complicated function of the displacement x. At the equilibrium position, $x = 0$, the force is zero. For small displacements from equilibrium, we can fit any F with a polynomial in x (this is known as a Taylor series, you should have seen it in your calculus class.)

$$F = A + Bx + Cx^2 + Dx^3 + \ldots$$

Since $F(0) = 0$ (equilibrium), $A = 0$. Therefore,

$$F = Bx + Cx^2 + Dx^3 + \ldots$$

Now, assume that the displacement x is small. If that is the case, higher powers of x will be smaller still, and we can neglect them to a good approximation. We therefore end up with the approximate force

$$F \approx Bx$$

Now if B is positive, the force will push in the direction of the displacement, and the mass will accelerate away from the equilibrium point. This is called ***unstable equilibrium*** (an example is a ball balanced on top of a hill). If B is negative, the force will try restore the system to the equilibrium point, and we have a ***stable equilibrium***. This is the case we're interested in, so we will make B negative and define $B = -k$, so that k is positive. This gives us Hooke's Law:

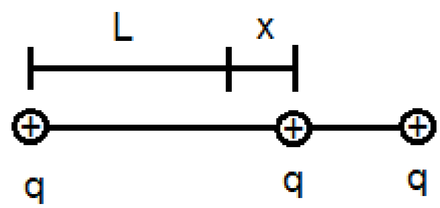
$$F = -kx$$

This discussion makes clear a few points:

(1) Hooke's Law is an approximation, there are corrections from higher powers of x
(2) If the displacement becomes big enough, Hooke's Law is no longer valid
(3) It's fairly universal - many systems near equilibrium obey Hooke's Law

Example of how Hooke's Law arises for a simple system:

Consider a charge $+q$, free to move in on a line segment of length $2L$, trapped between two other charges $+q$ that are held in place at the ends of the segment:



The equilibrium position is clearly right in the middle. In general, the force on the central charge is the sum of the electrostatic repulsive forces from the charges on both sides:

$$F = \frac{k_e q^2}{(L+x)^2} - \frac{k_e q^2}{(L-x)^2}$$

This, of course, looks nothing like Hooke's Law. However, for $x << L$, we can use the binomial theorem (or a Taylor expansion) as follows:

$$\frac{1}{(L+x)^2} = \frac{1}{L^2}\frac{1}{(1+x/L)^2} \approx \frac{1}{L^2}\left(1 - \frac{2x}{L}\right)$$
$$\frac{1}{(L-x)^2} \approx \frac{1}{L^2}\left(1 + \frac{2x}{L}\right)$$

Plugging this into the equation for the force, we obtain

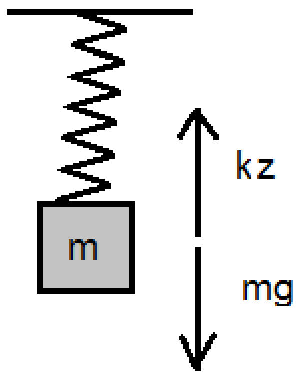
$$F \approx \frac{k_e q^2}{L^2}\left(\left(1 - \frac{2x}{L}\right) - \left(1 + \frac{2x}{L}\right)\right)$$
$$F \approx -\frac{4k_e q^2}{L^3}x$$

This is exactly the same as Hooke's Law, with an effective “spring constant” of

$$k = \frac{4k_e q^2}{L^3}$$

Mass Suspended from a Spring:

Consider a slightly more complicated spring system, where the mass hangs under gravity:



The equilibrium position is one where the net force is zero, so

$$z_0 = \frac{mg}{k}$$

This means the spring constant can be measured by measuring how much the spring stretches with a given mass. Suppose a 0.1kg mass stretches the spring by 5 cm. Then, the spring constant is

$$k = \frac{mg}{z_0} = \frac{0.1kg \times 9.8m/s^2}{0.05m} = 19.6kg/s^2$$

Hooke's Law: Now that we've found the new equilibrium position, let us define $x = z - z_0$, so that x measures the displacement from the new equilibrium. Plugging this into our equation for the force:

$$F = -kz + mg = -k(x + z_0) + mg = -kx - k\frac{mg}{k} + mg = -kx$$

$$F = -kx$$

The spring constant is thus unchanged. The only effect that a constant force like gravity has is shifting the equilibrium position.

We can look at the potential energy of this system, as well:

$$
\begin{aligned}
U = \frac{1}{2}kz^2 - mgz = \frac{1}{2}k\left(x + z_0\right)^2 - mg(x + z_0) = \\
= \frac{1}{2}kx^2 + (kz_0 - mg)x + \frac{1}{2}kz_0^2 - mgz_0 = \\
= \frac{1}{2}kx^2 - \frac{m^2g^2}{2k^2}
\end{aligned}
$$

The last term is just a constant. Recall that adding a constant to the potential energy doesn't change the physics, so we can simply drop the last term and get

$$U = \frac{1}{2}kx^2$$

Kinematics of Harmonic Oscillators

Now that we know how to get the force law for systems near equilibrium, we can write down and solve the equations of motion. From Newton's second law,

$$ma = F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

This is a *differential equation* - an equation relating *x* to its derivatives. We don't expect you to know how to solve this kind of equation in general (it's covered in a second-year math class), but this one is simple enough that we can easily guess the answer. What kind of function gives something proportional to minus itself when you take the derivative twice? A sine or a cosine. Therefore, we guess the following solution:

$$
\begin{aligned}
x(t) &= Acos(\omega t + \phi) \\
v(t) &= \frac{dx}{dt} = -A\omega sin(\omega t + \phi) \\
a(t) &= \frac{dv}{dt} = -A\omega^2 cos(\omega t + \phi)
\end{aligned}
$$

