Chapter 2

Mesoscopia

2.1 References

- Y. Imry, *Introduction to Mesoscopic Physics*
- M. Janssen, *Fluctuations and Localization*
- D. Ferry and S. M. Goodnick, *Transport in Nanostructures*

2.2 Introduction

Current nanofabrication technology affords us the remarkable opportunity to study condensed matter systems on an unprecedented small scale. For example, small electron boxes known as *quantum dots* have been fabricated, with characteristic size ranging from 10 nm to 1 μm; the smallest quantum dots can hold as few as one single electron, while larger dots can hold thousands. In systems such as these, one can probe discrete energy level spectra associated with quantization in a finite volume. Oftentimes systems are so small that Bloch's theorem and the theoretical apparatus of Boltzmann transport are of dubious utility.

2.3 The Landauer Formula

Consider a disordered one-dimensional wire connected on each end to reservoirs at fixed chemical potential $\mu_{\rm L}$ and $\mu_{\rm R}$. For the moment, let us consider only a single spin species, or imagine that the spins are completely polarized. Suppose further that $\mu_{\rm L} > \mu_{\rm R}$, so that a current I flows from the left reservoir (L) to the right reservoir (R). Next, consider a cross-sectional surface Σ just to the right of the disordered region. We calculate the current

flowing past this surface as a sum over three terms:

$$I_\Sigma = -e \int d\varepsilon \, \mathcal{N}(\varepsilon) \, v(\varepsilon) \left\{ T(\varepsilon) \, f(\varepsilon - \mu_{\rm L}) + R'(\varepsilon) \, f(\varepsilon - \mu_{\rm R}) - f(\varepsilon - \mu_{\rm R}) \right\} . \tag{2.1}$$

Here, $\mathcal{N}(\varepsilon)$ is the density of states in the leads per spin degree of freedom, and corresponding to motion in a given direction (right or left but not both); $v(\varepsilon)$ is the velocity, and $f(\varepsilon - \mu_{\rm L,R})$ are the respective Fermi distributions. $T(\varepsilon)$ is the *transmission probability* that electrons of energy ε emerging from the left reservoir will end up in right reservoir; $R'(\varepsilon)$ is the *reflection probability* that electrons emerging from right reservoir will return to the right reservoir. The three terms on the right hand side of (2.1) correspond, respectively, to: (i) electrons emerging from L which make it through the wire and are deposited in R, (ii) electrons emerging from R which fail to 'swim upstream' to L and are instead reflected back into R, and (iii) all electrons emerging from reservoir R (note this contribution is of opposite sign). The transmission and reflection probabilities are obtained by solving for the quantum mechanical scattering due to the disordered region. If the incoming *flux amplitudes* from the left and right sides are i and i', respectively, and the outgoing flux amplitudes on those sides o' and o, linearity of the Schrödinger equation requires that

$$\begin{pmatrix} o' \\ o \end{pmatrix} = \mathcal{S} \begin{pmatrix} i \\ i' \end{pmatrix} \qquad ; \qquad \mathcal{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} . \tag{2.2}$$

The matrix $\mathcal{S}$ is known as the *scattering matrix* (or $\mathcal{S}$-matrix, for short). The S-matrix elements r, t, *etc.* are reflection and transmission *amplitudes*. The reflection and transmission *probabilities* are given by

$$R = |r|^2 \qquad T' = |t'|^2 \tag{2.3}$$

$$T = |t|^2 \qquad R' = |r'|^2 . \tag{2.4}$$

Going back to (2.1), let us assume that we are close to equilibrium, so the difference $\mu_{\rm R} - \mu_{\rm L}$ in chemical potentials is slight. We may then expand

$$f(\varepsilon - \mu_{\rm R}) = f(\varepsilon - \mu_{\rm L}) + f'(\varepsilon - \mu_{\rm L}) \, (\mu_{\rm L} - \mu_{\rm R}) + \ldots \tag{2.5}$$

and obtain the result

$$\begin{aligned} I &= e(\mu_{\rm R} - \mu_{\rm L}) \int d\varepsilon \, \mathcal{N}(\varepsilon) \, v(\varepsilon) \left(-\frac{\partial f^0}{\partial \varepsilon} \right) T(\varepsilon) \\ &= \frac{e}{h} (\mu_{\rm R} - \mu_{\rm L}) \int d\varepsilon \, \left(-\frac{\partial f^0}{\partial \varepsilon} \right) T(\varepsilon) \, , \end{aligned} \tag{2.6}$$

valid to lowest order in $(\mu_{\rm R} - \mu_{\rm L})$. We have invoked here a very simple, very important result for the one-dimensional density of states. Considering only states moving in a definite direction (left or right) and with a definite spin polarization (up or down), we have

$$\mathcal{N}(\varepsilon) \, d\varepsilon = \frac{dk}{2\pi} \quad \Longrightarrow \quad \boxed{\mathcal{N}(\varepsilon) = \frac{1}{2\pi} \frac{dk}{d\varepsilon} = \frac{1}{h v(\varepsilon)}} \tag{2.7}$$

where $h = 2\pi\hbar$ is Planck's constant. Thus, there is a remarkable cancellation in the product $\mathcal{N}(\varepsilon)\, v(\varepsilon) = h^{-1}$. Working at $T = 0$, we therefore obtain

$$I = \frac{e}{h}(\mu_{\rm R} - \mu_{\rm L})\, T(\varepsilon_{\rm F}) \ , \tag{2.8}$$

where $T(\varepsilon_{\rm F})$ is the transmission probability at the Fermi energy. The chemical potential varies with voltage according to $\mu(V) = \mu(0) - eV$, hence the *conductance* $G = I/V$ is found to be

$$G = \frac{e^2}{h}\, T(\varepsilon_{\rm F}) \qquad \text{(per spin channel)} \tag{2.9}$$

$$= \frac{2e^2}{h}\, T(\varepsilon_{\rm F}) \qquad \text{(spin degeneracy included)} \tag{2.10}$$

The quantity h/e^2 is a conveniently measurable $25,813\,\Omega$.

We conclude that *conductance is transmission* - G is e^2/h times the transmission probability $T(\varepsilon_{\rm F})$ with which an electron at the Fermi level passes through the wire. This has a certain intuitive appeal, since clearly if $T(\varepsilon_{\rm F}) = 0$ we should expect $G = 0$. However, two obvious concerns should be addressed:

- The power dissipated should be $P = I^2R = V^2G$. Yet the scattering in the wire is assumed to be purely elastic. Hence no dissipation occurs within the wire at all, and the Poynting vector immediately outside the wire must vanish. What, then, is the source of the dissipation?
- For a perfect wire, $T(\varepsilon_{\rm F}) = 1$, and $G = e^2/h$ (per spin) is finite. Shouldn't a perfect (*i.e.* not disordered) wire have *zero* resistance, and hence *infinite* conductance?

The answer to the first of these riddles is simple – all the dissipation takes place in the R reservoir. When an electron makes it through the wire from L to R, it deposits its excess energy $\mu_{\rm L} - \mu_{\rm R}$ in the R reservoir. The mechanism by which this is done is not our concern – we only need assume that there is *some* inelastic process (*e.g.* electron-phonon scattering, electron-electron scattering, *etc.*) which acts to equilibrate the R reservoir.

The second riddle is a bit more subtle. One solution is to associate the resistance h/e^2 of a perfect wire with the *contact resistance* due to the leads. The intrinsic conductance of the wire $G_{\rm i}$ is determined by assuming the wire resistance and contact resistances are in series:

$$G^{-1} = G_{\rm i}^{-1} + \frac{h}{e^2} \quad \Longrightarrow \quad G_{\rm i} = \frac{e^2}{h}\,\frac{T(\varepsilon_{\rm F})}{1 - T(\varepsilon_{\rm F})} = \frac{e^2}{h}\,\frac{T(\varepsilon_{\rm F})}{R(\varepsilon_{\rm F})} \ , \tag{2.11}$$

where $G_{\rm i}$ is the intrinsic conductance of the wire, per spin channel. Now we see that when $T(\varepsilon_{\rm F}) \to 1$ the intrinsic conductance diverges: $G_{\rm i} \to \infty$. When $T \ll 1$, $G_{\rm i} \approx G = (e^2/h)T$. This result (2.11) is known as the *Landauer Formula*.

To derive this result in a more systematic way, let us assume that the disordered segment is connected to the left and right reservoirs by perfect leads, and that the leads are not in

equilibrium at chemical potentials $\mu_{\rm L}$ and $\mu_{\rm R}$ but instead at $\tilde\mu_{\rm L}$ and $\tilde\mu_{\rm R}$. To determine $\tilde\mu_{\rm L}$ and $\tilde\mu_{\rm R}$, we compute the number density (per spin channel) in the leads,

$$n_{\rm L} = \int d\varepsilon\,\mathcal{N}(\varepsilon)\Big\{\big[1+R(\varepsilon)\big]f(\varepsilon-\mu_{\rm L}) + T'(\varepsilon)\,f(\varepsilon-\mu_{\rm R})\Big\} \tag{2.12}$$

$$n_{\rm R} = \int d\varepsilon\,\mathcal{N}(\varepsilon)\Big\{T(\varepsilon)f(\varepsilon-\mu_{\rm L}) + \big[1+R'(\varepsilon)\big]\,f(\varepsilon-\mu_{\rm R})\Big\} \tag{2.13}$$

and associate these densities with chemical potentials $\tilde\mu_{\rm L}$ and $\tilde\mu_{\rm R}$ according to

$$n_{\rm L} = 2\int d\varepsilon\,\mathcal{N}(\varepsilon)\,f(\varepsilon-\tilde\mu_{\rm L}) \tag{2.14}$$

$$n_{\rm R} = 2\int d\varepsilon\,\mathcal{N}(\varepsilon)\,f(\varepsilon-\tilde\mu_{\rm R})\ , \tag{2.15}$$

where the factor of two accounts for both directions of motion. To lowest order, then, we obtain

$$2(\mu_{\rm L}-\tilde\mu_{\rm L}) = (\mu_{\rm L}-\mu_{\rm R})\,T' \quad\Longrightarrow\quad \tilde\mu_{\rm L} = \mu_{\rm L} + \tfrac{1}{2}\,T'\,(\mu_{\rm R}-\mu_{\rm L}) \tag{2.16}$$

$$2(\mu_{\rm R}-\tilde\mu_{\rm R}) = (\mu_{\rm R}-\mu_{\rm L})\,T \quad\Longrightarrow\quad \tilde\mu_{\rm R} = \mu_{\rm R} + \tfrac{1}{2}\,T\,(\mu_{\rm L}-\mu_{\rm R}) \tag{2.17}$$

and therefore

$$\begin{aligned}(\tilde\mu_{\rm L}-\tilde\mu_{\rm R}) &= \big(1-\tfrac{1}{2}T-\tfrac{1}{2}T'\big)\,(\mu_{\rm L}-\mu_{\rm R}) \\ &= (1-T)\,(\mu_{\rm L}-\mu_{\rm R})\ ,\end{aligned} \tag{2.18}$$

where the last equality follows from unitarity ($\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = 1$). There are two experimental configurations to consider:

- *Two probe measurement* – Here the current leads are also used as voltage leads. The voltage difference is $\Delta V = (\mu_{\rm R}-\mu_{\rm L})/e$ and the measured conductance is $G_{2-\rm probe} = (e^2/h)\,T(\varepsilon_{\rm F})$.
- *Four probe measurement* – Separate leads are used for current and voltage probes. The observed voltage difference is $\Delta V = (\tilde\mu_{\rm R}-\tilde\mu_{\rm L})/e$ and the measured conductance is $G_{4-\rm probe} = (e^2/h)\,T(\varepsilon_{\rm F})/R(\varepsilon_{\rm F})$.

2.3.1 Example: Potential Step

Perhaps the simplest scattering problem is one-dimensional scattering from a potential step, $V(x) = V_0\,\Theta(x)$. The potential is piecewise constant, hence the wavefunction is piecewise a plane wave:

$$x<0: \quad \psi(x) = I\,e^{ikx} + O'\,e^{-ikx} \tag{2.19}$$

$$x>0: \quad \psi(x) = O\,e^{ik'x} + I'\,e^{-ik'x}\ , \tag{2.20}$$

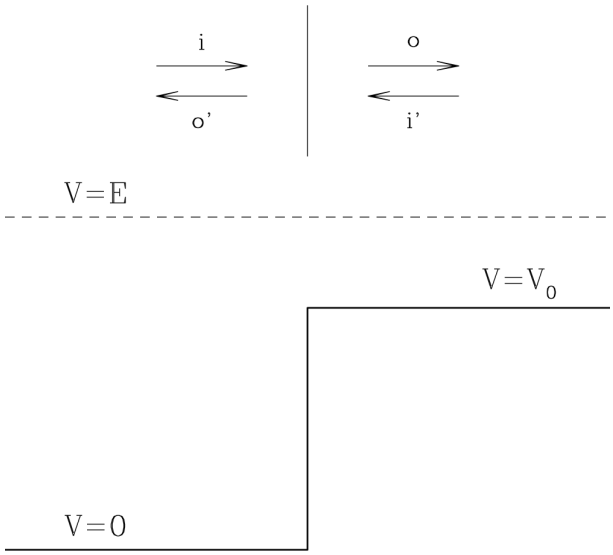

Figure 2.1: Scattering at a potential step.

with

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k'^2}{2m} + V_0 \ . \tag{2.21}$$

The requirement that $\psi(x)$ and its derivative $\psi'(x)$ be continuous at $x = 0$ gives us two equations which relate the four wavefunction amplitudes:

$$I + O' = O + I' \tag{2.22}$$

$$k(I - O') = k'(O - I') \ . \tag{2.23}$$

As emphasized earlier, the $\mathcal{S}$-matrix acts on flux amplitudes. We have

$$\begin{pmatrix} i \\ o' \end{pmatrix} = \sqrt{v} \begin{pmatrix} I \\ O' \end{pmatrix} \qquad , \qquad \begin{pmatrix} o \\ i' \end{pmatrix} = \sqrt{v'} \begin{pmatrix} O \\ I' \end{pmatrix} \ , \tag{2.24}$$

with $v = \hbar k/m$ and $v' = \hbar k'/m$. One easily finds the $\mathcal{S}$-matrix, defined in eqn. 2.2, is given by

$$\mathcal{S} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{\epsilon}}{1+\epsilon} & \frac{\epsilon-1}{1+\epsilon} \\ \frac{1-\epsilon}{1+\epsilon} & \frac{2\sqrt{\epsilon}}{1+\epsilon} \end{pmatrix} \ , \tag{2.25}$$

where $\epsilon \equiv v'/v = k'/k = \sqrt{1 - \frac{V_0}{E}}$, where $E = \varepsilon_{\rm F}$ is the Fermi energy. The two- and four-terminal conductances are then given by

$$G_{2-\rm probe} = \frac{e^2}{h} \, |t|^2 = \frac{e^2}{h} \cdot \frac{4\epsilon}{(1+\epsilon)^2} \tag{2.26}$$

$$G_{4-\rm probe} = \frac{e^2}{h} \, \frac{|t|^2}{|r|^2} = \frac{e^2}{h} \cdot \frac{4\epsilon}{(1-\epsilon)^2} \ . \tag{2.27}$$

Both are maximized when the transmission probability $T = |t|^2 = 1$ is largest, which occurs for $\epsilon = 1$, *i.e.* $k' = k$.

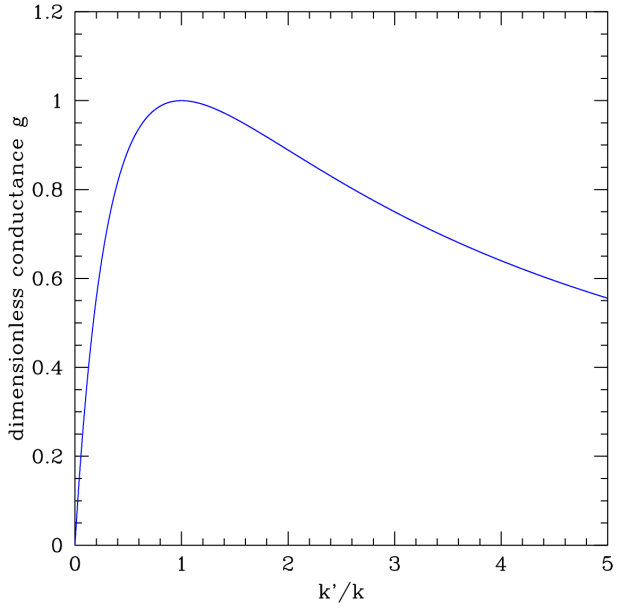

Figure 2.2: Dimensionless two-terminal conductance g *versus* k'/k for the potential step. The conductance is maximized when $k' = k$.

