PHYSICS 4C QUIZ 2 FALL QUARTER 2007
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Formulas:

$\vec{F}_2 = \frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$ Coulomb's law ; $\vec{E} = \vec{F}/q_0$ electric field ; $\vec{E}(x,y,z) = \int \frac{\rho(x',y',z')(\hat{r} - \hat{r}')}{|\vec{r} - \vec{r}'|^2} dx'dy'dz'$

$\oint \vec{E}\cdot d\vec{a} = 4\pi q_{enc} = 4\pi \int \rho dv$ Gauss' law 1 charge at the origin: $\vec{E}(\vec{r}) = \frac{q}{r^2}\hat{r}$

Linear, surface, volume charge density : $dq = \lambda\, ds$, $dq = \sigma\, dA$, $dq = \rho\, dV$

Electric field of : charge : $E = \frac{q}{r^2}$; line of charge : $E = \frac{2\lambda}{r}$; sheet of charge : $E = 2\pi\sigma$

Potential of single charge q : $\phi(\vec{r}) = \frac{q}{r}$; charge distribution : $\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dx'dy'dz'$

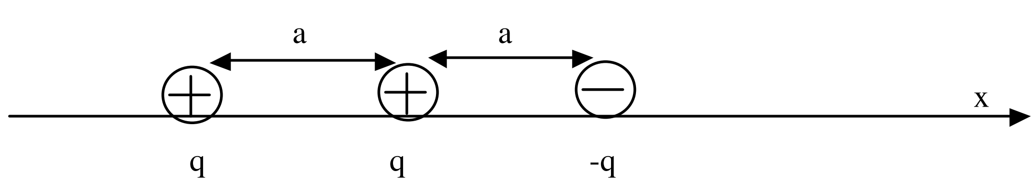
$\phi(x,y,z) - \phi(x_0,y_0,z_0) = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{E}\cdot d\vec{s}$; $\vec{E} = -\nabla\phi$; $\nabla^2\phi = -4\pi\rho$; $\text{div}\vec{E} = 4\pi\rho$

$\text{div}\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$; $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

energy of 3 charges: $U = \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}$; energy of q in potential ϕ : $U = q\,\phi(x,y,z)$

Problem 1 (10 pts)

Consider the 3 charges along the x-axis at distance a from each other of same magnitude and sign as shown in the figure:

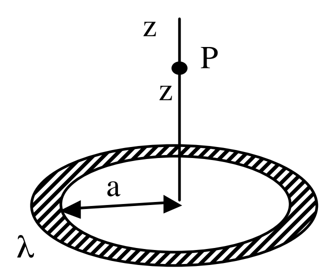


assume the center charge is at x=0.
(a) Make a plot of the potential $\phi(x)$ versus x extending from large negative x to large positive x.
(b) Locate 2 points in the graph where if you put a test charge q_0 it will be in equilibrium. Will the equilibrium be stable or unstable? Answer separately for $q_0>0$ and $q_0<0$. (assume the test charge can only move along the x-ais.
(c) Locate a point on the x axis to which you can bring a test charge q_0 from infinity without doing any net work.
Justify all your answers.

Problem 2 (10 pts)

An infinitely long cylinder of radius R has a non-uniform charge distribution ρ in its interior. The potential for r<R (r is the distance to the cylinder axis) is given by
$\phi(x,y,z) = x^4 + 2x^2y^2 + y^4$
(a) Find the charge density ρ(x,y,z) inside the cylinder. Show that it can be expressed as ρ(r).
(b) Find the electric field at r=R using the charge density found in (a) and Gauss' law. Show all the steps.
Hint: $\int dxdy \cdot f(\sqrt{x^2+y^2}) = 2\pi \int dr \cdot r \cdot f(r)$
(c) Find the electric field $\vec{E}(x = R, y = 0, z = 0)$ directly from the potential ϕ. Explain why your result agrees or disagrees with the result in (b).

