Lecture 3

• Weak Interactions (continued)
  • muon decay
  • Pion decay
Muon Decay Overview (1)

• Feynman diagram:

\[ \mu^- \equiv u(p) \]

\[ e^- \equiv \bar{u}(p') \]

\[ \nu \equiv \bar{u}(k) \]

\[ \bar{\nu} \equiv \nu(k') \]

• Matrix Element:

\[ M = \frac{G}{\sqrt{2}} \left[ \bar{u}(k) \gamma^\mu (1 - \gamma^5) u(p) \right] \left[ \bar{u}(p') \gamma_\mu (1 - \gamma^5) \nu(k') \right] \]

\( G = \) effective coupling of a 4-fermion interaction

Structure of 4-fermion interaction is \((V-A) \times (V-A)\)

We clearly want to test this experimentally, e.g. compare against \((V-A) \times (V+A)\), S, P, T, etc.
Muon Decay Overview (2)

- Differential width:
  \[ d\Gamma = \frac{1}{2E} |M|^2 dQ \]

- Phase space differential:
  \[ dQ = \frac{d^3 p'}{(2\pi)^3 2E'} \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 k'}{(2\pi)^3 2\omega'} (2\pi)^4 \delta^{(4)}(p - p' - k - k') \]
Calculational Challenges

• There’s a spin averaged matrix element involved, requiring the use of some trace theorems.

• The phase space integral is not trivial.

*I’ll provide you with an outline of how these are done, and leave it up to you to go through the details in H&M.*
Phase Space Integration (1)

• We have:

\[ dQ = \frac{d^3 p'}{(2\pi)^3 2 E'} \frac{d^3 k}{(2\pi)^3 2 \omega} \frac{d^3 k'}{(2\pi)^3 2 \omega'} (2\pi)^4 \delta^{(4)}(p - p' - k - k') \]

• We can get rid of one of the three \( d^3 p/E \) by using:

\[ \int \frac{d^3 k}{\omega} = \int d^4 k \ \theta(\omega) \delta(k^2) \]

• And eliminating the \( d^4 k \) integral against the 4d delta-function. This leads to:

\[ dQ = \frac{1}{(2\pi)^5} \frac{d^3 p'}{2 E'} \frac{d^3 k'}{2 \omega'} (2\pi)^4 \theta(E - E' - \omega') \delta\left(\left(p - p' - k\right)^2\right) \]

This means \( E - E' - \omega' > 0 \)
Phase Space Integration (2)

- As was done for beta-decay, we replace:

$$d^3 p' \, d^3 k' = 4\pi E'^2 \, dE' \, 2\pi \omega'^2 \, d\omega' \, d\cos\theta$$

- And evaluate delta-fct argument in muon restframe:

$$\delta\left((p - p' - k')^2\right) = \delta\left(m^2 - 2mE' - 2m\omega' + 2E'\omega'(1 - \cos\theta)\right)$$

Recall: primed variables are from second W vertex.
Spin Average Matrix Element

- We neglect the mass of the electron and neutrinos.
- And use the trace theorem H&M (12.29) to arrive at:

\[
|M|^2 = \frac{1}{2} \sum_{\text{spins}} |M|^2 = 64G^2 (k \cdot p')(k' \cdot p)
\]
Muon Restframe

• To actually do the phase space integral, we go into the muon restframe where we find:
  • \(2 \, k'p' = (k+p')^2 = (p-k')^2 = m^2 - 2pk' = m^2 - 2m \, \omega'\)
  • \(k'p = \omega'm\)

• And as a result we get:

\[
|M|^2 = 32G^2 \left( m^2 - 2m \omega' \right) m \omega'
\]
Putting the pieces together and doing the integration over $\cos \theta$, the opening angle of $e$ and its anti-neutrino, we arrive at:

$$d\Gamma = \frac{G^2}{2\pi^3} dE' d\omega' m\omega(m - 2\omega')$$

\[
\frac{1}{2} m - E' \leq \omega' \leq \frac{1}{2} m
\]

\[
0 \leq E' \leq \frac{1}{2} m
\]

The inequality come from the requirement that $\cos \theta$ is physical. And are easily understood from the allowed 3-body phasespace where one of the 3 is at rest.
Electron Energy Spectrum