2.4 Multichannel Systems

The single channel scenario described above is obtained as a limit of a more general multichannel case, in which there are transverse degrees of freedom (due *e.g.* to finite cross-sectional area of the wire) as well. We will identify the transverse states by labels i. Within the perfect leads, the longitudinal and transverse energies are decoupled, and we may write

$$\varepsilon = \varepsilon_{\perp i} + \varepsilon_{\parallel}(k) \ , \tag{2.28}$$

where $\varepsilon_{\parallel}(k)$ is the one-dimensional dispersion due to motion along the wire (*e.g.* $\varepsilon_{\parallel}(k) = \hbar^2 k^2/2m^*$, $\varepsilon_{\parallel}(k) = -2t\cos ka$, *etc.*). k is the component of the wavevector along the axis of the wire. We assume that the transverse dimensions are finite, so fixing the Fermi energy $\varepsilon_{\rm F}$ in turn fixes the total number of transverse channels, $N_{\rm c}$, which contribute to the transport:

$$N_{\rm c}(\varepsilon) = \sum_i \Theta(\varepsilon - \varepsilon_{\perp i}) \qquad \text{(continuum)} \tag{2.29}$$

$$= \sum_i \Theta\big(2t - |\varepsilon - \varepsilon_{\perp i}|\,\big) \qquad \text{(tight binding)} \ . \tag{2.30}$$

Equivalently, an electron with energy ε in transverse state i has wavevector k_i which satisfies

$$\varepsilon_{\parallel}(k_i) = \varepsilon - \varepsilon_{\perp i} \ . \tag{2.31}$$

$N_{\rm c}$ is the number of real positive roots of (2.31). Typically $N_{\rm c} \approx k_{\rm F}^{d-1} A$, where A is the cross-sectional area and $k_{\rm F}$ is the Fermi wavevector. The velocity v_i is

$$v_i(\varepsilon) = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial k}\bigg|_{k_i} = \frac{1}{\hbar}\frac{\partial \varepsilon_\parallel(k)}{\partial k}\bigg|_{k=k_i} \quad . \tag{2.32}$$

The density of states $\mathcal{N}_i(\varepsilon)$ (per unit spin, per direction) for electrons in the $i^{\rm th}$ transverse channel is

$$\mathcal{N}_i(\varepsilon) = \int\limits_{\hat\Omega} \frac{dk}{2\pi}\,\Theta\big(v(k)\big)\,\delta\big(\varepsilon - \varepsilon_{\perp i} - \varepsilon_\parallel(k)\big) = \frac{1}{2\pi}\frac{dk}{d\varepsilon_\parallel}\bigg|_{k=k_i} \quad , \tag{2.33}$$

so once again we have for the product $h\,v_i(\varepsilon)\mathcal{N}_i(\varepsilon) = 1$.

Consider now a section of disordered material connected to perfect leads on the left and right. The solution to the Schrödinger equation on either side of the disordered region is

$$\psi_{\rm left}(\boldsymbol{x}_\perp, z) = \sum_{j=1}^{N_{\rm c}^{\rm L}} \Big\{ I_j\, e^{+ik_j z} + O'_i\, e^{-ik_j z} \Big\}\, \varphi_j^{\rm L}(\boldsymbol{x}_\perp) \tag{2.34}$$

$$\psi_{\rm right}(\boldsymbol{x}_\perp, z) = \sum_{a=1}^{N_{\rm c}^{\rm R}} \Big\{ O_a\, e^{+ik_a z} + I'_a\, e^{-ik_a z} \Big\}\, \varphi_a^{\rm R}(\boldsymbol{x}_\perp) \ . \tag{2.35}$$

Here, we have assumed a general situation in which the number of transverse channels $N_{\rm c}^{\rm L,R}$ may differ between the left and right lead. The quantities $\{I_j, O'_j, O_a, I'_a\}$ are *wave function amplitudes*. The $\mathcal{S}$-matrix, on the other hand, acts on *flux amplitudes* $\{i_j, o'_j, o_a, i'_a\}$, which are related to the wavefunction amplitudes as follows:

$$i_i = v_i^{1/2}\, I_i \qquad o_a = v_a^{1/2}\, O_a \tag{2.36}$$

$$o'_i = v_i^{1/2}\, O'_i \qquad i'_a = v_a^{1/2}\, I'_a \ . \tag{2.37}$$

The $\mathcal{S}$-matrix is a $(N_{\rm c}^{\rm R} + N_{\rm c}^{\rm L}) \times (N_{\rm c}^{\rm R} + N_{\rm c}^{\rm L})$ matrix,

$$\mathcal{S} = \begin{pmatrix} r_{N_{\rm c}^{\rm L} \times N_{\rm c}^{\rm L}} & t'_{N_{\rm c}^{\rm L} \times N_{\rm c}^{\rm R}} \\ t_{N_{\rm c}^{\rm R} \times N_{\rm c}^{\rm L}} & r'_{N_{\rm c}^{\rm R} \times N_{\rm c}^{\rm R}} \end{pmatrix} \tag{2.38}$$

which relates outgoing and incoming flux amplitudes:

$$\begin{pmatrix} o' \\ o \end{pmatrix} = \overbrace{\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}}^{\mathcal{S}} \begin{pmatrix} i \\ i' \end{pmatrix} \quad . \tag{2.39}$$

Unitarity of $\mathcal{S}$ means that $\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = \mathbb{I}$, where

$$\mathcal{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \qquad \Longrightarrow \qquad \mathcal{S}^\dagger = \begin{pmatrix} r^\dagger & t^\dagger \\ t'^\dagger & r'^\dagger \end{pmatrix} \ , \tag{2.40}$$

and hence unitarity says

$$r_{ik}\,r^*_{jk} + t'_{ic}\,t'^*_{jc} = \delta_{ij} \qquad\qquad r^*_{ki}\,r_{kj} + t^*_{ci}\,t_{cj} = \delta_{ij} \tag{2.41}$$

$$t_{ak}\,t^*_{bk} + r'_{ac}\,r'^*_{bc} = \delta_{ab} \qquad\qquad t'^*_{ka}\,t'_{kb} + r'^*_{ca}\,r'_{cb} = \delta_{ab} \tag{2.42}$$

$$r_{ik}\,t^*_{ak} + t'_{ic}\,r'^*_{ac} = 0 \qquad\qquad r^*_{ki}\,t'_{ka} + t^*_{ci}\,r'_{ca} = 0\ , \tag{2.43}$$

or, in matrix notation,

$$r\,r^\dagger + t'\,t'^\dagger \;=\; r^\dagger\,r + t^\dagger t \;=\; \mathbb{I}_{N_{\rm c}^{\rm L}\times N_{\rm c}^{\rm L}} \tag{2.44}$$

$$t\,t^\dagger + r'\,r'^\dagger \;=\; t'^\dagger\,t' + r'^\dagger\,r' \;=\; \mathbb{I}_{N_{\rm c}^{\rm R}\times N_{\rm c}^{\rm R}} \tag{2.45}$$

$$r\,t^\dagger + t'\,r'^\dagger \;=\; r^\dagger\,t' + t^\dagger\,r' \;=\; \mathbb{O}_{N_{\rm c}^{\rm L}\times N_{\rm c}^{\rm R}} \tag{2.46}$$

$$t\,r^\dagger + r'\,t'^\dagger \;=\; t'^\dagger\,r + r'^\dagger\,t \;=\; \mathbb{O}_{N_{\rm c}^{\rm R}\times N_{\rm c}^{\rm L}}\ . \tag{2.47}$$

We define the probabilities

$$R_i = \sum_{k=1}^{N_{\rm c}^{\rm L}} r_{ik}\,r^*_{ik} \qquad\qquad T_a = \sum_{k=1}^{N_{\rm c}^{\rm L}} t_{ak}\,t^*_{ak} \tag{2.48}$$

$$T'_i = \sum_{c=1}^{N_{\rm c}^{\rm R}} t'_{ic}\,t'^*_{ic} \qquad\qquad R'_a = \sum_{c=1}^{N_{\rm c}^{\rm R}} r'^*_{ac}\,r'_{ac}\ , \tag{2.49}$$

for which it follows that

$$R_i + T'_i = 1 \qquad , \qquad R'_a + T_a = 1 \tag{2.50}$$

for all $i \in \{1, \ldots, N_{\rm c}^{\rm L}\}$ and $a \in \{1, \ldots, N_{\rm c}^{\rm R}\}$. Unitarity of the $\mathcal{S}$-matrix preserves particle flux:

$$|i|^2 - |i'|^2 = |o|^2 - |o'|^2\ , \tag{2.51}$$

which is shorthand for

$$\sum_{j=1}^{N_{\rm c}^{\rm L}} |i_j|^2 + \sum_{a=1}^{N_{\rm c}^{\rm R}} |i'_a|^2 = \sum_{a=1}^{N_{\rm c}^{\rm R}} |o_a|^2 + \sum_{j=1}^{N_{\rm c}^{\rm L}} |o'_j|^2\ . \tag{2.52}$$

Onsager reciprocity demands that $\mathcal{S}(-\boldsymbol{H}) = \mathcal{S}^{\rm t}(\boldsymbol{H})$.

Let us now compute the current in the right lead flowing past the imaginary surface Σ

$$\begin{aligned} I_\Sigma &= -e \sum_{a=1}^{N_{\rm c}^{\rm R}} \int d\varepsilon \, \mathcal{N}_a(\varepsilon) \, v_a(\varepsilon) \Bigg\{ \overbrace{\sum_{i=1}^{N_{\rm c}^{\rm L}} \big| t_{ai}(\varepsilon) \big|^2}^{T_a(\varepsilon)} \, f(\varepsilon - \mu_{\rm L}) \\ &\qquad + \Bigg[\overbrace{\sum_{b=1}^{N_{\rm c}^{\rm R}} \big| r'_{ab}(\varepsilon) \big|^2}^{R'_a(\varepsilon)} - 1 \Bigg] f(\varepsilon - \mu_{\rm R}) \Bigg\} \\ &= \frac{e}{h} \, (\mu_{\rm R} - \mu_{\rm L}) \int d\varepsilon \left(-\frac{\partial f^0}{\partial \varepsilon} \right) \sum_{a=1}^{N_{\rm c}^{\rm R}} T_a(\varepsilon) \ . \end{aligned} \tag{2.53}$$

Thus, the result of a two-probe measurement would be

$$G_{2-{\rm probe}} = \frac{eI}{\mu_{\rm R} - \mu_{\rm L}} = \frac{e^2}{h} \int d\varepsilon \left(-\frac{\partial f^0}{\partial \varepsilon} \right) \sum_{a=1}^{N_{\rm c}^{\rm R}} T_a(\varepsilon) \ . \tag{2.54}$$

At zero temperature, then,

$$\boxed{G_{2-{\rm probe}} = \frac{e^2}{h} \, {\rm Tr} \, t t^\dagger} \tag{2.55}$$

where

$${\rm Tr} \, t t^\dagger = {\rm Tr} \, t^\dagger t = \sum_{i=1}^{N_{\rm c}^{\rm L}} \sum_{a=1}^{N_{\rm c}^{\rm R}} \big| t_{ai} \big|^2 \ . \tag{2.56}$$

To determine $G_{4-{\rm probe}}$, we must compute the effective chemical potentials $\tilde{\mu}_{\rm L}$ and $\tilde{\mu}_{\rm R}$ in the leads. We again do this by equating expressions for the electron number density. In the left lead,

$$n_{\rm L} = \sum_{i=1}^{N_{\rm c}^{\rm L}} \int d\varepsilon \, \mathcal{N}_i(\varepsilon) \Big\{ \big[1 + R_i(\varepsilon) \big] f(\varepsilon - \mu_{\rm L}) + T'_i(\varepsilon) \, f(\varepsilon - \mu_{\rm R}) \Big\} \tag{2.57}$$

$$= 2 \sum_i \int d\varepsilon \, \mathcal{N}_i(\varepsilon) \, f(\varepsilon - \tilde{\mu}_{\rm L}) \tag{2.58}$$

$$\Longrightarrow \quad \tilde{\mu}_{\rm L} = \mu_{\rm L} - \tfrac{1}{2} \overline{T'} \, (\mu_{\rm L} - \mu_{\rm R}) \tag{2.59}$$

where $\overline{T'}$ is a weighted average,

$$\overline{T}' \equiv \frac{\sum_i v_i^{-1} \, T'_i}{\sum_i v_i^{-1}} \ . \tag{2.60}$$

Similarly, one obtains for the right lead,

$$n_{\rm R} = \sum_{a=1}^{N_{\rm c}^{\rm R}} \int d\varepsilon\, \mathcal{N}_a(\varepsilon) \Big\{ T_a(\varepsilon)\, f(\varepsilon - \mu_{\rm L}) + \big[1 + R'_a(\varepsilon)\big]\, f(\varepsilon - \mu_{\rm R}) \Big\} \tag{2.61}$$

$$= 2 \sum_a \int d\varepsilon\, \mathcal{N}_a(\varepsilon)\, f(\varepsilon - \tilde{\mu}_{\rm R}) \tag{2.62}$$

$$\implies \quad \tilde{\mu}_{\rm R} = \mu_{\rm R} + \tfrac{1}{2}\overline{T}\,(\mu_{\rm L} - \mu_{\rm R}) \tag{2.63}$$

where

$$\overline{T} \equiv \frac{\sum_a v_a^{-1}\, T_a}{\sum_a v_a^{-1}} \ . \tag{2.64}$$

(We have assumed zero temperature throughout.) The difference in lead chemical potentials is thus

$$(\tilde{\mu}_{\rm L} - \tilde{\mu}_{\rm R}) = \Big(1 - \tfrac{1}{2}\overline{T} - \tfrac{1}{2}\overline{T'}\Big) \cdot (\mu_{\rm L} - \mu_{\rm R}) \ . \tag{2.65}$$

Hence, we obtain the 4-probe conductance,

$$\boxed{G_{4-\rm probe} = \frac{e^2}{h}\, \frac{\sum_a T_a}{1 - \frac{1}{2}\Big(\sum_i T'_i\, v_i^{-1} \Big/ \sum_i v_i^{-1}\Big) - \frac{1}{2}\Big(\sum_a T_a\, v_a^{-1} \Big/ \sum_a v_a^{-1}\Big)}} \tag{2.66}$$

2.4.1 Transfer Matrices: The Pichard Formula

The transfer matrix $\mathcal{S}$ acts on incoming flux amplitudes to give outgoing flux amplitudes. This linear relation may be recast as one which instead relates flux amplitudes in the right lead to those in the left lead, *i.e.*

$$\begin{pmatrix} o' \\ o \end{pmatrix} = \overbrace{\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}}^{\mathcal{S}} \begin{pmatrix} i \\ i' \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} o \\ i' \end{pmatrix} = \overbrace{\begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}}^{\mathcal{M}} \begin{pmatrix} i \\ o' \end{pmatrix} \ . \tag{2.67}$$

$\mathcal{M}$ is known as the *transfer matrix*. Note that each of the blocks of $\mathcal{M}$ is of dimension $N_{\rm c}^{\rm R} \times N_{\rm c}^{\rm L}$, and $\mathcal{M}$ itself is a rectangular $2N_{\rm c}^{\rm R} \times 2N_{\rm c}^{\rm L}$ matrix. The individual blocks of $\mathcal{M}$ are readily determined:

$$o' = r\, i + t'\, i' \qquad \Longrightarrow \qquad i' = -t'^{-1}\, r\, i + t'^{-1}\, o' \tag{2.68}$$

$$o = t\, i + r'\, i' \qquad \Longrightarrow \qquad o = (t - r'\, t'^{-1}\, r)\, i + r'\, t'^{-1}\, o' \ , \tag{2.69}$$

so we conclude

$$\mathcal{M}_{11} = t^{\dagger - 1} \qquad\qquad \mathcal{M}_{12} = r'\, t'^{-1} \tag{2.70}$$

$$\mathcal{M}_{21} = -t'^{-1}\, r \qquad\qquad \mathcal{M}_{22} = t'^{-1} \ . \tag{2.71}$$

WARNING: *None of this makes any sense if* $N_{\rm c}^{\rm L} \neq N_{\rm c}^{\rm R}$ *!* The reason is that it is problematic to take the inverse of a rectangular matrix such as t or t', as was blithely done

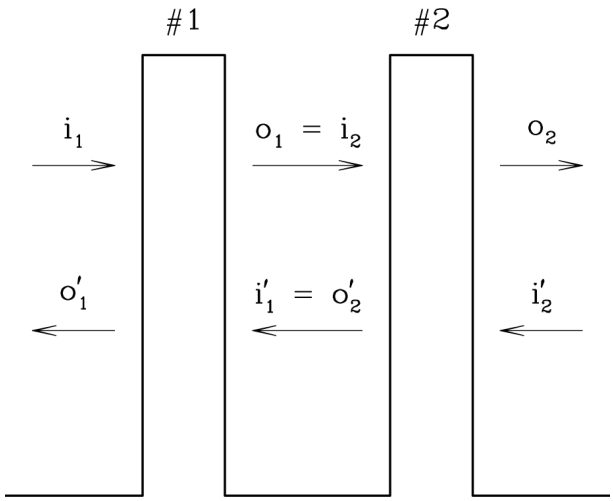

Figure 2.3: Two quantum scatterers in series. The right side data for scatterer #1 become the left side data for scatterer #2.

above in (2.68,2.69). We therefore must assume $N_{\rm c}^{\rm R} = N_{\rm c}^{\rm L} = N_{\rm c}$, and that the scatterers are separated by identical perfect regions. Practically, this imposes no limitations at all, since the width of the perfect regions can be taken to be arbitrarily small.