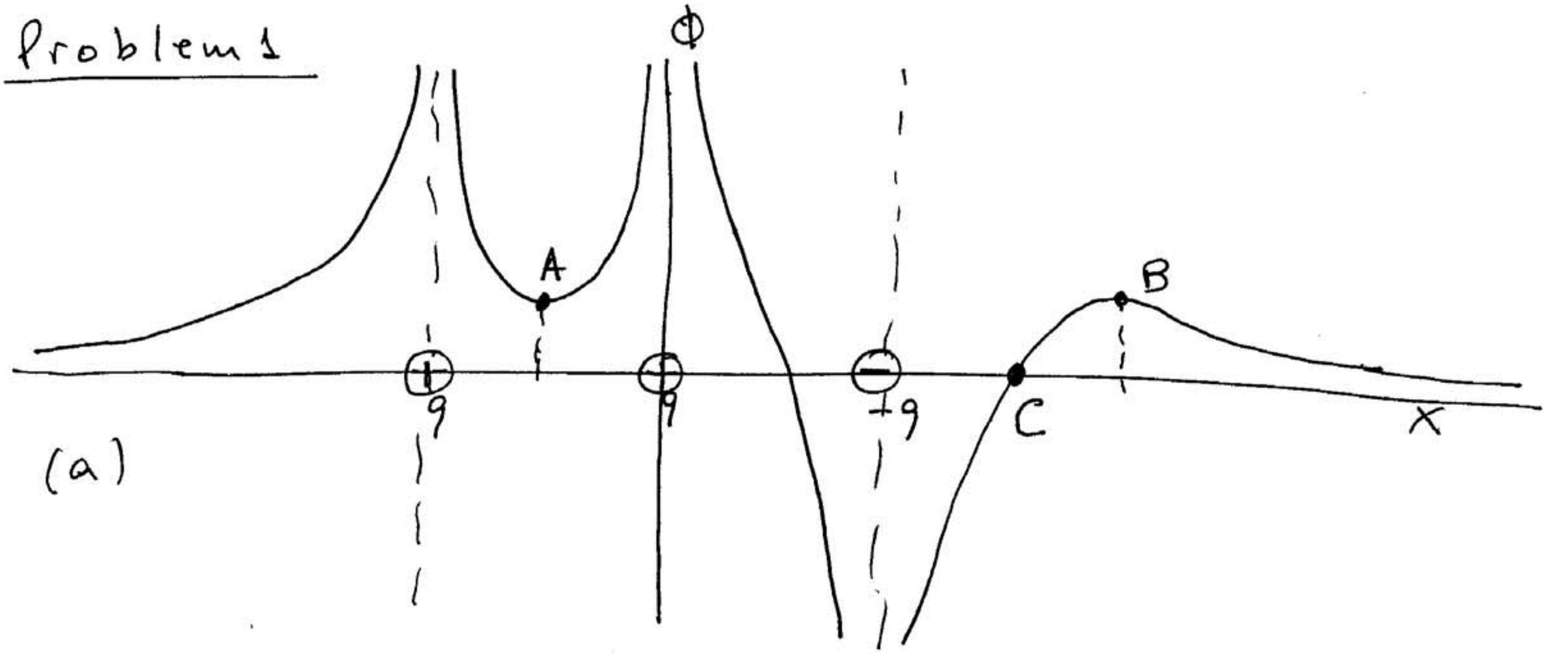
Problem 3 (10 pts)



Consider a ring of radius a and total charge q, i.e. linear charge density λ=q/(2πa).
(a) Find the potential at point P a distance z along the axis from the center.
(b) Calculate the electric field (in the z direction) at point P from the potential. Make a plot of the electric field E_z versus z that includes both positive and negative z.
(c) Calculate the electric field directly from its definition, without using the potential. Explain all steps. Explain why your result does or does not agree with the result of (b).

Physics 4C — Quiz 2 Solutions — Fall 2007

Problem 1



(a)

$$\Phi(x) = \frac{q}{|x-x_1|} + \frac{q}{|x-x_2|} - \frac{q}{|x-x_3|} \quad \text{with } x_1 = -a,\ x_2 = 0,\ x_3 = +a$$

The electric field is $E\hat{x} = -\frac{d\Phi}{dx}$. A test charge is in equilibrium where $E = 0 \Rightarrow \Phi$ is max or min.