• Integrate over electron antineutrino energy:

\[
\frac{d\Gamma}{dE'} = \frac{G^2}{2\pi^3} \int_{\frac{1}{2}m-E'}^{\frac{1}{2}m} d\omega' \omega'(m - 2\omega')
\]

\[
\frac{d\Gamma}{dE'} = \frac{G^2}{2\pi^3} m^2 E'^2 \left( 3 - \frac{4E'}{m} \right)
\]

The spectrum can be used to test V-A. This is discussed as “measurement of Michel Parameters” in the literature.
Michel Parameters

- It can be shown that any 4-fermion coupling will lead to an electron spectrum like the one we derived here, once we allow a “Michel Parameter” $\rho$, as follows:

$$\frac{d\Gamma}{\Gamma dx} = 12x^2 \left[ 1 - x + \frac{2}{3} \rho \left( \frac{4}{3} x - 1 \right) \right]$$

Measured Value:

$$\rho\mu = 0.7509 \pm 0.0010$$
$$\rho\tau = 0.745 \pm 0.008$$

- $\rho = 0$ for (V-A)x(V+A), S, P; $\rho = 1$ for T
- $\rho = 0.75$ for (V-A)x(V-A)
- With polarized muon beams and measurement of electron polarization, other “Michel Parameters” come into play.
Total Decay Width of Muon

• Integrate over electron energy:

\[ \Gamma = \frac{1}{\tau} = \frac{G^2}{2\pi^3} m^2 \int_0^m dE' E'^2 \left( 3 - \frac{4E'}{m} \right) \]

Note: Comparing muon and tau decays, as well as
tau decays to electron and muon, allows for stringent
tests of lepton universality to better than 1%.
Pion Decay

\[
\begin{array}{c}
\bar{u} \\
\bar{d}
\end{array}
\to
\begin{array}{c}
W^- \\
V_{ud}
\end{array}
\to
\begin{array}{c}
\mu^- \equiv \bar{u}(p) \\
\bar{\nu} \equiv \nu(k)
\end{array}
\]

- Leptonic vertex is identical to the leptonic current vertex in muon decay.
- Hadronic vertex needs to be parametrized as it can NOT be treated as a current composed of free quarks.
Parametrization of Hadronic Current

- Matrix element is Lorentz invariant scalar.
  - Hadronic current must be vector or axial vector
- Pion is spinless
  - Q is the only vector to construct a current from.
- The current at the hadronic vertex thus must be of the form:
  \[ q^\mu f_\pi (q^2) = q^\mu f_\pi (m_\pi^2) = q^\mu f_\pi \]
- However, as \( q^2 = m_\pi^2 = \text{constant} \), we refer to \( f_\pi \) simply as the “pion decay constant”.
- All other purely leptonic decays of weakly decaying mesons can be calculated in the same way. There are thus “decay constants” for \( B^0, B_s^0, D^+, D_s^+, K^+ \), etc.
Aside:

• This sort of parametrization is “reused” also when extrapolating from semileptonic to hadronic decays at fixed $q^2$
  – E.g. Using $B \rightarrow D \lnu$ to predict $B \rightarrow D X$ where $X$ is some hadron.
  – This is crude, but works reasonably well in some cases.
Matrix Element for Pion Decay

\[ M = \frac{GV_{ud}}{\sqrt{2}} (p^\mu + k^\mu) f_\pi \left[ \bar{u}(p) \gamma_\mu (1 - \gamma^5) v(k) \right] \]

Now use the Dirac Equation for muon and neutrino:

\[ \bar{u}(p) (p^\mu \gamma_\mu - m_\mu) = 0 \]

\[ \Rightarrow \bar{u}(p) p^\mu \gamma_\mu (1 - \gamma^5) v(k) = \bar{u}(p) m_\mu (1 - \gamma^5) v(k) \]

\[ k^\mu \gamma_\mu v(k) = 0 \]

\[ \Rightarrow \bar{u}(p) k^\mu \gamma_\mu (1 - \gamma^5) v(k) = 0 \]

\[ \Rightarrow M = \frac{G}{\sqrt{2}} m_\mu f_\pi \left[ \bar{u}(p) (1 - \gamma^5) v(k) \right] \]