EXERCISE: Show that $\mathcal{M}_{11} = t - r'\,t'^{-1}\,r = t^{\dagger -1}$.

The virtue of transfer matrices is that they are *multiplicative*. Consider, for example, two disordered regions connected by a region of perfect conductor. The outgoing flux o from the first region becomes the incoming flux i for the second, as depicted in fig. 2.3. Thus, if $\mathcal{M}_1$ is the transfer matrix for scatterer #1, and $\mathcal{M}_2$ is the transfer matrix for scatterer #2, the transfer matrix for the two scatterers in succession is $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1$:

$$\begin{pmatrix} o_2 \\ i'_2 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_2^{11} & \mathcal{M}_2^{12} \\ \mathcal{M}_2^{21} & \mathcal{M}_2^{22} \end{pmatrix} \begin{pmatrix} i_2 \\ o'_2 \end{pmatrix} = \overbrace{\begin{pmatrix} \mathcal{M}_2^{11} & \mathcal{M}_2^{12} \\ \mathcal{M}_2^{21} & \mathcal{M}_2^{22} \end{pmatrix} \begin{pmatrix} \mathcal{M}_1^{11} & \mathcal{M}_1^{12} \\ \mathcal{M}_1^{21} & \mathcal{M}_1^{22} \end{pmatrix}}^{\mathcal{M}=\mathcal{M}_2\,\mathcal{M}_1} \begin{pmatrix} i_1 \\ o'_1 \end{pmatrix} \ . \tag{2.72}$$

Clearly, then, if we have many scatterers in succession, this result generalizes to

$$\mathcal{M} = \mathcal{M}_N\,\mathcal{M}_{N-1}\cdots\mathcal{M}_1 \ . \tag{2.73}$$

Unitarity of the $\mathcal{S}$-matrix means that the transfer matrix is *pseudo-unitary* in that it satisfies

$$\mathcal{M}^\dagger\,\Sigma\,\mathcal{M} = \Sigma \qquad \text{where} \quad \Sigma = \begin{pmatrix} \mathbb{I}_{N_{\rm c}\times N_{\rm c}} & \mathbb{O}_{N_{\rm c}\times N_{\rm c}} \\ \mathbb{O}_{N_{\rm c}\times N_{\rm c}} & -\mathbb{I}_{N_{\rm c}\times N_{\rm c}} \end{pmatrix} \ . \tag{2.74}$$

This, in turn, implies conservation of the pseudo-norm,

$$|o|^2 - |i'|^2 = |i|^2 - |o'|^2 \ , \tag{2.75}$$

which is simply a restatement of (2.51).

We now assert that

$$\Big[\mathcal{M}^\dagger\mathcal{M} + (\mathcal{M}^\dagger\mathcal{M})^{-1} + 2\cdot\mathbb{I}\Big]^{-1} = \tfrac{1}{4}\begin{pmatrix} t^\dagger t & 0 \\ 0 & t't'^\dagger \end{pmatrix} \ . \tag{2.76}$$

This result is in fact easily derived once one notes that

$$\mathcal{M}^{-1} = \Sigma\,\mathcal{M}^\dagger\,\Sigma = \begin{pmatrix} \mathcal{M}^\dagger_{11} & -\mathcal{M}^\dagger_{21} \\ -\mathcal{M}^\dagger_{12} & \mathcal{M}^\dagger_{22} \end{pmatrix} \ . \tag{2.77}$$

EXERCISE: Verify eqn. (2.76).

The 2-probe conductance (per spin channel) may now be written in terms of the transfer matrix as

$$\boxed{G_{2-\rm probe} = \frac{2e^2}{h}\,{\rm Tr}\,\Big[\mathcal{M}^\dagger\mathcal{M} + (\mathcal{M}^\dagger\mathcal{M})^{-1} + 2\cdot\mathbb{I}\Big]^{-1}} \tag{2.78}$$

This is known as the *Pichard Formula*.

2.4.2 Discussion of the Pichard Formula

It is convenient to work in an eigenbasis of the Hermitian matrix $\mathcal{M}^\dagger\mathcal{M}$. The eigenvalues of $\mathcal{M}^\dagger\mathcal{M}$ are roots of the characteristic polynomial

$$p(\lambda) = \det\left(\lambda - \mathcal{M}^\dagger\mathcal{M}\right) \ . \tag{2.79}$$

Owing to the pseudo-unitarity of $\mathcal{M}$, we have

$$\begin{aligned} p(\lambda) &= \det\big(\lambda - \mathcal{M}^\dagger\mathcal{M}\big) \\ &= \det\big(\lambda - \Sigma\,\mathcal{M}^{-1}\,\Sigma\cdot\Sigma\,\mathcal{M}^{\dagger-1}\,\Sigma\big) \\ &= \det\big(\lambda - \Sigma\,\mathcal{M}^{-1}\,\mathcal{M}^{\dagger-1}\,\Sigma\big) \\ &= \lambda^{2N_{\rm c}}\det\big(\lambda^{-1} - \mathcal{M}^\dagger\mathcal{M}\big)/\det\big(\mathcal{M}^\dagger\mathcal{M}\big) \ , \end{aligned} \tag{2.80}$$

from which we conclude that $p(\lambda) = 0$ implies $p(\lambda^{-1}) = 0$, and the eigenvalues of $\mathcal{M}^\dagger\mathcal{M}$ come in (λ, λ^{-1}) pairs. We can therefore write Pichard's formula as

$$G_{2-\rm probe} = \frac{e^2}{h}\sum_{i=1}^{N_{\rm c}} \frac{4}{\lambda_i + \lambda_i^{-1} + 2} \ , \tag{2.81}$$

where without loss of generality we assume $\lambda_i \geq 1$ for each $i \in \{1, \ldots, N_{\rm c}\}$. We define the $i^{\rm th}$ *localization length* ξ_i through

$$\lambda_i \equiv \exp\left(\frac{2L}{\xi_i}\right) \qquad \Longrightarrow \qquad \xi_i = \frac{2L}{\ln\lambda_i} \ , \tag{2.82}$$

where L is the length of the disordered region. We now have

$$\boxed{G_{2-\text{probe}} = \frac{e^2}{h} \sum_{i=1}^{N_c} \frac{2}{1+\cosh(2L/\xi_i)}} \tag{2.83}$$

If N_c is finite, then as $L \to \infty$ the $\{\xi_i\}$ converge to definite values, for a wide range of distributions $P(\mathcal{M}_n)$ for the individual scatterer transfer matrices. This follows from a version of the central limit theorem as applied to nonabelian multiplicative noise (*i.e.* products of random matrices), known as *Oseledec's theorem*. We may choose to order the eigenvalues such that $\lambda_1 < \lambda_2 < \cdots < \lambda_{N_c}$, and hence $\xi_1 > \xi_2 > \cdots > \xi_{N_c}$. In the $L \to \infty$ limit, then, the conductance is dominated by the largest localization length, and

$$G(L) \simeq \frac{4e^2}{h}\, e^{-2L/\xi_1} \qquad (N_c \text{ finite}, L \to \infty) \ . \tag{2.84}$$

(We have dropped the label '2-probe' on G.) The quantity $\xi \equiv \xi_1$ is called the *localization length*, and it is dependent on the (Fermi) energy: $\xi = \xi(\varepsilon)$.

Suppose now that L is finite, and furthermore that $\xi_1 > 2L > \xi_{N_c}$. Channels for which $2L \ll \xi_j$ give $\cosh(2L/\xi_j) \approx 1$, and therefore contribute a quantum of conductance e^2/h to G. These channels are called *open*. Conversely, when $2L \gg \xi_j$, we have $\cosh(2L/\xi_j) \sim \frac{1}{2}\exp(2L/\xi_j) \gg 1$, and these *closed* channels each contribute $\Delta G_j = (2e^2/h)e^{-2L/\xi_j}$ to the conductance, a negligible amount. Thus,

$$G(L) \simeq \frac{e^2}{h}\, N_c^{\text{open}} \qquad , \qquad N_c^{\text{open}} \equiv \sum_{j=1}^{N_c} \Theta(\xi_j - 2L) \ . \tag{2.85}$$

Of course, $N_c^{\text{closed}} = N_c - N_c^{\text{open}}$, although there is no precise definition for open *vs.* closed for channels with $\xi_j \sim 2L$. This discussion naturally leads us to the following classification scheme:

- When $L > \xi_1$, the system is in the *localized regime*. The conductance vanishes exponentially with L according to $G(L) \approx (4e^2/h)\exp(-2L/\xi)$, where $\xi(\varepsilon) = \xi_1(\varepsilon)$ is the *localization length*. In the localized regime, there are no open channels: $N_c^{\text{open}} = 0$.
- When $N_c^{\text{open}} = \ell\, N_c/L$, where ℓ is the *elastic scattering length*, one is in the *Ohmic regime*. In the Ohmic regime, for a d-dimensional system of length L and ($(d-1)$-dimensional) cross-sectional area A,

  $$G_{\text{Ohmic}} \approx \frac{e^2}{h}\frac{\ell}{L}\, k_F^{d-1} A = \frac{e^2}{h}\, k_F^{d-1}\, \ell \cdot \frac{A}{L} \ . \tag{2.86}$$

  Note that G is proportional to the cross sectional area A and inversely proportional to the length L, which is the proper Ohmic behavior: $G = \sigma\, A/L$, where

  $$\sigma \approx \frac{e^2}{h}\, k_F^{d-1}\, \ell \tag{2.87}$$

  is the conductivity.

- When $L < \xi_{N_c}$, all the channels are open: $N_c^{\rm open} = N_c$. The conductance is

$$G(L) = \frac{e^2}{h}\, N_c \approx \frac{e^2}{h}\, k_F^{d-1}\, A \ . \tag{2.88}$$

  This is the *ballistic regime*.

If we keep $N_c \propto (k_F L)^{d-1}$, then for $L \to \infty$ Oseledec's theorem does not apply, because the transfer matrix is ∞-dimensional. If $\xi_1(\varepsilon)$ nonetheless remains finite, then $G(L) \approx (4e^2/h)\exp(-L/\xi) \to 0$ and the system is in the localized regime. If, on the other hand, $\xi_1(\varepsilon)$ diverges as $L \to \infty$ such that $\exp(L/\xi_1)$ is finite, then $G > 0$ and the system is a conductor.

If we define $\nu_i \equiv \ln \lambda_i$, the dimensionless conductance $g = (h/e^2)\, G$ is given by

$$g = 2 \int\limits_0^\infty d\nu\, \frac{\sigma(\nu)}{1 + \cosh \nu} \ , \tag{2.89}$$

where

$$\sigma(\nu) = \sum_{i=1}^{N_c} \delta(\nu - \nu_i) \tag{2.90}$$

is the density of ν values. This distribution is normalized so that $\int_0^\infty d\nu\, \sigma(\nu) = N_c$. Spectral properties of the $\{\nu_i\}$ thus determine the statistics of the conductance. For example, averaging over disorder realizations gives

$$\langle g \rangle = 2 \int\limits_0^\infty d\nu\, \frac{\big\langle \sigma(\nu) \big\rangle}{1 + \cosh \nu} \ . \tag{2.91}$$

The average of g^2, though, depends on the two-point correlation function, *viz.*

$$\langle g^2 \rangle = 4 \int\limits_0^\infty d\nu \int\limits_0^\infty d\nu'\, \frac{\big\langle \sigma(\nu)\, \sigma(\nu') \big\rangle}{(1 + \cosh \nu)(1 + \cosh \nu')} \ . \tag{2.92}$$

2.4.3 Two Quantum Resistors in Series

Let us consider the case of two scatterers in series. For simplicity, we will assume that $N_c = 1$, in which case the transfer matrix for a single scatterer may be written as

$$\mathcal{M} = \begin{pmatrix} 1/t^* & -r^*/t^* \\ -r/t' & 1/t' \end{pmatrix} \ . \tag{2.93}$$

A pristine segment of wire of length L has a diagonal transfer matrix

$$\mathcal{N} = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \ , \tag{2.94}$$

where $\beta = kL$. Thus, the composite transfer matrix for two scatterers joined by a length L of pristine wire is $\mathcal{M} = \mathcal{M}_2 \mathcal{N} \mathcal{M}_1$, *i.e.*

$$\mathcal{M} = \begin{pmatrix} 1/t_2^* & -r_2^*/t_2^* \\ -r_2/t_2' & 1/t_2' \end{pmatrix} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \begin{pmatrix} 1/t_1^* & -r_1^*/t_1^* \\ -r_1/t_1' & 1/t_1' \end{pmatrix} \ . \tag{2.95}$$

In fact, the inclusion of the transfer matrix $\mathcal{N}$ is redundant; the phases $e^{\pm i\beta}$ can be completely absorbed via a redefinition of $\{t_1, t_1', r_1, r_1'\}$.

Extracting the upper left element of $\mathcal{M}$ gives

$$\frac{1}{t^*} = \frac{e^{i\beta} - e^{-i\beta}\, r_1'^* r_2^*}{t_1^* t_2^*} \ , \tag{2.96}$$

hence the transmission coefficient T for the composite system is

$$T = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\,\cos\delta} \tag{2.97}$$

where $\delta = 2\beta + \arg(r_1' r_2)$. The dimensionless Landauer resistance is then

$$\begin{aligned} \mathcal{R} = \frac{R}{T} &= \frac{R_1 + R_2 - 2\sqrt{R_1 R_2}\,\cos\delta}{T_1 T_2} \\ &= \mathcal{R}_1 + \mathcal{R}_2 + 2\,\mathcal{R}_1\mathcal{R}_2 - 2\sqrt{\mathcal{R}_1\mathcal{R}_2(1+\mathcal{R}_1)(1+\mathcal{R}_2)}\,\cos\delta \ . \end{aligned} \tag{2.98}$$

If we average over the random phase δ, we obtain

$$\langle \mathcal{R} \rangle = \mathcal{R}_1 + \mathcal{R}_2 + 2\,\mathcal{R}_1\mathcal{R}_2 \ . \tag{2.99}$$

The first two terms correspond to Ohm's law. The final term is unfamiliar and leads to a divergence of resistivity as a function of length. To see this, imagine that that $\mathcal{R}_2 = \varrho\, dL$ is small, and solve (2.99) iteratively. We then obtain a differential equation for the dimensionless resistance $\mathcal{R}(L)$:

$$d\mathcal{R} = (1 + 2\mathcal{R})\,\varrho\, dL \qquad \Longrightarrow \qquad \mathcal{R}(L) = \tfrac{1}{2}\big(e^{2\varrho L} - 1\big) \ . \tag{2.100}$$

In fact, the distribution $P_L(\mathcal{R})$ is extremely broad, and it is more appropriate to average the quantity $\ln(1 + \mathcal{R})$. Using

$$\int_0^{2\pi} \frac{d\delta}{2\pi}\, \ln(a - b\cos\delta) = \ln\left(\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - b^2}\right) \tag{2.101}$$

with

$$a = 1 + \mathcal{R}_1 + \mathcal{R}_2 + 2\,\mathcal{R}_1\,\mathcal{R}_2 \tag{2.102}$$

$$b = 2\sqrt{\mathcal{R}_1\mathcal{R}_2(1+\mathcal{R}_1)(1+\mathcal{R}_2)} \ , \tag{2.103}$$

we obtain the result

$$\langle \ln(1+\mathcal{R}) \rangle = \ln(1+\mathcal{R}_1) + \ln(1+\mathcal{R}_2) \ . \tag{2.104}$$

We define the quantity

$$x(L) \equiv \ln\big\{1+\mathcal{R}(L)\big\} \ , \tag{2.105}$$

and we observe

$$\big\langle x(L) \big\rangle = \varrho\, L \tag{2.106}$$

$$\big\langle e^{x(L)} \big\rangle = \tfrac{1}{2}\big(e^{2\varrho L}+1\big) \ . \tag{2.107}$$

Note that $\langle e^x \rangle \neq e^{\langle x \rangle}$. The quantity $x(L)$ is an appropriately *self-averaging* quantity in that its root mean square fluctuations are small compared to its average, *i.e.* it obeys the central limit theorem. On the other hand, $\mathcal{R}(L)$ is *not* self-averaging, *i.e.* it is not normally distributed.