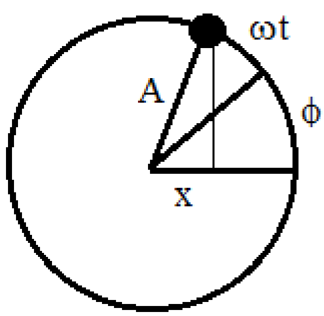
Plug this into our equation:

$$-mA\omega^2 cos(\omega t + \phi) = -kAcos(\omega t + \phi)$$

$$m\omega^2 = k \qquad\qquad \omega = \sqrt{\frac{k}{m}}$$

The constants A and ϕ are unconstrained. A is known as the ***amplitude***, and gives the maximum displacement from equilibrium achieved by the system. In reality, it should be small enough that Hooke's Law is still valid at the maximum displacement. ϕ is called the ***phase angle***, and determines what the system is doing at $t = 0$. For example, if the system starts at $x = A$, the phase angle is zero. If the system starts at $x = 0$, then $\phi = \pi/2$.

Analogy to circular motion: Consider a particle moving at a constant speed along a circular orbit of radius A, starting at an angle ϕ and moving with angular frequency of ω radians / second:



The horizontal displacement of the mass from the origin, x is then given by

$$x = Acos(\omega t + \phi)$$

This is exactly the same equation as for the harmonic oscillator. For this reason, we use terms such as “phase angle” and “angular frequency”, borrowed from periodic circular motion.

The ***period*** of the motion is the amount of time it takes the system to oscillate through a complete cycle. In terms of angle, that's the time it takes the system to go through 2π radians. The amount of time taken to do that is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The ***frequency*** of the motion is the number of complete oscillations the system goes through per second. It is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This gives yet another method of measuring the spring constant. Suppose we have a 1kg mass oscillating on a spring with a frequency of 1 hz. What is the spring constant?

$$k = (2\pi)^2(1s^{-1})^2(1kg) = 39.5kg/s^2$$

Conservation of energy: Does our solution conserve energy? Let's check:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega^2 sin^2(\omega t + \phi) + \frac{1}{2}kA^2 cos^2(\omega t + \phi) =$$
$$= \frac{1}{2}kA^2\left(sin^2(\omega t + \phi) + cos^2(\omega t + \phi)\right) = \frac{1}{2}kA^2 = constant$$

So, yes, the energy is conserved, and is equal to potential energy at maximum displacement. This makes sense, since the particle isn't moving at the turning point, so kinetic energy is zero there, and the total energy is equal to the potential energy.

Sample problem 1:
A mass is suspended from a spring at equilibrium in an elevator moving down at 1 m/s. The mass has m = 1 kg, while the spring has k = 100 kg/s^2. The elevator comes to a sudden stop. What is the amplitude of the mass's oscillations after the elevator stops?

At first, the mass keeps moving at 1 m/s due to inertia. It is at the equilibrium position. So, its energy is

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}(1.0kg)(1.0m/s)^2 = 0.5J$$

$$E = \frac{1}{2}kA^2 \qquad A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2 \times 0.5J}{100kg/s^2}} = 0.1m = 10cm$$

The amplitude of oscillations is thus 10 cm.

Sample problem 2:
What is the speed of the harmonic oscillator when the displacement from equilibrium is 1/4 the amplitude, compared to the maximum speed?

The speed at 1/4 the amplitude can be obtained from conservation of energy:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \sqrt{\frac{k}{m}(A^2 - (A/4)^2)} = 0.968A\sqrt{\frac{k}{m}}$$

$$v_{max} = v(x=0) = A\sqrt{\frac{k}{m}} \qquad v(x = \frac{1}{4}A) = 0.968v_{max}$$

Sample problem 3:
Suppose we have a 1kg mass attached to a spring with $k = 100 kg/s^2$. The mass slides on a horizontal table with a coefficient of kinetic friction $\mu_k = 0.2$. The mass is released from a displacement $x = 0.1$m. How far to the other side of the equilibrium point does it get before it stops?

This is a problem where some of the energy is lost to a constant frictional force. Recall that the energy lost is simply frictional force times distance traveled, so the total remaining energy is

$$E = \frac{1}{2}kA^2 - (A - x)\mu_k mg = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

When the mass stops, $v = 0$. So, "conservation" of energy becomes

$$\begin{aligned}
&\frac{1}{2}kA^2 - (A - x)\mu_k mg = \frac{1}{2}kx^2 \\
&\frac{1}{2}kx^2 - \mu_k mgx - \left(\frac{1}{2}kA^2 - \mu_k mgA\right) = 0 \\
&x = \frac{\mu_k mg}{k} \pm \frac{1}{k}\sqrt{\mu_k^2 m^2 g^2 + 2k\left(\frac{1}{2}kA^2 - \mu_k mgA\right)} = \\
&= 0.0196m \pm 0.0804m = -0.0608m = -6.08cm
\end{aligned}$$

We picked the negative sign since the mass stops on the other side of the equilibrium point; the positive sign simply gives $x = 10cm$, the initial position of the mass (where it is indeed at rest).