For $q_0 > 0$: A is stable equilibrium, B is unstable (b)
$q_0 < 0$: " unstable " " stable

Note: for large x, $\Phi(x) > 0$ because there are two positive charges, one negative one $\Rightarrow$ total charge is positive.

(c) Work done = energy of particle at point $= q\,\Phi$
(since we define $\Phi = 0$ at $x = \infty$)
So find a point where $\Phi = 0$, i.e. point C on the graph

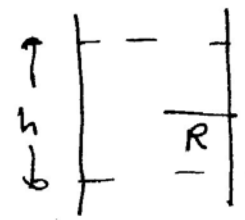
Problem 2

Use $\nabla^2 \Phi = -4\pi \rho$, $\Phi = x^4 + 2x^2y^2 + y^4$

$$\nabla^2 \Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi = 12x^2 + 4y^2 + 4x^2 + 12y^2 =$$

$$= 16(x^2+y^2) = -4\pi\rho \Rightarrow \boxed{\rho = -\frac{4}{\pi}(x^2+y^2)}$$

(b) $$\int \vec{E}\cdot d\vec{a} = 4\pi \int \rho\, dv$$



we can write $\rho = \rho(r) = -\frac{4}{\pi} r^2$

$$\int \rho\, dv = 2\pi h \int_0^R dr \cdot r \cdot \rho(r) = -2\cancel{\pi}\cdot\frac{4}{\cancel{\pi}} h \int_0^R dr\, r^3 = -2R^4 h$$

$$\int \vec{E}\cdot d\vec{a} = E\cdot h \cdot 2\pi R \Rightarrow$$

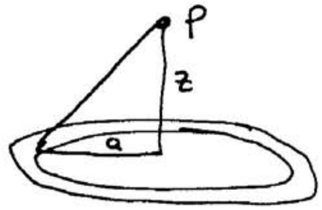
$$\Rightarrow E\cdot \cancel{h}\cdot \cancel{2}\cancel{\pi} R = 4\cancel{\pi}\cdot(-\cancel{2}R^4\cancel{h}) \Rightarrow \boxed{E = -4R^3}$$

(c) $$E(x=R,0,0) = -\frac{\partial \Phi}{\partial x}\hat{x} - \frac{\partial \Phi}{\partial y}\hat{y} - \frac{\partial \Phi}{\partial z}\hat{z} =$$

$$= -4x^3\Big)_{x=R} = \boxed{-4R^3}$$

agrees with (b), as it should.

Problem 3

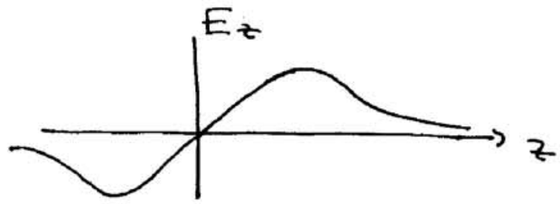


$$\phi(\vec{r}) = \int dv \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

here, $|\vec{r}-\vec{r}'|$ is $\sqrt{a^2+z^2}$, constant ; $\rho \to \lambda$, $\int dv = 2\pi a$

$$\boxed{\phi(z) = \frac{2\pi a \lambda}{\sqrt{a^2+z^2}} = \frac{q}{\sqrt{a^2+z^2}}}$$

(b) $E_z = -\dfrac{d\phi}{dz} = \dfrac{qz}{\sqrt{(a^2+z^2)^3}}$



(c) Direct calculation:

$d\vec{E} = \dfrac{dq}{r^2}\hat{r} = \dfrac{dq}{a^2+z^2}\hat{r}$; the horizontal components will cancel out. The z component is

$$dE_z = dE\cos\theta = dE\frac{z}{\sqrt{a^2+z^2}} = \frac{dq\, z}{(a^2+z^2)^{3/2}}$$

$$\Rightarrow \boxed{E_z = \int dE_z = \frac{qz}{(a^2+z^2)^{3/2}}}$$

same as (b).