Abelian Multiplicative Random Processes

Let $p(x)$ be a distribution on the nonnegative real numbers, normalized according to

$$\int\limits_0^\infty dx\, p(x) = 1 \ , \tag{2.108}$$

and define

$$X \equiv \prod_{i=1}^N x_i \qquad , \qquad Y \equiv \ln X = \sum_{i=1}^N \ln x_i \ . \tag{2.109}$$

The distribution for Y is

$$\begin{aligned} P_N(Y) &= \int\limits_0^\infty dx_1 \int\limits_0^\infty dx_2 \cdots \int\limits_0^\infty dx_N\, p(x_1)\, p(x_2) \cdots p(x_N)\, \delta\left(Y - \sum_{i=1}^N \ln x_i\right) \\ &= \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi}\, e^{i\omega Y} \left\{ \int\limits_0^\infty dx\, p(x)\, e^{-i\omega \ln x} \right\}^N \\ &= \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi}\, e^{i\omega Y} \Big[1 - i\omega \langle \ln x \rangle - \tfrac{1}{2}\omega^2 \langle \ln^2 x \rangle + \mathcal{O}(\omega^3) \Big]^N \\ &= \int\limits_{-\infty}^\infty \frac{d\omega}{2\pi}\, e^{i\omega (Y - N\langle \ln x \rangle)}\, e^{-\frac{1}{2} N\omega^2 \left(\langle \ln^2 x \rangle - \langle \ln x \rangle^2 \right) + \mathcal{O}(\omega^3)} \\ &= \frac{1}{\sqrt{2\pi N \sigma^2}}\, e^{-(Y-N\mu)^2/2N\sigma^2} \ \cdot \Big\{ 1 + \mathcal{O}(N^{-1}) \Big\} \ , \end{aligned} \tag{2.110}$$

with

$$\mu = \langle \ln x \rangle \qquad , \qquad \sigma^2 = \langle \ln^2 x \rangle - \langle \ln x \rangle^2 \tag{2.111}$$

and

$$\big\langle f(x)\big\rangle \equiv \int\limits_0^\infty dx\, p(x)\, f(x)\ . \tag{2.112}$$

Thus, Y is normally distributed with mean $\langle Y\rangle = N\mu$ and standard deviation $\langle (Y-N\mu)^2\rangle = N\sigma^2$. This is typical for extensive self-averaging quantities: the average is proportional to the size N of the system, and the root mean square fluctuations are proportional to $\sqrt{N}$. Since $\lim_{N\to\infty} Y_{\rm rms}/\langle Y\rangle \sim \sigma/\sqrt{N}\mu \to 0$, we have that

$$P_{N\to\infty}(Y) \simeq \delta\big(Y - N\mu\big)\ . \tag{2.113}$$

This is the central limit theorem (CLT) at work. The quantity Y is a sum of independent random variables: $Y = \sum_i \ln x_i$, and is therefore normally distributed with a mean $\bar Y = N\mu$ and standard deviation $\sqrt{N}\sigma$, as guaranteed by the CLT. On the other hand, $X = \exp(Y)$ is *not* normally distributed. Indeed, one readily computes the moments of X to be

$$\langle X^k\rangle = e^{kN\mu}\, e^{Nk^2/2\sigma^2}\ , \tag{2.114}$$

hence

$$\frac{\langle X^k\rangle}{\langle X\rangle^k} = e^{Nk(k-1)/2\sigma^2}\ , \tag{2.115}$$

which increases exponentially with N. In particular, one finds

$$\frac{\sqrt{\langle X^2\rangle - \langle X\rangle^2}}{\langle X\rangle} = \Big(e^{N/\sigma^2} - 1\Big)^{1/2}\ . \tag{2.116}$$

The multiplication of random transfer matrices is a more difficult problem to analyze, owing to its essential nonabelian nature. However, as we have seen in our analysis of series quantum resistors, a similar situation pertains: it is the logarithm $\ln(1+\mathcal{R})$, and not the dimensionless resistance $\mathcal{R}$ itself, which is an appropriate self-averaging quantity.

2.4.4 Two Quantum Resistors in Parallel

The case of parallel quantum resistors is more difficult than that of series resistors. The reason for this is that the conduction path for parallel resistances is multiply connected, *i.e.* electrons can get from start to finish by traveling through either resistor #1 or resistor #2.

Consider electrons with wavevector $k > 0$ moving along a line. The wavefunction is

$$\psi(x) = I\, e^{ikx} + O'\, e^{-ikx}\ , \tag{2.117}$$

hence the transfer matrix $\mathcal{M}$ for a length L of pristine wire is

$$\mathcal{M}(L) = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix}\ . \tag{2.118}$$

Now let's bend our wire of length L into a ring. We therefore identify the points $x = 0$ and $x = L = 2\pi R$, where R is the radius. In order for the wavefunction to be single-valued we must have

$$\Big[\mathcal{M}(L) - \mathbb{I}\Big] \begin{pmatrix} I \\ O' \end{pmatrix} = 0 \ , \tag{2.119}$$

and in order to have a nontrivial solution (*i.e.* I and O' not both zero), we must demand $\det(\mathcal{M} - \mathbb{I}) = 0$, which says $\cos kL = 1$, *i.e.* $k = 2\pi n/L$ with integer n. The energy is then quantized: $\varepsilon_n = \varepsilon_{\scriptscriptstyle\parallel}(k = 2\pi n/L)$.

Next, consider the influence of a vector potential on the transfer matrix. Let us assume the vector potential A along the direction of motion is nonzero over an interval from $x = 0$ to $x = d$. The Hamiltonian is given by the *Peierls substitution*,

$$\mathcal{H} = \varepsilon_{\scriptscriptstyle\parallel}\Big(-i\partial_x + \tfrac{e}{\hbar c}A(x)\Big) \ . \tag{2.120}$$

Note that we can write

$$\mathcal{H} = \Lambda^\dagger(x)\,\varepsilon_{\scriptscriptstyle\parallel}(-i\partial_x)\,\Lambda(x) \tag{2.121}$$

$$\Lambda(x) = \exp\left\{\frac{ie}{\hbar c}\int\limits_0^x dx'\,A(x')\right\} \ . \tag{2.122}$$

Hence the solutions $\psi(x)$ to $\mathcal{H}\psi = \varepsilon\psi$ are given by

$$\psi(x) = I\,\Lambda^\dagger(x)\,e^{ikx} + O'\,\Lambda^\dagger(x)\,e^{-ikx} \ . \tag{2.123}$$

The transfer matrix for a segment of length d is then

$$\mathcal{M}(d, A) = \begin{pmatrix} e^{ikd}\,e^{-i\gamma} & 0 \\ 0 & e^{-ikd}\,e^{-i\gamma} \end{pmatrix} \tag{2.124}$$

with

$$\gamma = \frac{e}{\hbar c}\int\limits_0^d dx\,A(x) \ . \tag{2.125}$$

We are free to choose any gauge we like for $A(x)$. The only constraint is that the gauge-invariant content, which is encoded in he magnetic fluxes through every closed loop $\mathcal{C}$,

$$\Phi_{\mathcal{C}} = \oint\limits_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{l} \ , \tag{2.126}$$

must be preserved. On a ring, there is one flux Φ to speak of, and we define the dimensionless flux $\phi = e\Phi/\hbar c = 2\pi\Phi/\phi_0$, where $\phi_0 = hc/e = 4.137\times10^{-7}\,\mathrm{G{\cdot}cm^2}$ is the Dirac flux quantum. In a field of $B = 1\,\mathrm{kG}$, a single Dirac quantum is enclosed by a ring of radius $R = 0.11\,\mu\mathrm{m}$. It is convenient to choose a gauge in which A vanishes everywhere along our loop except for

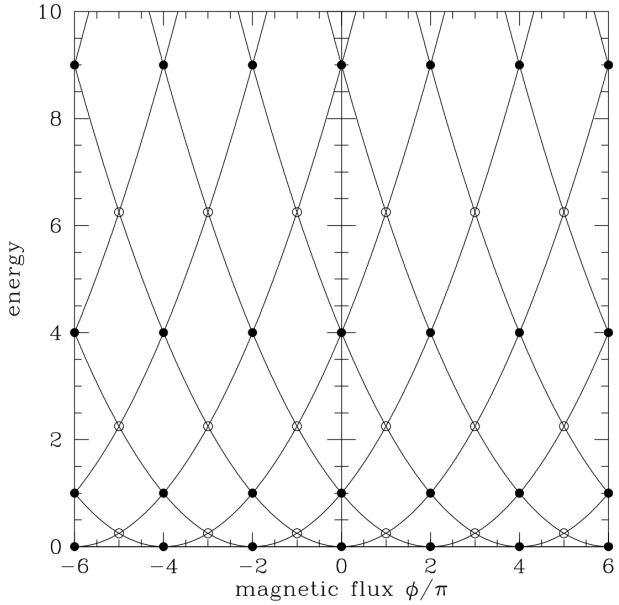

Figure 2.4: Energy *versus* dimensionless magnetic flux for free electrons on a ring. The degeneracies are lifted in the presence of a crystalline potential.

a vanishingly small region, in which all the accrued vector potential piles up in a δ-function of strength Φ. The transfer matrix for this infinitesimal region is then

$$\mathcal{M}(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} = e^{i\phi} \cdot \mathbb{I} \; . \tag{2.127}$$

If $k > 0$ corresponds to clockwise motion around the ring, then the phase accrued is $-\gamma$, which explains the sign of ϕ in the above equation.

For a pristine ring, then, combining the two transfer matrices gives

$$\mathcal{M} = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{ikL}\, e^{i\phi} & 0 \\ 0 & e^{-ikL}\, e^{i\phi} \end{pmatrix} \; , \tag{2.128}$$

and thus $\det(\mathcal{M} - \mathbb{I}) = 0$ gives the solutions,

$$\begin{aligned} kL &= 2\pi n - \phi \qquad \text{(right-movers)} \\ kL &= 2\pi n + \phi \qquad \text{(left-movers)} \; . \end{aligned}$$

Note that different n values are allowed for right- and left-moving branches since by assumption $k > 0$. We can simplify matters if we simply write $\psi(x) = A\, e^{ikx}$ with k unrestricted in sign, in which case $k = (2\pi n - \phi)/L$ with n chosen from the entire set of integers. The allowed energies for free electrons are then

$$\varepsilon_n(\phi) = \frac{2\pi^2\hbar^2}{mL^2} \cdot \big(n - \frac{\phi}{2\pi}\big)^2 \; , \tag{2.129}$$

which are plotted in fig. 2.4.

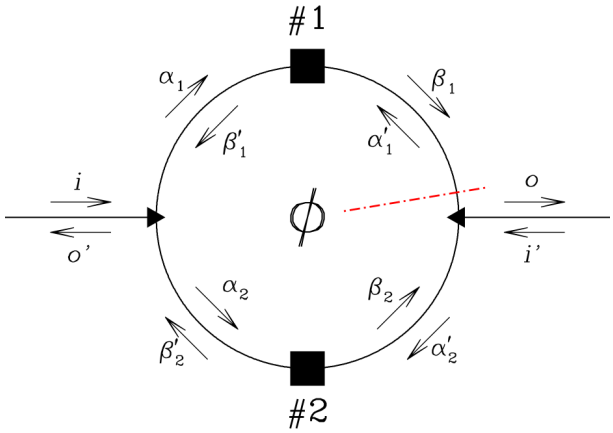

Figure 2.5: Scattering problem for a ring enclosing a flux Φ. The square and triangular blocks represent scattering regions and are describes by 2×2 and 3×3 $\mathcal{S}$-matrices, respectively. The dot-dash line represents a cut across which the phase information due to the enclosed flux is accrued discontinuously.

Now let us add in some scatterers. This problem was first considered in a beautiful paper by Büttiker, Imry, and Azbel, *Phys. Rev. A* **30**, 1982 (1984). Consider the ring geometry depicted in fig. 2.5. We want to compute the $\mathcal{S}$-matrix for the ring. We now know how to describe the individual quantum resistors #1 and #2 in terms of $\mathcal{S}$-matrices (or, equivalently, $\mathcal{M}$-matrices). Assuming there is no magnetic field penetrating the wire (or that the wire itself is infinitesimally thin), we have $\mathcal{S} = \mathcal{S}^{\rm t}$ for each scatterer. In this case, we have $t = t' = \sqrt{T}\, e^{i\alpha}$. We know $|r|^2 = |r'|^2 = 1 - |t|^2$, but in general r and r' may have different phases. The most general 2×2 transfer matrix, under conditions of time-reversal symmetry, depends on three parameters, which may be taken to be the overall transmission probability T and two phases:

$$\mathcal{M}(T, \alpha, \beta) = \frac{1}{\sqrt{T}} \begin{pmatrix} e^{i\alpha} & \sqrt{1-T}\, e^{i\beta} \\ \sqrt{1-T}\, e^{-i\beta} & e^{-i\alpha} \end{pmatrix} \ . \tag{2.130}$$

We can include the effect of free-particle propagation in the transfer matrix $\mathcal{M}$ by multiplying $\mathcal{M}$ on the left and the right by a free propagation transfer matrix of the form

$$\mathcal{N} = \begin{pmatrix} e^{i\theta/4} & 0 \\ 0 & e^{-i\theta/4} \end{pmatrix} \ , \tag{2.131}$$

where $\theta = kL = 2\pi kR$ is the phase accrued by a particle of wavevector k freely propagating once around the ring. $\mathcal{N}$ is the transfer matrix corresponding to one quarter turn around the ring. One easily finds

$$\mathcal{M} \to \mathcal{N}\,\mathcal{M}(T, \alpha, \beta)\,\mathcal{N} = \mathcal{M}(T, \alpha + \tfrac{1}{2}\theta, \beta). \tag{2.132}$$

For pedagogical reasons, we will explicitly account for the phases due to free propagation, and write

$$\begin{pmatrix} \beta_1 \\ \alpha'_1 \end{pmatrix} = \mathcal{N}\,\mathcal{M}_1\,\mathcal{N} \begin{pmatrix} \alpha_1 \\ \beta'_1 \end{pmatrix} \qquad , \qquad \begin{pmatrix} \beta'_2 \\ \alpha_2 \end{pmatrix} = \mathcal{N}\,\widetilde{\mathcal{M}}_2\,\mathcal{N} \begin{pmatrix} \alpha'_2 \\ \beta_2 \end{pmatrix} \ , \tag{2.133}$$

where $\widetilde{\mathcal{M}}_2$ is the transfer matrix for scatterer #2 going from right to left.