Note: this works same way for any aV+bA.
Trace and Spin averaging

- The spin average matrix element squared is then given by:

\[ |M|^2 = |V_{ud}|^2 \frac{G^2}{2} f_\pi^2 m_\mu^2 \text{Tr} \left[ (p^\mu \gamma_\mu + m_\mu)(1 - \gamma^5)k^\mu \gamma_\mu (1 + \gamma^5) \right] \]

\[ |M|^2 = 4G^2 |V_{ud}|^2 f_\pi^2 m_\mu^2 (p \cdot k) \]

- You can convince yourself that this trace is correct by going back to H&M (6.19), (6.20). The only difference is the “+” sign. This comes from “pulling” a gamma matrix past gamma5.
Going into the pion restframe

- We get:
  \[ p \cdot k = E \omega - p \vec{k} = \omega(E + \omega) \]

- Where we used that muon and neutrino are back to back in the pion restframe.
Pion leptonic decay width

- Putting it all together, we then get:

\[
\frac{1}{2m_\pi} |M|^2 \frac{d^3 p}{(2\pi)^3 2E} \frac{d^3 k}{(2\pi)^3 2\omega} (2\pi)^4 \delta(q - p - k)
\]

\[
\Gamma = \frac{G^2 |V_{ud}|^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi} \int \frac{d^3 p}{E} \frac{d^3 k}{\omega} \delta(m_\pi - E - \omega) \delta^{(3)}(\vec{k} + \vec{p}) \omega(E + \omega)
\]

Energy conservation

3-momentum conservation

Use this to kill int over \(d^3 p\)
Pion leptonic width

• I’ll spare you the details of the integrations. They are discussed in H&M p.265f
• The final result is:

\[
\Gamma = \frac{G^2 |V_{ud}|^2}{8\pi} f_{\pi}^2 m_{\mu}^2 m_{\pi} \left(1 - \frac{m_{\mu}^2}{m_{\pi}^2}\right)^2
\]
Helicity Suppression

• The pion has spin=0.
• Angular momentum is conserved.
  ⇒ Electron and anti-neutrino have same helicity.
  ⇒ However, weak current does not couple to J=0 electron & antineutrino pair.
  ⇒ Rate is suppressed by a factor:

\[ \frac{m_\mu^2}{m_\pi^2} \]

\[
\Gamma_\pi = \frac{G^2 |V_{ud}|^2}{8\pi} f_\pi^2 m_\pi^3 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \times \frac{m_\mu^2}{m_\pi^2}
\]

\[
\Gamma_\mu = \frac{G^2}{192\pi^3} m_\mu^5
\]

Helicity suppression
Experimentally

• As the pion decay constant is not known, it is much more powerful to form the ratio of partial widths:

\[
\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \left( \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2 = 1.233 \times 10^{-4}
\]

Experimentally, we find: \((1.230 \pm 0.004) \times 10^{-4}\)

Aside: Theory number here includes radiative corrections !!!
I.e., this is not just the mass ratio as indicated !!!
Experimental Relevance

• We’ve encountered this a few times already, and now we have actually shown the size of the helicity suppression, and where it comes from.

\[
\frac{\Gamma (\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma (\pi^- \rightarrow \mu^- \bar{\nu}_\mu)} = \frac{m_e^2}{m_\mu^2} \left( \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2 = 1.23 \times 10^{-4}
\]

Accordingly, pion decay produces a rather pure muon neutrino beam, with the charge of the pion determining neutrino or anti-neutrino in the beam.
Origin of Helicity Suppression Recap

• The muon mass entered because of the vector nature of the leptonic current.
  ⇒ Either V or A or some combination of $aV+bA$ will all lead to helicity suppression.
  ⇒ In particular, a charged weak current with S,P, or T instead of V,A is NOT consistent with experiment.

• In addition, we used:
  – Neutrinos are massless
  – Electron-muon universality
Window for New Physics via leptonic decays

Example $B^+$ decay

\[ \Gamma = \frac{G^2 |V_{ub}|^2 f_B^2 m^2_B}{8\pi} m^2_\mu \left(1 - \frac{m^2_\mu}{m^2_B}\right)^2 \]

The smallness of $V_{ub}$ and muon mass allows for propagators other than $W$ to compete, especially if they do not suffer from helicity suppression $\Rightarrow$ e.g. charged Higgs