EXERCISE: Show that the right-to-left transfer matrix $\widetilde{\mathcal{M}}$ is related to the left-to-right transfer matrix $\mathcal{M}$ according to

$$\widetilde{\mathcal{M}} = \Lambda\,\Sigma\,\mathcal{M}^\dagger\,\Sigma\,\Lambda\ , \tag{2.134}$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad , \qquad \Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ . \tag{2.135}$$

We now have to model the connections between the ring and the leads, which lie at the confluence of three segments. Accordingly, these regions are described by 3×3 $\mathcal{S}$-matrices. The constraints $\mathcal{S} = \mathcal{S}^\dagger$ (unitarity) and $\mathcal{S} = \mathcal{S}^{\rm t}$ (time-reversal symmetry) reduce the number of independent real parameters in $\mathcal{S}$ from 18 to 5. Further assuming that the scattering is symmetric with respect to the ring branches brings this number down to 3, and finally assuming $\mathcal{S}$ is real reduces the dimension of the space of allowed $\mathcal{S}$-matrices to one. Under these conditions, the most general 3×3 $\mathcal{S}$-matrix may be written

$$\begin{pmatrix} -(a+b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & a & b \\ \sqrt{\epsilon} & b & a \end{pmatrix} \tag{2.136}$$

where

$$(a+b)^2 + 2\epsilon = 1 \tag{2.137}$$

$$a^2 + b^2 + \epsilon = 1\ . \tag{2.138}$$

The parameter ϵ, which may be taken as a measure of the coupling between the ring and the leads ($\epsilon = 0$ means ring and leads are decouped) is restricted to the range $0 \le \epsilon \le \frac{1}{2}$. There are four solutions for each allowed value of ϵ:

$$a = \pm\tfrac{1}{2}\big(\sqrt{1-2\epsilon} - 1\big) \quad , \quad b = \pm\tfrac{1}{2}\big(\sqrt{1-2\epsilon} + 1\big) \tag{2.139}$$

and

$$a = \pm\tfrac{1}{2}\big(\sqrt{1-2\epsilon} + 1\big) \quad , \quad b = \pm\tfrac{1}{2}\big(\sqrt{1-2\epsilon} - 1\big)\ . \tag{2.140}$$

We choose the first pair, since it corresponds to the case $|b| = 1$ when $\epsilon = 0$, *i.e.* perfect transmission through the junction. We choose the top sign in (2.139).

We therefore have at the left contact,

$$\begin{pmatrix} o' \\ \alpha_2 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} -(a_{\rm L} + b_{\rm L}) & \sqrt{\epsilon}_{\rm L} & \sqrt{\epsilon}_{\rm L} \\ \sqrt{\epsilon}_{\rm L} & a_{\rm L} & b_{\rm L} \\ \sqrt{\epsilon}_{\rm L} & b_{\rm L} & a_{\rm L} \end{pmatrix} \begin{pmatrix} i \\ \beta_2' \\ \beta_1' \end{pmatrix}\ , \tag{2.141}$$

and at the right contact

$$\begin{pmatrix} o \\ \widetilde{\alpha}_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} -(a_{\rm R} + b_{\rm R}) & \sqrt{\epsilon}_{\rm R} & \sqrt{\epsilon}_{\rm R} \\ \sqrt{\epsilon}_{\rm R} & a_{\rm R} & b_{\rm R} \\ \sqrt{\epsilon}_{\rm R} & b_{\rm R} & a_{\rm R} \end{pmatrix} \begin{pmatrix} i' \\ \widetilde{\beta}_1 \\ \beta_2 \end{pmatrix}\ , \tag{2.142}$$

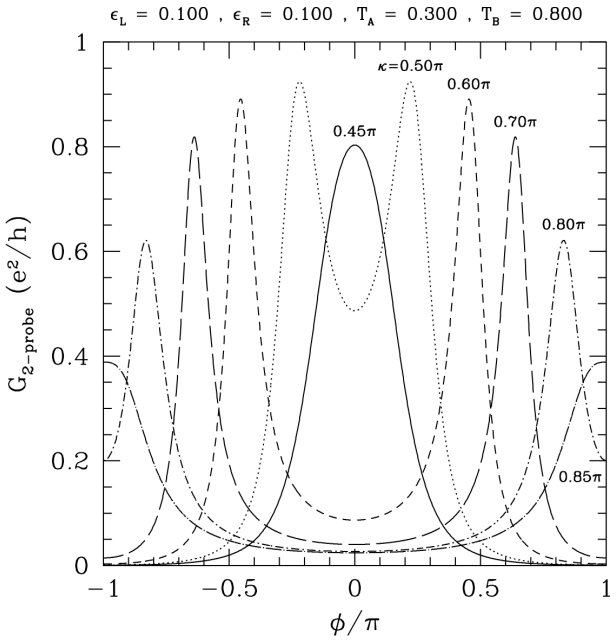

Figure 2.6: Two-probe conductance $G(\phi, \kappa)$ of a model ring with two scatterers. The enclosed magnetic flux is $\phi\hbar c/e$, and $\kappa = 2\pi kR$ (ring radius R). G *vs.* ϕ curves for various values of κ are shown.

where accounting for the vector potential gives us

$$\widetilde{\beta}_1 = e^{i\phi}\,\beta_1 \quad , \quad \widetilde{\alpha}'_1 = e^{i\phi}\,\alpha'_1 \ . \tag{2.143}$$

We set $i = 1$ and $i' = 0$, so that the transmission and reflection amplitudes are obtained from $t = o$ and $r = o'$.

From (2.141), we can derive the relation

$$\begin{pmatrix} \alpha_1 \\ \beta'_1 \end{pmatrix} = \overbrace{\frac{1}{b_{\rm L}} \begin{pmatrix} b_{\rm L}^2 - a_{\rm L}^2 & a_{\rm L} \\ -a_{\rm L} & 1 \end{pmatrix}}^{\mathcal{Q}_{\rm L}} \begin{pmatrix} \beta'_2 \\ \alpha_2 \end{pmatrix} + \frac{\sqrt{\epsilon_{\rm L}}}{b_{\rm L}} \begin{pmatrix} b_{\rm L} - a_{\rm L} \\ -1 \end{pmatrix} \ . \tag{2.144}$$

Similarly, from (2.142), we have

$$\begin{pmatrix} \alpha'_2 \\ \beta_2 \end{pmatrix} = \overbrace{\frac{1}{b_{\rm R}} \begin{pmatrix} b_{\rm R}^2 - a_{\rm R}^2 & a_{\rm R} \\ -a_{\rm R} & 1 \end{pmatrix}}^{\mathcal{Q}_{\rm R}} e^{i\phi} \begin{pmatrix} \beta_1 \\ \alpha'_1 \end{pmatrix} \ . \tag{2.145}$$

The matrices $\mathcal{Q}_{\rm L}$ and $\mathcal{Q}_{\rm R}$ resemble transfer matrices. However, they are not pseudo-unitary: $\mathcal{Q}^\dagger\,\Sigma\,\mathcal{Q} \neq \Sigma$. This is because some of the flux can leak out along the leads. Indeed, when

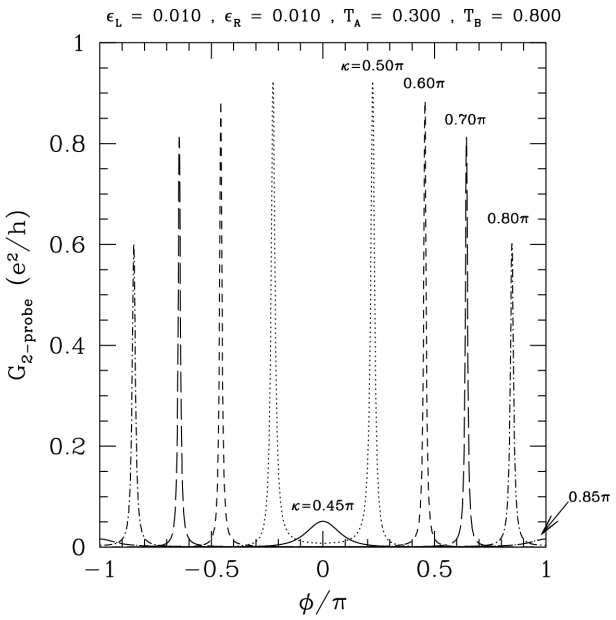

Figure 2.7: Two-probe conductance $G(\phi,\kappa)$ of a model ring with two scatterers. The enclosed magnetic flux is $\phi\hbar c/e$, and $\kappa = 2\pi kR$ (ring radius R). G *vs.* ϕ curves for various values of κ are shown. The coupling between leads and ring is one tenth as great as in fig. 2.6, and accordingly the resonances are much narrower. Note that the resonances at $\kappa = 0.45\pi$ and $\kappa = 0.85\pi$ are almost completely suppressed.

$\epsilon = 0$, we have $b = 1$ and $a = 0$, hence $\mathcal{Q} = 1$, which *is* pseudo-unitary (*i.e.* flux preserving). Combining these results with those in (2.133), we obtain the solution

$$\begin{pmatrix}\alpha_1\\ \beta_1'\end{pmatrix} = \Big\{\mathbb{I} - e^{i\phi}\,\mathcal{Q}_{\rm L}\,\mathcal{N}\,\widetilde{\mathcal{M}}_2\,\mathcal{N}\,\mathcal{Q}_{\rm R}\,\mathcal{N}\,\mathcal{M}_1\,\mathcal{N}\Big\}^{-1}\frac{\sqrt{\epsilon_{\rm L}}}{b_{\rm L}}\begin{pmatrix}b_{\rm L}-a_{\rm L}\\ -1\end{pmatrix}\ . \tag{2.146}$$

From this result, using (2.133), all the flux amplitudes can be obtained.

We can define the effective ring transfer matrix $\mathcal{P}$ as

$$\mathcal{P} \equiv e^{i\phi}\,\mathcal{Q}_{\rm L}\,\mathcal{N}\,\widetilde{\mathcal{M}}_2\,\mathcal{N}\,\mathcal{Q}_{\rm R}\,\mathcal{N}\,\mathcal{M}_1\,\mathcal{N}\ , \tag{2.147}$$

which has the following simple interpretation. Reading from right to left, we first move $\frac{1}{4}$-turn clockwise around the ring ($\mathcal{N}$). Then we encounter scatterer #1 ($\mathcal{M}_1$). After another quarter turn ($\mathcal{N}$), we encounter the right T-junction ($\mathcal{Q}_{\rm R}$). Then it's yet another quarter turn ($\mathcal{N}$) until scatterer #2 ($\mathcal{M}_2$), and one last quarter turn ($\mathcal{N}$) brings us to the left T-junction ($\mathcal{Q}_{\rm L}$), by which point we have completed one revolution. As the transfer matrix acts on both right-moving and left-moving flux amplitudes, it accounts for both clockwise as well as counterclockwise motion around the ring. The quantity

$$\big\{\mathbb{I} - \mathcal{P}\big\}^{-1} = 1 + \mathcal{P} + \mathcal{P}^2 + \mathcal{P}^3 + \dots\ , \tag{2.148}$$

then sums up over all possible integer windings around the ring. In order to properly account for the effects of the ring, an infinite number of terms must be considered; these may be

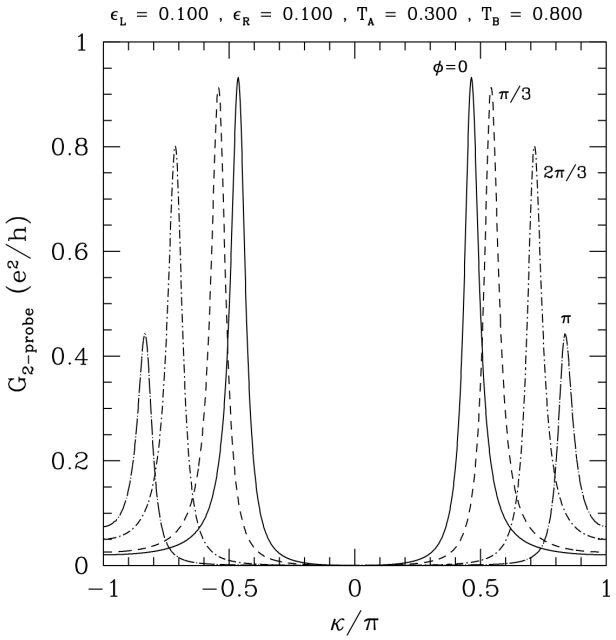

Figure 2.8: Two-probe conductance $G(\phi, \kappa)$ of a model ring with two scatterers. The enclosed magnetic flux is $\phi\hbar c/e$, and $\kappa = 2\pi kR$ (ring radius R). G *vs.* κ curves for various values of ϕ are shown.

resummed into the matrix inverse in (2.147). The situation is analogous to what happens when an electromagnetic wave reflects off a thin dielectric slab. At the top interface, the wave can reflect. However, it can also refract, entering the slab, where it may undergo an arbitrary number of internal reflections before exiting.

The transmission coefficient t is just the outgoing flux amplitude: $t = o$. We have from (2.142,2.133) that

$$\begin{aligned} t &= \sqrt{\epsilon_{\rm R}}\,\big(\beta_1\, e^{i\phi} + \beta_2\big) \\ &= \frac{\sqrt{\epsilon_{\rm R}}}{b_{\rm R}}\,\begin{pmatrix} b_{\rm R} - a_{\rm R} & 1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \alpha_1' \end{pmatrix} \\ &= \frac{\sqrt{\epsilon_{\rm L}\,\epsilon_{\rm R}}}{b_{\rm L}\, b_{\rm R}}\,\big(\mathcal{Y}_{12}^{-1} + \mathcal{Y}_{22}^{-1} - \mathcal{Y}_{11}^{-1} - \mathcal{Y}_{21}^{-1}\big) \end{aligned} \tag{2.149}$$

where

$$\mathcal{Y} = \mathcal{Q}_{\rm L}\,\mathcal{N}\,\widetilde{\mathcal{M}}_2\,\mathcal{N}\,\mathcal{Q}_{\rm R} - e^{-i\phi}\,\mathcal{N}^{-1}\mathcal{M}_1^{-1}\mathcal{N}^{-1} \quad . \tag{2.150}$$

It is straightforward to numerically implement the above calculation. Sample results are shown in figs. 2.6 and 2.8.

2.5 Universal Conductance Fluctuations in Dirty Metals

The conductance of a disordered metal is a function of the strength and location of the individual scatterers. We now ask, how does the conductance fluctuate when the position or strength of a scatterer or a group of scatterers is changed. From the experimental point of view, this seems a strange question to ask, since we generally do not have direct control over the position of individual scatterers within a bulk system. However, we can imagine changing some external parameter, such as the magnetic field B or the chemical potential μ (via the density n). Using computer modeling, we can even 'live the dream' of altering the position of a single scatterer to investigate its effect on the overall conductance. Naïvely, we would expect there to be very little difference in the conductance if we were to, say, vary the position of a single scatterer by a distance ℓ, or if we were to change the magnetic field by $\Delta B = \phi_0/A$, where A is the cross sectional area of the system. Remarkably, though, what is found both experimentally and numerically is that the conductance exhibits fluctuations with varying field B, chemical potential μ, or impurity configuration (in computer models). The root-mean-square magnitude of these fluctuations for a given sample is the same as that between different samples, and is on the order $\delta G \sim e^2/h$. These *universal conductance fluctuations* (UCF) are independent on the degree of disorder, the sample size, the spatial dimensions, so long as the inelastic mean free path (or *phase breaking length*) satisfies $L_\phi > L$, *i.e.* the system is mesoscopic.

Theoretically the phenomenon of UCF has a firm basis in diagrammatic perturbation theory. Here we shall content ourselves with understanding the phenomenon on a more qualitative level, following the beautiful discussion of P. A. Lee in *Physica* **140A**, 169 (1986). We begin with the multichannel Landauer formula,

$$G = \frac{e^2}{h}\,\mathrm{Tr}\,tt^\dagger = \frac{e^2}{h}\sum_{a,j=1}^{N_c} |t_{aj}|^2 \ . \tag{2.151}$$

The transmission amplitudes t_{aj} can be represented as a quantum mechanical sum over paths γ,

$$t_{aj} = \sum_\gamma \mathcal{A}_{aj}(\gamma) \ , \tag{2.152}$$

where $\mathcal{A}_{aj}(\gamma)$ is the probability amplitude for Feynman path γ to connect channels j and a. The sum is over all such Feynman paths, and ultimately we must project onto a subspace of definite energy – this is, in Lee's own words, a 'heuristic argument'. Now assume that the $\mathcal{A}_{aj}(\gamma)$ are independent complex random variables. The fluctuations in $|t_{aj}|^2$ are computed from

$$\begin{aligned}\big\langle |t_{aj}|^4 \big\rangle &= \sum_{\substack{\gamma_1,\gamma_2\\ \gamma_3,\gamma_4}} \big\langle \mathcal{A}_{aj}(\gamma_1)\,\mathcal{A}^*_{aj}(\gamma_2)\,\mathcal{A}_{aj}(\gamma_3)\,\mathcal{A}^*_{aj}(\gamma_4)\big\rangle \\ &= 2\Big\langle \sum_\gamma \big|\mathcal{A}_{aj}(\gamma)\big|^2\Big\rangle^2 + \big\langle \sum_\gamma \big|\mathcal{A}_{aj}(\gamma)\big|^4\big\rangle \\ &= 2\big\langle |t_{aj}|^2\big\rangle^2 \cdot \Big\{1 + \mathcal{O}(M^{-1})\Big\} \ ,\end{aligned} \tag{2.153}$$

where M is the (extremely large) number of paths in the sum over γ. As the $\mathcal{O}(M^{-1})$ term is utterly negligible, we conclude

$$\frac{\big\langle |t_{aj}|^4 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle^2}{\big\langle |t_{aj}|^2 \big\rangle^2} = 1 \ . \tag{2.154}$$

Now since

$$\mathrm{var}(G) = \langle G^2 \rangle - \langle G \rangle^2 \tag{2.155}$$

$$= \frac{e^4}{h^2} \sum_{\substack{a,a' \\ j,j'}} \Big\{ \big\langle |t_{aj}|^2 \, |t_{a'j'}|^2 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle \big\langle |t_{aj}|^2 \big\rangle \Big\} \ , \tag{2.156}$$

we also need to know about the correlation between $|t_{aj}|^2$ and $|t_{a'j'}|^2$. The simplest assumption is to assume they are uncorrelated unless $a = a'$ and $j = j'$, *i.e.*

$$\big\langle |t_{aj}|^2 \, |t_{a'j'}|^2 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle \big\langle |t_{aj}|^2 \big\rangle = \Big(\big\langle |t_{aj}|^4 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle^2 \Big) \delta_{aa'} \, \delta_{jj'} \ , \tag{2.157}$$

in which case the conductance is a given by a sum of $N_{\rm c}^2$ independent real random variables, each of which has a standard deviation equal to its mean, *i.e.* equal to $\big\langle |t_{aj}|^2 \big\rangle$. According to the central limit theorem, then, the rms fluctuations of G are given by

$$\Delta G = \sqrt{\mathrm{var}(G)} = \frac{e^2}{h} N_{\rm c} \, \big\langle |t_{aj}|^2 \big\rangle \ . \tag{2.158}$$

Further assuming that we are in the Ohmic regime where $G = \sigma \, L^{d-2}$, with $\sigma \approx (e^2/h) \, k_{\rm F}^{d-1} \ell$, and $N_{\rm c} \approx (k_{\rm F} L)^{d-1}$, we finally conclude

$$\big\langle |t_{aj}|^2 \big\rangle \approx \frac{1}{N_{\rm c}} \cdot \frac{\ell}{L} \quad \Longrightarrow \quad \Delta G \approx \frac{e^2}{h} \, \frac{\ell}{L} \ . \tag{2.159}$$

This result is much smaller than the correct value of $\Delta G \sim (e^2/h)$.

To reiterate the argument in terms of the dimensionless conductance g,

$$g = \sum_{a,j} |t_{aj}|^2 \simeq N_{\rm c} \cdot \frac{\ell}{L} \quad \Longrightarrow \quad \big\langle |t_{aj}|^2 \big\rangle = \frac{g}{N_{\rm c}^2} \tag{2.160}$$

$$\mathrm{var}(g) = \sum_{\substack{a,a' \\ j,j'}} \Big\{ \big\langle |t_{aj}|^2 \, |t_{a'j'}|^2 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle \big\langle |t_{aj}|^2 \big\rangle \Big\}$$

$$\approx \sum_{aj} \Big\{ \big\langle |t_{aj}|^4 \big\rangle - \big\langle |t_{aj}|^2 \big\rangle^2 \Big\}$$

$$= \sum_{aj} \big\langle |t_{aj}|^2 \big\rangle^2 \approx N_{\rm c}^2 \cdot \Big(\frac{g}{N_{\rm c}^2} \Big)^2 = \frac{g^2}{N_{\rm c}^2} \tag{2.161}$$

$$\Longrightarrow \quad \boxed{ \sqrt{\mathrm{var}(g)} \approx \frac{g}{N_{\rm c}} = \frac{\ell}{L} \quad \text{(WRONG!)} } \ . \tag{2.162}$$

What went wrong? The problem lies in the assumption that the contributions $\mathcal{A}_{aj}(\gamma)$ are independent for different paths γ. The reason is that in disordered systems there are certain preferred channels within the bulk along which the conduction paths run. Different paths γ will often coincide along these channels. A crude analogy: whether you're driving from La Jolla to Burbank, or from El Cajon to Malibu, eventually you're going to get on the 405 freeway – anyone driving from the San Diego area to the Los Angeles area must necessarily travel along one of a handful of high-volume paths. The same is not true of reflection, though! Those same two hypothetical drivers executing local out-and-back trips from home will in general travel along completely different, hence uncorrelated, routes. Accordingly, let us compute $\mathrm{var}(N_{\rm c} - g)$, which is identical to $\mathrm{var}(g)$, but is given in terms of a sum over reflection coefficients. We will see that making the same assumptions as we did in the case of the transmission coefficients produces the desired result. We need only provide a sketch of the argument:

$$N_{\rm c} - g = \sum_{i,j} |r_{ij}|^2 \simeq N_{\rm c} \cdot \Big(1 - \frac{\ell}{L}\Big) \quad \Longrightarrow \quad \big\langle |r_{ij}|^2 \big\rangle = \frac{N_{\rm c} - g}{N_{\rm c}^2} \ , \tag{2.163}$$

so

$$\begin{aligned} \mathrm{var}(N_{\rm c} - g) &= \sum_{\substack{i,i' \\ j,j'}} \Big\{ \big\langle |r_{ij}|^2 \, |r_{i'j'}|^2 \big\rangle - \big\langle |r_{ij}|^2 \big\rangle \big\langle |r_{i'j'}|^2 \big\rangle \Big\} \\ &\approx \sum_{ij} \Big\{ \big\langle |r_{ij}|^4 \big\rangle - \big\langle |r_{ij}|^2 \big\rangle^2 \Big\} \\ &= \sum_{ij} \big\langle |r_{ij}|^2 \big\rangle^2 \approx N_{\rm c}^2 \cdot \Big(\frac{N_{\rm c} - g}{N_{\rm c}^2} \Big)^2 = \Big(1 - \frac{\ell}{L}\Big)^2 \end{aligned} \tag{2.164}$$

$$\Longrightarrow \quad \boxed{ \sqrt{\mathrm{var}(N_{\rm c} - g)} = \sqrt{\mathrm{var}(g)} = \Big(1 - \frac{\ell}{L}\Big) } \tag{2.165}$$

The assumption of uncorrelated *reflection* paths is not as problematic as that of uncorrelated *transmission* paths. Again, this is due to the existence of preferred internal channels within the bulk, along which transmission occurs. In reflection, though there is no need to move along identical segments.

There is another bonus to thinking about reflection *versus* transmission. Let's express the reflection probability as a sum over paths, *viz.*

$$|r_{ij}|^2 = \sum_{\gamma,\gamma'} \mathcal{A}_{ij}(\gamma) \, \mathcal{A}^*_{ij}(\gamma') \ . \tag{2.166}$$

Each path γ will have a time-reversed mate $\gamma^{\rm T}$ for which, in the absence of external magnetic fields,

$$\mathcal{A}_{ij}(\gamma) = \mathcal{A}_{ij}(\gamma^{\rm T}) \ . \tag{2.167}$$

This is because the action functional,

$$S[\boldsymbol{r}(t)] = \int\limits_{t_1}^{t_2} dt \, \Big\{ \tfrac{1}{2} m \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}) - \frac{e}{c} \boldsymbol{A}(\boldsymbol{r}) \cdot \dot{\boldsymbol{r}} \Big\} \tag{2.168}$$

satisfies

$$S[\boldsymbol{r}(t)] = S[\boldsymbol{r}(-t)] \qquad \text{if } \boldsymbol{B} = 0 \ . \tag{2.169}$$

There is, therefore, an extra *negative* contribution to the conductance G arising from *phase coherence of time-reversed paths*. In the presence of an external magnetic field, the path γ and its time-reversed mate $\gamma^{\rm T}$ have a relative phase $\eta = 4\pi\Phi_\gamma/\phi_0$, where Φ_γ is the magnetic flux enclosed by the path γ. A magnetic field, then, tends to destroy the phase coherence between time-reversed paths, and hence we expect a *positive magnetoconductance* (*i.e.* negative magnetoresistance) in mesoscopic disordered metals.

Conductance Fluctuations in Metallic Rings

The conductance of a ring must be periodic under $\Phi \to \Phi + n\phi_0$ for any integer n – rings with flux differing by an integer number of Dirac quanta are gauge-equivalent, provided no magnetic field penetrates the ring itself. The conductance as a function of the enclosed flux Φ must be of the form

$$G(\Phi) = G_{\rm cl} + \sum_{m=1}^{\infty} G_m \cos\Big(\frac{2\pi m\Phi}{\phi_0} + \alpha_m\Big) \tag{2.170}$$

where $G_{\rm cl}$ is the classical (Boltzmann) conductance of the ring. The second harmonic $G_{m=2}$ is usually detectable and is in many cases much larger than the $m = 1$ term. The origin of the $m = 2$ term, which is periodic under $\Phi \to \Phi + \frac{1}{2}\phi_0$, lies in the interference between time-reversed paths of winding number ± 1. The $m = 1$ fundamental is easily suppressed, *e.g.* by placing several rings in series.

2.5.1 Weak Localization

A more rigorous discussion of enhanced backscattering was first discussed by Altshuler, Aronov, and Spivak (AAS) in 1981. AAS showed that there are corrections to Boltzmann transport of the form $\sigma = \sigma_0 + \delta\sigma$, where $\sigma_0 = ne^2\tau/m^*$ is the Drude conductivity and (including a factor of 2 for spin),

$$\delta\sigma = -\frac{2e^2}{h}\,\ell^2 \cdot \frac{1}{V}\int d^dr\, \mathcal{C}(\boldsymbol{r},\boldsymbol{r}) \ , \tag{2.171}$$

where ℓ is the elastic mean free path and $\mathcal{C}(\boldsymbol{r},\boldsymbol{r}')$ is the *Cooperon propagator*, which satisfies

$$\left\{ -\ell^2\Big(\boldsymbol{\nabla} + \frac{2ie}{\hbar c}\,\boldsymbol{A}(\boldsymbol{r})\Big)^2 + \frac{\tau}{\tau_\phi}\right\} \mathcal{C}(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \ , \tag{2.172}$$

where τ_ϕ is the inelastic collision time, $\tau_\phi = L_\phi^2/D$, where $D = v_{\rm F}\ell = \ell^2/\tau$ is the diffusion constant. The linear differential operator

$$\mathcal{L} = -\ell^2\Big(\boldsymbol{\nabla} + \frac{2ie}{\hbar c}\,\boldsymbol{A}(\boldsymbol{r})\Big)^2 \tag{2.173}$$

bears a strong resemblance to the Hamiltonian of a particle of charge $e^* = 2e$ in an external magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Expanding in eigenfunctions of L, we obtain the solution

$$\left(\mathcal{L} + \frac{\tau}{\tau_\phi}\right) \mathcal{C}(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{2.174}$$

$$\mathcal{L}\,\psi_\alpha(\boldsymbol{r}) = \lambda_\alpha \psi_\alpha(\boldsymbol{r}) \tag{2.175}$$

$$\mathcal{C}(\boldsymbol{r}, \boldsymbol{r}') = \sum_\alpha \frac{\psi_\alpha(\boldsymbol{r})\,\psi_\alpha^*(\boldsymbol{r}')}{\lambda_\alpha + \frac{\tau}{\tau_\phi}} \ , \tag{2.176}$$

which resembles a Green's function from quantum mechanics, where the energy parameter is identified with $-\tau/\tau_\phi$. There is accordingly a path integral representation for $\mathcal{C}(\boldsymbol{r}, \boldsymbol{r}')$:

$$\mathcal{C}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int ds\, e^{-is\tau/\tau_\phi} \int\limits_{\substack{\boldsymbol{r}(0)=\boldsymbol{r}_1 \\ \boldsymbol{r}(s)=\boldsymbol{r}_2}} \mathcal{D}\boldsymbol{r}[u]\, \exp\left\{ i \int\limits_0^s du \left[\frac{1}{4\ell^2}\left(\frac{\partial \boldsymbol{r}}{\partial u}\right)^2 - \frac{2e}{\hbar c}\boldsymbol{A}(\boldsymbol{r}) \cdot \frac{\partial \boldsymbol{r}}{\partial u} \right] \right\} . \tag{2.177}$$

Notice that there is no potential term $V(\boldsymbol{r})$ in the action of (2.177). The effect of the static random potential here is to provide a 'step length' ℓ for the propagator. According to the AAS result (2.171), the corrections to the conductivity involve paths which begin and end at the same point $\boldsymbol{r}$ in space. The charge $e^* = 2e$ which appears in the Cooperon action arises from adding up contributions due to time-reversed paths, as we saw earlier.

Let's try to compute $\delta\sigma$, which is called the *weak localization* correction to the conductivity. In the absence of an external field, the eigenvalues of $\mathcal{L}$ are simply $\boldsymbol{k}^2\ell^2$, where $k_\mu = 2\pi n_\mu / L$. We then have

$$\mathcal{C}(\boldsymbol{r}, \boldsymbol{r}) = \int \frac{d^d k}{(2\pi)^d}\, \frac{1}{\boldsymbol{k}^2\ell^2 + \frac{\tau}{\tau_\phi}} \ . \tag{2.178}$$

A cutoff Λ is needed in dimensions $d \geq 2$ in order to render the integral convergent in the ultraviolet. This cutoff is the inverse step size for diffusion: $\Lambda \sim \ell^{-1}$. This gives, for $\tau \ll \tau_\phi$,

$$\delta\sigma_d \sim \frac{e^2}{h} \cdot \begin{cases} -\ell^{-1} & (d = 3) \\ -\ln(\tau_\phi/\tau) & (d = 2) \\ -\ell\sqrt{\tau_\phi/\tau} & (d = 1) \ . \end{cases} \tag{2.179}$$

Let us now compute the magnetoconductance in $d = 2$. In the presence of a uniform magnetic field, $\mathcal{L}$ has evenly spaced eigenvalues

$$\lambda_n = (n + \tfrac{1}{2}) \left(\frac{2\ell}{\ell_B}\right)^2 , \tag{2.180}$$

where $\ell_B = \sqrt{\hbar c/eB}$ is the magnetic length (for a charge $q = -e$ electron). Each of these

Landau levels has a macroscopic degeneracy of $2eBA/hc = A/\pi\ell_B^2$, Thus,

$$\begin{aligned}\delta\sigma(B) &= -\frac{2e^2}{h}\,\frac{\ell^2}{\pi\ell_B^2}\sum_{n=0}^{\ell_B^2/4\ell^2}\frac{1}{(n+\frac{1}{2})\frac{4\ell^2}{\ell_B^2}+\frac{\tau}{\tau_\phi}} \\ &= -\frac{e^2}{2\pi h}\sum_{n=0}^{\ell_B^2/4\ell^2}\frac{1}{n+\frac{1}{2}+\frac{\ell_B^2}{L_\phi^2}}\ ,\end{aligned} \tag{2.181}$$

where we have invoked $\tau/\tau_\phi = \ell^2/L_\phi^2$. The magnetoconductance is then

$$\delta\sigma(B) - \delta\sigma(0) = -\frac{1}{2\pi}\,\frac{e^2}{h}\left\{\Psi\Big(\frac{1}{2}+\frac{\ell_B^2}{4\ell^2}\Big) - \Psi\Big(\frac{1}{2}+\frac{\ell_B^2}{4L_\phi^2}\Big)\right\}\ , \tag{2.182}$$

where

$$\Psi(z) = \frac{1}{\Gamma(z)}\,\frac{d\Gamma(z)}{dz} = \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \ldots \tag{2.183}$$

is the digamma function. If the field is weak, so that $\ell_B \gg \ell$, then

$$\delta\sigma(B) - \delta\sigma(0) = +\frac{1}{6\pi}\,\frac{e^2}{h}\left(\frac{L_\phi}{\ell_B}\right)^2\ , \tag{2.184}$$

which is positive, as previously discussed. The magnetic field suppresses phase coherence between time-reversed paths, and thereby promotes diffusion by suppressing the resonant backscattering contributions to $\delta\sigma$. At large values of the field, the behavior is logarithmic. Generally, we can write

$$\delta\sigma(B) - \delta\sigma(0) = \frac{1}{2\pi}\,\frac{e^2}{h}\,f\Big(\frac{B}{B_\phi}\Big)\ , \tag{2.185}$$

where

$$\frac{2\pi B_\phi}{B} \equiv \frac{\ell_B^2}{4L_\phi^2} \quad\Longrightarrow\quad B_\phi = \frac{\phi_0}{8\pi L_\phi^2} \tag{2.186}$$

and

$$f(x) = \ln x + \Psi\big(\tfrac{1}{2}+\tfrac{1}{x}\big) = \begin{cases}\frac{x^2}{24} & \text{as } x\to 0\\ \ln x - 1.96351\ldots & \text{as } x\to\infty\ .\end{cases} \tag{2.187}$$

2.6 Anderson Localization

In 1958, P. W. Anderson proposed that static disorder could lead to localization of electronic eigenstates in a solid. Until this time, it was generally believed that disorder gave rise to an elastic scattering length ℓ and a diffusion constant $D = v_{\rm F}\ell$. The diffusion constant is related to the electrical conductivity through the Einstein relation: $\sigma = \frac{1}{2}e^2 D(\varepsilon_{\rm F})\mathcal{N}(\varepsilon_{\rm F})$. If the states at the Fermi level are *localized*, then $D(\varepsilon_{\rm F}) = 0$.

Anderson considered an electron propagating in a random potential:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}) \ , \tag{2.188}$$

where $V(\boldsymbol{r})$ is chosen from an ensemble of random functions. Physically, $V(\boldsymbol{r})$ is bounded and smooth, although often theorists often study uncorrelated 'white noise' potentials where the ensemble is described by the distribution functional

$$P[V(\boldsymbol{r})] = \exp\Big\{ -\frac{1}{2\gamma}\int d^d r\, V^2(\boldsymbol{r})\Big\} \ , \tag{2.189}$$

for which

$$\big\langle V(\boldsymbol{r})\, V(\boldsymbol{r}')\big\rangle = \gamma\,\delta(\boldsymbol{r}-\boldsymbol{r}') \ . \tag{2.190}$$

A tight binding version of this model would resemble

$$\mathcal{H} = -t\sum_{\langle ij\rangle}\big(c_i^\dagger\, c_j + c_j^\dagger\, c_i\big) + \sum_i \varepsilon_i\, c_i^\dagger\, c_i \ , \tag{2.191}$$

where the single site energies $\{\varepsilon_i\}$ are independently distributed according to some function $p(\varepsilon)$. The first term can be diagonalized by Fourier transform:

$$\mathcal{H}_0 = -t\sum_{\langle ij\rangle}\big(c_i^\dagger\, c_j + c_j^\dagger\, c_i\big) = -zt\sum_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}\, c_{\boldsymbol{k}}^\dagger\, c_{\boldsymbol{k}} \ , \tag{2.192}$$

where $\gamma_{\boldsymbol{k}} = z^{-1}\sum_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}$, where δ is a nearest neighbor direct lattice vector and z is the lattice coordination number; the bandwidth is $2zt$. What happens when we add the random potential term to (2.192)? Suppose the width of the distribution $p(\varepsilon)$ is W, *e.g.* $p(\varepsilon) = W^{-1}\,\Theta(\frac{1}{4}W^2 - \varepsilon^2)$, with $W \ll zt$. We expect that the band edges shift from $\pm zt$ to $\pm(zt+\frac{1}{2}W)$. The density of states must vanish for $|\varepsilon| > zt+\frac{1}{2}W$; the regions in the vicinity of $\pm(zt+\frac{1}{2}W)$ are known as *Lifshitz tails*.

Aside from the formation of Lifshitz tails, the density of states doesn't change much. What does change is the character of the eigenfunctions. Suppose we can find a region of contiguous sites all of which have energies $\varepsilon_i \approx \frac{1}{2}W$. Then we could form an approximate eigenstate by concentrating the wavefunction in this region of sites, setting its phase to be constant throughout. This is an example of a *localized state*. We can think of such a state as a particle in a box – the electron binds itself to local fluctuations in the potential. Outside this region, the wavefunction decays, typically exponentially. Scattering states are then extended states, and are associated with 'average' configurations of the $\{\varepsilon_i\}$. The typical spatial extent of the localized states is given by the localization length $\xi(\varepsilon)$. The localization length diverges at the *mobility edges* as

$$\xi(\varepsilon) \sim |\varepsilon - \varepsilon_{\rm c}|^{-\nu} \ . \tag{2.193}$$

There is no signature of the mobility edge in the density of states itself – $\mathcal{N}(\varepsilon)$ is completely smooth through the mobility edge $\varepsilon_{\rm c}$.

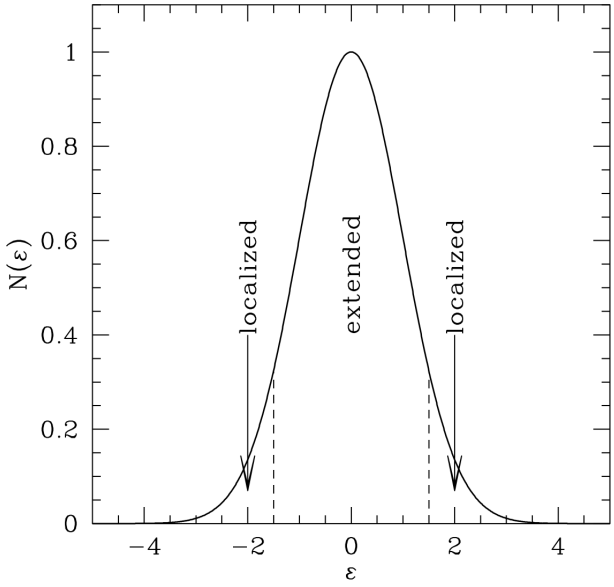

Figure 2.9: Schematic picture of density of states in a disordered system showing mobility edges (dashed lines) and localized states in band tails.

2.6.1 Characterization of Localized and Extended States

One way of characterizing the localization properties of quantum mechanical eigenstates (of $\mathcal{H}$) is to compute the *participation ratio* Q. The inverse participation ratio for the state α is given by

$$Q_\alpha^{-1} = \sum_i \left|\psi_\alpha(i)\right|^4 \Big/ \left(\sum_i \left|\psi_\alpha(i)\right|^2\right)^2 \qquad \text{(discrete)} \tag{2.194}$$

$$= \int d^dx \left|\psi_\alpha(\boldsymbol{x})\right|^4 \Big/ \left(\int d^dx \left|\psi_\alpha(\boldsymbol{x})\right|^2\right)^2 \qquad \text{(continuous)} \tag{2.195}$$

Consider the discrete case. If ψ_α is localized on a single site, then we have $\sum_i \left|\psi_\alpha(i)\right|^k = 1$ for all k, *i.e.* $Q_\alpha = 1$. But if ψ_α is spread evenly over N sites, then $Q_\alpha^{-1} = N/N^2 = N^{-1}$, and $Q_\alpha = N$. Hence, Q_α tells us approximately how many states $\big|\, i \,\big\rangle$ *participate* in the state $\big|\, \psi_\alpha \,\big\rangle$. The dependence of Q_α on the system size (linear dimension) L can be used as a diagnostic:

$$\big|\, \psi_\alpha \,\big\rangle \quad \text{localized} \quad \Longrightarrow \quad Q_\alpha \propto L^0 \tag{2.196}$$

$$\big|\, \psi_\alpha \,\big\rangle \quad \text{extended} \quad \Longrightarrow \quad Q_\alpha \propto L^\beta \ , \tag{2.197}$$

where $\beta > 0$.

Another way to distinguish extended from localized states is to examine their sensitivity to boundary conditions:

$$\psi(x_1\,,\,\ldots\,,\,x_\mu + L\,,\,\ldots\,,\,x_d) = e^{i\theta_\mu}\,\psi(x_1\,,\,\ldots\,,\,x_\mu\,,\,\ldots\,,\,x_d)\ , \tag{2.198}$$

where $\mu \in \{1, \ldots, d\}$. For periodic boundary conditions $\theta_\mu = 0$, while antiperiodic boundary conditions have $\theta_\mu = \pi$. For plane wave states, this changes the allowed wavevectors, so that $k_\mu \to k_\mu + \pi/L$. Thus, for an extended state,

$$\delta\varepsilon_{\rm ext} = \varepsilon^{\rm pbc} - \varepsilon^{\rm apbc} \simeq \frac{\partial\varepsilon}{\partial k}\,\delta k \propto L^{-1} \ . \tag{2.199}$$

For a localized state,

$$\delta\varepsilon_{\rm loc} \propto e^{-L/\xi(\varepsilon)} \ . \tag{2.200}$$

One defines the dimensionless *Thouless number*

$$\mathrm{Th}(\varepsilon, L) = \left|\varepsilon^{\rm pbc} - \varepsilon^{\rm apbc}\right| \cdot \mathcal{N}(\varepsilon) \ . \tag{2.201}$$

In the vicinity of a mobility edge, a scaling hypothesis suggests

$$\mathrm{Th}(\varepsilon, L) = f\big(L/\xi(\varepsilon)\big) \ , \tag{2.202}$$

where $f(x)$ is a *universal scaling function*.

As the Fermi level passes through the mobility edge into a region of localized states, the conductivity vanishes as

$$\sigma(\varepsilon_{\rm F}) \sim |\varepsilon_{\rm F} - \varepsilon_{\rm c}|^s \ , \tag{2.203}$$

where $s > 0$. Since the density is a continuous function of $\varepsilon_{\rm F}$, this can also be turned into a statement about the behavior of $\sigma(n)$:

$$\sigma(n) \sim (n - n_{\rm c})^s\,\Theta(n - n_{\rm c}) \ . \tag{2.204}$$

2.6.2 Numerical Studies of the Localization Transition

Pioneering work in numerical studies of the localization transition was performed by MacKinnon and Kramer in the early 1980's. They computed the localization length $\xi_M(W/t, E)$ for systems of dimension $M^{d-1} \times N$, from the formula

$$\xi_M^{-1}\Big(\frac{W}{t}, E\Big) = -\lim_{N\to\infty}\,\frac{1}{2N}\Big\langle \ln \sum_{i,j=1}^{M^{d-1}} \big|G_{1i,Nj}(E)\big|^2 \Big\rangle \ , \tag{2.205}$$

where $G(E) = (E + i\epsilon - \mathcal{H})^{-1}$ is the Green's function, and i, j label transverse sites. The average $\langle\cdots\rangle$ is over disorder configurations. It is computationally very convenient to compute the localization length in this manner, rather than from exact diagonalization, because the Green's function can be computed recursively. For details, see A. MacKinnon and B. Kramer, *Phys. Rev. Lett.* **47**, 1546 (1981). Note also that it is $\langle|G|^2\rangle$ which is computed, rather than $\langle G\rangle$. The reason for this is that the Green's function itself carries a complex phase which when averaged over disorder configurations results in a decay of $\langle G_{\boldsymbol{R},\boldsymbol{R}'}\rangle$ on the scale of the elastic mean free path ℓ.

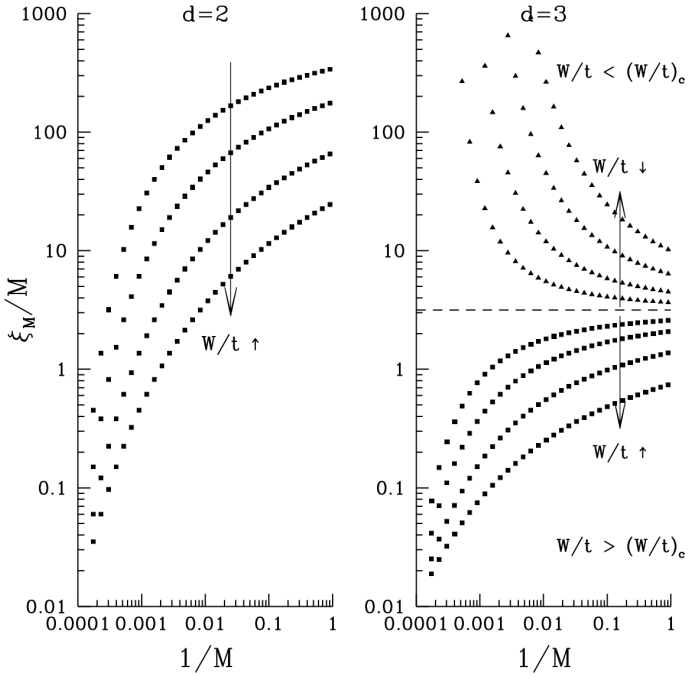

Figure 2.10: Mock-up of typical raw data from numerical study of localization length for $d = 2$ (left panel) and $d = 3$ (right panel) systems. For $d = 2$, $\xi_\infty(W/t)$ is finite and monotonically decreasing with increasing W/t. For $d = 3$, there is a critical value of W/t. For $W/t < (W/t)_c$, $\xi_M(W/t)$ diverges as $M \to \infty$; this is the extended phase. For $W/t > (W/t)_c$, $\xi_M(W/t)$ remains finite as $M \to \infty$; this is the localized phase.

MacKinnon and Kramer computed the localization length by employing the *Ansatz* of *finite size scaling*. They assumed that

$$\xi_M\Big(\frac{W}{t}, E\Big) = M f\bigg(\xi_\infty\Big(\frac{W}{t}, E\Big)\Big/M\bigg) \ , \tag{2.206}$$

where $f(x)$ is a universal scaling function which depends only on the dimension d. MK examined the band center, at $E = 0$; for $d > 2$ this is the last region to localize as W/t is increased from zero. A mock-up of typical raw data is shown in fig. 2.10.

In the $d = 2$ case, all states are localized. Accordingly, $\xi_M/M \to 0$ as $M \to \infty$, and $\xi_M(W/t)$ decreases with increasing W/t. In the $d = 3$ case, states at the band center are extended for weak disorder. As W/t increases, $\xi_M(W/t)$ decreases, but with $\xi_\infty(W/t)$ still divergent. At the critical point, $(W/t)_c$, this behavior changes. The band center states localize, and $\xi_\infty(W/t)$ is finite for $W/t > (W/t)_c$. If one rescales and plots ξ_M/M *versus* ξ_∞/M, the scaling function $f(x)$ is revealed. This is shown in fig. 2.11, which is from the paper by MacKinnon and Kramer. Note that there is only one branch to the scaling function for $d = 2$, but two branches for $d = 3$. MacKinnon and Kramer found $(W/t)_c \simeq 16.5$ for a disordered tight binding model on a simple cubic lattice.

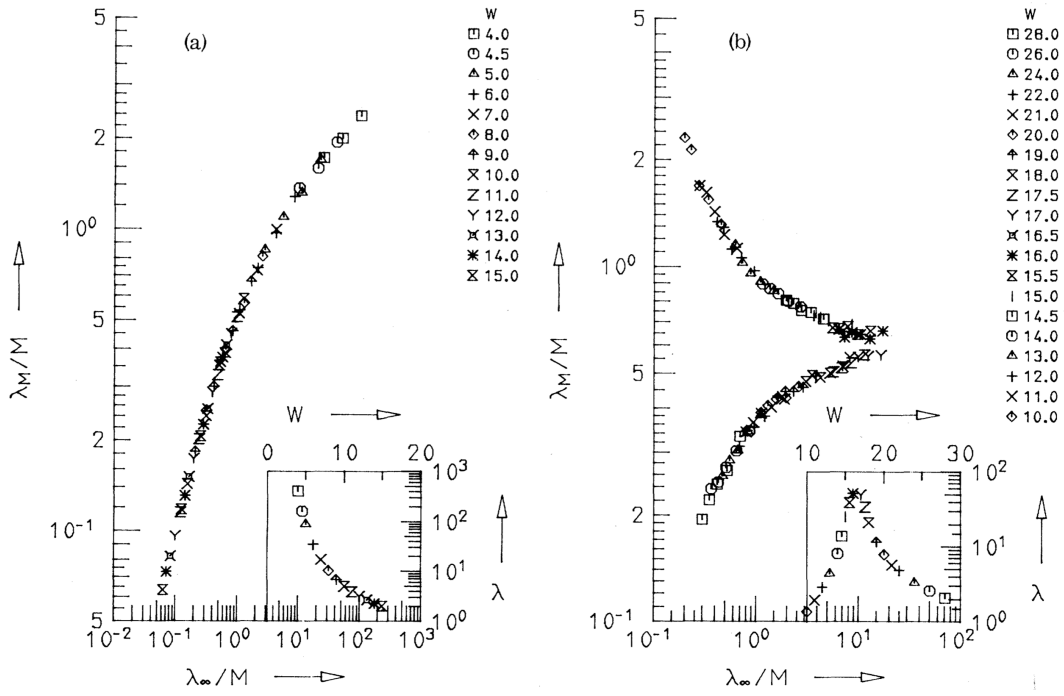

Figure 2.11: Scaling function λ_M/M *versus* λ_∞/M for the localization length λ_M of a system of thickness M for (a) $d = 2$, and (b) $d = 3$. Insets show the scaling parameter λ_∞ as a function of the disorder W. From A. MacKinnon and B. Kramer, *Phys. Rev. Lett.* **47**, 1546 (1981).

2.6.3 Scaling Theory of Localization

In the metallic limit, the dimensionless conductance of a L^d hypercube is given by the Ohmic result

$$g(L) = \frac{h\sigma}{e^2}\, L^{d-2} \ , \tag{2.207}$$

whereas in the localized limit we have, from Pichard's formula,

$$g(L) = 4\, e^{-2L/\xi} \ . \tag{2.208}$$

It is instructive to consider the function,

$$\beta(g) \equiv \frac{d\ln g}{d\ln L} \ , \tag{2.209}$$

which describes the change of g when we vary the size of the system. We now know the limiting values of $\beta(g)$ for small and large g:

$$\text{metallic } (g \gg 1) \quad \Longrightarrow \quad \beta(g) = d - 2 \tag{2.210}$$

$$\text{localized } (g \ll 1) \quad \Longrightarrow \quad \beta(g) = -\frac{2L}{\xi} = \ln g + \text{const.} \tag{2.211}$$

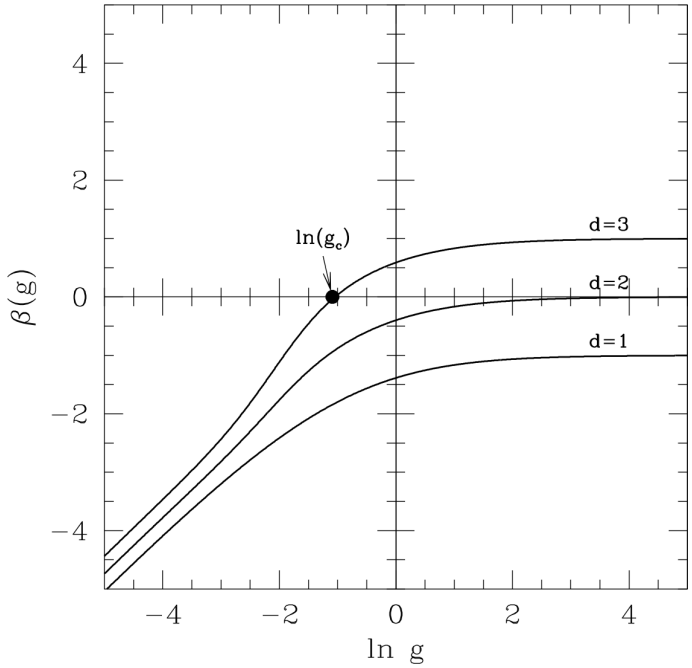

Figure 2.12: Sketch of the β-function for the localization problem for $d = 1, 2, 3$. A critical point exists at $g = g_c$ for the $d = 3$ case.

If we assume that $\beta(g)$ is a smooth monotonic function of g, we arrive at the picture in fig. 2.12. Note that in $d = 1$, we can compute $\beta(g)$ exactly, using the Landauer formula,

$$g = \frac{T}{1-T} \ , \tag{2.212}$$

where $T \propto \exp(-L/\xi)$. From this, we obtain

$$\beta_{d=1}(g) = -(1+g)\,\ln(1+g^{-1}) \ . \tag{2.213}$$

It should be stressed that the very existence of a β-function is hardly clear. If it does exist, it says that the conductance of a system of size L is uniquely determined by its conductance at some other length scale, typically chosen to be microscopic, *e.g.* $L_0 = \ell$. Integrating the β-function, we obtain an integral equation to be solved implicitly for $g(L)$:

$$\ln\left(\frac{L}{L_0}\right) = \int\limits_{\ln g(L_0)}^{\ln g(L)} \frac{d\ln g}{\beta(g)} \ . \tag{2.214}$$

A priori it seems more likely, though, that as L is increased the changes to the conductance may depend on more than g alone. *E.g.* the differential change dg might depend on the entire distribution function $P(W)$ for the disorder.

Integrating the β-function: $d = 3$

We know $\beta(g \to 0) \simeq \ln g$ and $\beta(g \to \infty) = 1$, hence by the intermediate value theorem there is at least one point were $\beta(g)$ vanishes. Whenever g satisfies $\beta(g) = 0$, the conductance g is *scale invariant* – it does not change with increasing (or decreasing) system size L. We will assume the situation is reflected by the sketch of fig. 2.12, and that there is one such point, $g_{\rm c}$.

We now apply (2.214). Not knowing the precise form of $\beta(g)$, we approximate it piecewise:

$$\beta(g) \simeq \begin{cases} 1 & \text{if } g \geq g_+ \\ \alpha \, \ln(g/g_{\rm c}) & \text{if } g_- < g < g_+ \\ \ln g & \text{if } g < g_- \ , \end{cases} \tag{2.215}$$

where $\alpha = g_{\rm c} \, \beta'(g_{\rm c})$. We determine g_+ and g_- by continuity:

$$\ln g_+ = \ln g_{\rm c} + \frac{1}{\alpha} \tag{2.216}$$

$$\ln g_- = \frac{\alpha}{\alpha - 1} \, \ln g_{\rm c} \ . \tag{2.217}$$

Now suppose we start with $g_0 = g_{\rm c} + \delta g$, where $|\delta g| \ll 1$. We integrate out to $g = g_+$ and then from g_+ to $g \gg 1$:

$$\ln\left(\frac{L_+}{L_0}\right) = \int\limits_{\ln g_0}^{\ln g_+} \frac{d \ln g}{\alpha \, \ln(g/g_{\rm c})} = \frac{1}{\alpha} \ln\left(\frac{\ln(g_+/g_{\rm c})}{\ln(g_0/g_{\rm c})}\right) \tag{2.218}$$

$$\ln\left(\frac{L}{L_+}\right) = \int\limits_{\ln g_+}^{\ln g} d \ln g = \ln(g/g_+) \ , \tag{2.219}$$

which together imply

$$g(L) = A_+ \, g_{\rm c} \cdot \frac{L}{L_0} \cdot (g_0 - g_{\rm c})^{1/\alpha} \ , \tag{2.220}$$

where $A_+ = (e\alpha/g_{\rm c})^{1/\alpha}$. The conductivity is

$$\sigma = \frac{e^2}{h} \cdot \frac{g}{L} = \frac{e^2}{h} \cdot \frac{A_+ \, g_{\rm c}}{L_0} \cdot (g_0 - g_{\rm c})^{1/\alpha} \ . \tag{2.221}$$

If instead we start with $g_0 = g_{\rm c} - \delta g$ and integrate out to large negative values of $\ln g$, then

$$\ln\left(\frac{L_-}{L_0}\right) = \int\limits_{\ln g_0}^{\ln g_-} \frac{d \ln g}{\alpha \, \ln(g/g_{\rm c})} = \frac{1}{\alpha} \ln\left(\frac{\ln(g_{\rm c}/g_-)}{\ln(g_{\rm c}/g_0)}\right) \tag{2.222}$$

$$\ln\left(\frac{L}{L_-}\right) = \int\limits_{\ln g_-}^{\ln g} \frac{d \ln g}{\ln g} = \ln\left(\frac{\ln g}{\ln g_-}\right) \ , \tag{2.223}$$

which says

$$g(L) = e^{-2L/\xi} \tag{2.224}$$

$$\xi = \frac{2L_0}{A_-} \cdot (g_{\rm c} - g_0)^{-1/\alpha} \ , \tag{2.225}$$

with

$$A_- = \frac{\ln(1/g_-)}{g_{\rm c} \cdot \left[\ln(g_{\rm c}/g_-)\right]^{1/\alpha}} \ . \tag{2.226}$$

On the metallic side of the transition, when $g_0 > g_{\rm c}$, we can identify a localization length through

$$g \equiv g_{\rm c}/\xi \ , \tag{2.227}$$

which says

$$\xi = \frac{L_0}{A_+} \, (g_0 - g_{\rm c})^{-1/\alpha} \ . \tag{2.228}$$

Finally, since g_0 is determined by the value of the Fermi energy $\varepsilon_{\rm F}$. we can define the *critical energy*, or *mobility edge* $\varepsilon_{\rm c}$, through

$$g(L_0, \varepsilon_{\rm c}) = g_{\rm c} \ , \tag{2.229}$$

in which case

$$\delta g \equiv g(L_0, \varepsilon_{\rm F}) - g_{\rm c} = \left.\frac{\partial g(L_0, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon = \varepsilon_{\rm c}} \cdot \big(\varepsilon_{\rm F} - \varepsilon_{\rm c}\big) \ . \tag{2.230}$$

Thus, $\delta g \propto \delta\varepsilon \equiv (\varepsilon_{\rm F} - \varepsilon_{\rm c})$.

Integrating the β-function: $d = 2$

In two dimensions, there is no fixed point. In the Ohmic limit $g \gg 1$, we have

$$\beta(g) = -\frac{c}{g} + \mathcal{O}(g^{-2}) \ , \tag{2.231}$$

where c is a constant. Thus,

$$\ln\left(\frac{L}{L_0}\right) = \int\limits_{\ln g_0}^{\ln g} \frac{d\ln g}{\beta(g)} = -\frac{g - g_0}{c} + \ldots \tag{2.232}$$

and

$$g(L) = k_{\rm F}\ell - c \, \ln(L/\ell) \ , \tag{2.233}$$

where we have used the Drude result $g = k_{\rm F}\ell$, valid for $L_0 = \ell$. We now see that the localization length ξ is the value of L for which the correction term is on the same order as g_0: $\xi = \ell \exp(k_{\rm F}\ell/c)$. A first principles treatment yields $c = \frac{2}{\pi}$. The metallic regime in $d = 2$ is often called the *weak localization* regime.

$2+\epsilon$ dimensions

At or below $d = 2$ dimensions, there is no mobility edge and all eigenstates are localized. $d = 2$ *is the lower critical dimension for the localization transition.* Consider now the problem in $d = 2 + \epsilon$ dimensions. One has

$$\beta(g) = \epsilon - \frac{c}{g} + \mathcal{O}(g^{-2}) \ . \tag{2.234}$$

The critical conductance lies at $g_{\rm c} = c/\epsilon$. For $\epsilon \to 0^+$, this is large enough that higher order terms in the expansion of the β-function can safely be ignored in the metallic limit. An analysis similar to that for $d = 3$ now yields

$$g > g_{\rm c} \quad \Longrightarrow \quad g(L) = \frac{h}{e^2}\,\sigma\,L^{\epsilon} \tag{2.235}$$

$$g < g_{\rm c} \quad \Longrightarrow \quad g(L) = e^{-2L/\xi} \ , \tag{2.236}$$

with

$$\sigma = \frac{e^2}{h}\,L_0^{\epsilon}\cdot(g_0 - g_{\rm c}) \tag{2.237}$$

$$\xi = \frac{2L_0}{A_-}\cdot(g_{\rm c} - g_0) \ . \tag{2.238}$$

Note that $\alpha = g_{\rm c}\,\beta'(g_{\rm c}) = +c/g_{\rm c} = \epsilon$. We thus obtain

$$\xi(\varepsilon) \propto |\varepsilon - \varepsilon_{\rm c}|^{-\nu} \tag{2.239}$$

with $\nu = 1 + \mathcal{O}(\epsilon)$. Close to the transition on the metallic side, the conductivity vanishes as

$$\sigma(\varepsilon) \propto |\varepsilon - \varepsilon_{\rm c}|^{s} \ . \tag{2.240}$$

The relation $s = (d-2)\nu$, which follows from the above treatment, may be used to relate the localization length and conductivity critical exponents. (In $d = 3$, MacKinnon and Kramer obtained $\nu = s \simeq 1.2$.)

2.6.4 Finite Temperature

In the metallic regime, one obtains from the scaling theory,

$$\sigma_{d=3}(L) = \frac{e^2}{h}\cdot\left\{\frac{2k_{\rm F}^2\ell}{3\pi} - \frac{2}{\pi^2}\left(\frac{1}{\ell} - \frac{1}{L}\right)\right\} \tag{2.241}$$

$$\sigma_{d=2}(L) = \frac{e^2}{h}\cdot\left\{k_{\rm F}\ell - \frac{2}{\pi}\ln\left(\frac{L}{\ell}\right)\right\} \tag{2.242}$$

$$\sigma_{d=1}(L) = \frac{e^2}{h}\cdot\Big\{4\ell - 2(L-\ell)\Big\} \ . \tag{2.243}$$

Clearly the $d = 1$ result must break down for even microscopic $L \gtrsim 3\ell$. The above results are computed using the β-function

$$\beta(g) = (d-2) - \frac{c_d}{g} + \mathcal{O}(g^{-2}) \ , \tag{2.244}$$

where the coefficients c_d are computed from perturbation theory in the disorder.

At finite temperature, the cutoff becomes $\min(L, L_\phi)$, where $L_\phi = \sqrt{D\tau_\phi}$ is the inelastic scattering length and $D = v_{\rm F}\ell$ is the diffusion constant. Suppose that $\tau_\phi(T) \propto T^{-p}$ as $T \to 0$, so that $L_\phi = a\,(T/T_0)^{-p/2}$, where T_0 is some characteristic temperature (*e.g.* the Debye temperature, if the inelastic mechanism is electron-phonon scattering). Then, for $L_\phi > L$,

$$\sigma_{d=3}(T) = \sigma^{\rm B}_{d=3} - \frac{2}{\pi^2}\,\frac{e^2}{h}\left\{\frac{1}{\ell} - \frac{1}{a}\left(\frac{T}{T_0}\right)^{p/2}\right\} \tag{2.245}$$

$$\sigma_{d=2}(T) = \sigma^{\rm B}_{d=2} - \frac{2}{\pi}\,\frac{e^2}{h}\left\{\ln\left(\frac{a}{\ell}\right) - \frac{p}{2}\ln\left(\frac{T}{T_0}\right)\right\} \tag{2.246}$$

$$\sigma_{d=1}(T) = \sigma^{\rm B}_{d=1} - 2\,\frac{e^2}{h}\left\{a\left(\frac{T_0}{T}\right)^{p/2} - \ell\right\} \ , \tag{2.247}$$

where $\sigma^{\rm B}_d$ is the Boltzmann conductivity. Note that $\sigma(T)$ decreases with decreasing temperature, unlike the classic low T result for metals, where $\rho(T) = \rho_0 + AT^2$. *I.e.* usually $\rho(T)$ increases as T increases due to a concomitant decrease in transport scattering time τ. Weak localization physics, though, has the opposite effect, as the enhanced backscattering is suppressed as T increases and L_ϕ decreases. The result is that $\rho(T)$ starts to decrease as T is lowered from high temperatures, but turns around at low T and starts increasing again. This behavior was first observed in 1979 by Dolan and Osheroff, who studied thin metallic PdAu films, observing a logarithmic increase in $\rho_{d=2}(T)$ at the lowest temperatures.