Establishing Relationships
Linear Least Squares Fitting

Lecture 5

Physics 2CL

Summer 2010

Outline

- Determining the relationship between measured values
- Review of experiment # 2
- Physics for experiment # 3
  - Oscillations & resonance

Important Reminder

- Entering the last 3 labs
- Need to sign-up for the setups
- Formal report on FIRST of the last set of labs
- CAPE evaluations

Schedule

Meeting	Experiment
1 (Aug. 3 or 4)	none
2 (Aug. 5 or 6)	0
3 (Aug. 10 or 11)	1
4 (Aug. 12 or 13)	1
5 (Aug. 17 or 18)	2
6 (Aug. 19 or 20)	3
7 (Aug. 24 or 25)	4,5,6
8 (Aug. 26 or 27)	4,5,6
9 (Aug. 31 or Sept. 1)	4,5,6

Relationships

- So far, we've talked about measuring a single quantity
- Often experiments measure two variables, both varying simultaneously
- Want to know mathematical relationship between them
- Want to compare to models
- How to analyze quantitatively?

Principle of Maximum Likelihood

- Best estimates of X and σ from N measurements (x_1 - x_N) are those for which $\text{Prob}_{X,\sigma}(x_i)$ is a maximum

The Principle of Maximum Likelihood

Recall the probability density for measurements of some quantity *x*
(distributed as a Gaussian with mean X and standard deviation σ)

$$P_{X,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

Normal distribution is one example of *P(x)*.

Now, lets make repeated measurements of *x* to help reduce our errors.

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n$$

We define the Likelihood as the product of the probabilities. The larger *L*, the more likely a set of measurements is.

$$L = P(x_1)P(x_2)P(x_3)\ldots P(x_n)$$

Is *L* a Probability?

Why does max *L* give the best estimate?

The best estimate for the parameters of *P(x)* are those that maximize *L*.

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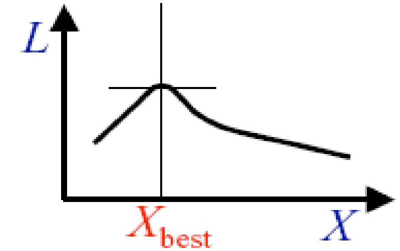
Using the Principle of Maximum Likelihood:

Prove the mean is best estimate of X

Assume X is a parameter of $P(x)$.
When L is maximum, we must have: $\frac{\partial L}{\partial X} = 0$



Lets assume a Normal error distribution and find the formula for the best value for X.

$$L = P(x_1)P(x_2)...P(x_n) = \prod_{i=1}^{n} P(x_i)$$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - X)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\sum_{i=1}^{n}\frac{(x_i - X)^2}{2\sigma^2}}$$

$$L = Ce^{-\chi^2/2}$$

$$\chi^2 = \sum_{i=1}^{n} \frac{(x_i - X)^2}{\sigma^2}$$ Defininition

$$\frac{\partial L}{\partial X} = 0 = Ce^{-\chi^2/2} \frac{-1}{2} \frac{\partial \chi^2}{\partial X}$$

$$\Rightarrow \quad \frac{\partial \chi^2}{\partial X} = 0$$

$$\frac{\partial \chi^2}{\partial X} = \frac{1}{\sigma^2} \sum_{i=1}^{n} 2(x_i - X) = 0$$

$$\sum_{i=1}^{n} (x_i - X) = 0$$

$$\sum_{i=1}^{n} x_i - nX = 0$$

$$X = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Q.E.D.
the mean

What is the Error on the Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Formula for mean of measurements. (We just proved that this is the best estimate of the true x.)

Now, use propagation of errors to get the error on the mean.

$$\sigma_{\bar{x}} = \frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1} \oplus \frac{\partial \bar{x}}{\partial x_2}\sigma_{x_2} \oplus \ldots \oplus \frac{\partial \bar{x}}{\partial x_n}\sigma_{x_n}$$

$$\frac{\partial \bar{x}}{\partial x_i} = \frac{1}{n}$$

$$\sigma_{\bar{x}} = \sqrt{\sum_{i=1}^{n}\left(\frac{\sigma_{x_i}}{n}\right)^2} = \sqrt{n\left(\frac{\sigma}{n}\right)^2} = \frac{\sigma}{\sqrt{n}}$$

What would you do if the x_i had different errors?

We got the error on the mean (SDOM) by propagating errors.

Weighted averages (Chapter 7)

We can use maximum Likelihood (χ^2) to average measurements with different errors.

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - X}{\sigma_i} \right)^2$$

We derived the result that: $$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Using error propagation, we can determine the error on the weighted mean:

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{\sum_{i=1}^{n} w_i}}$$

What does this give in the limit where all errors are equal?

$$\frac{\partial \chi^2}{\partial X} = 0 = -2 \sum_{i=1}^{n} \frac{x_i - X}{\sigma_i^2}$$

$$\sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} - X \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = 0$$

$$w_i \equiv \frac{1}{\sigma_i^2}$$

$$\sum_{i=1}^{n} w_i x_i = X \sum_{i=1}^{n} w_i$$

$$X = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

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Example: Weighted Average

Suppose 2 students measure the radius of Neptune.

- Student A gets r=80 Mm with an error of 10 Mm and
- Student B gets r=60 Mm with an error of 3 Mm

What is the best estimate of the true radius?

$$\bar{r} = \frac{w_A r_A + w_B r_B}{w_A + w_B} = \frac{\frac{1}{100}80 + \frac{1}{9}60}{\frac{1}{100} + \frac{1}{9}} = 61.65 \text{ Mm}$$

What does this tell us about the importance of error estimates?

Example:

Compatibility of measurements
Best estimate, Weighted Average

Two measurements of the speed of sound give the answers:

$u_A = 332 \pm 1$ and $u_B = 339 \cdot \pm 3$

(Both in m/s.)

- a) Are these measurements consistent at the 5% level? At the 1% level?
- b) What is your best estimate for the speed of sound and its uncertainty?

a) To check if the two measurements are consistent, we compute:

$$q = u_A - u_B = 339\text{-}332 = 7 \text{ m/s}$$

and: $$\sigma_q = \sqrt{\sigma_{uA}^2 + \sigma_{uB}^2} = 3.16 \text{ m/s}$$

so that: $$t = \frac{q}{\sigma_q} = \frac{339 - 332}{3.16} = 2.21$$

From Table A we get that 2.21 sigma corresponds to: 97.21%

Therefore the probability to get a worse result is 1-97% ~3%.

The results are not consistent at the 5% and are consistent at the 1% confidence level.

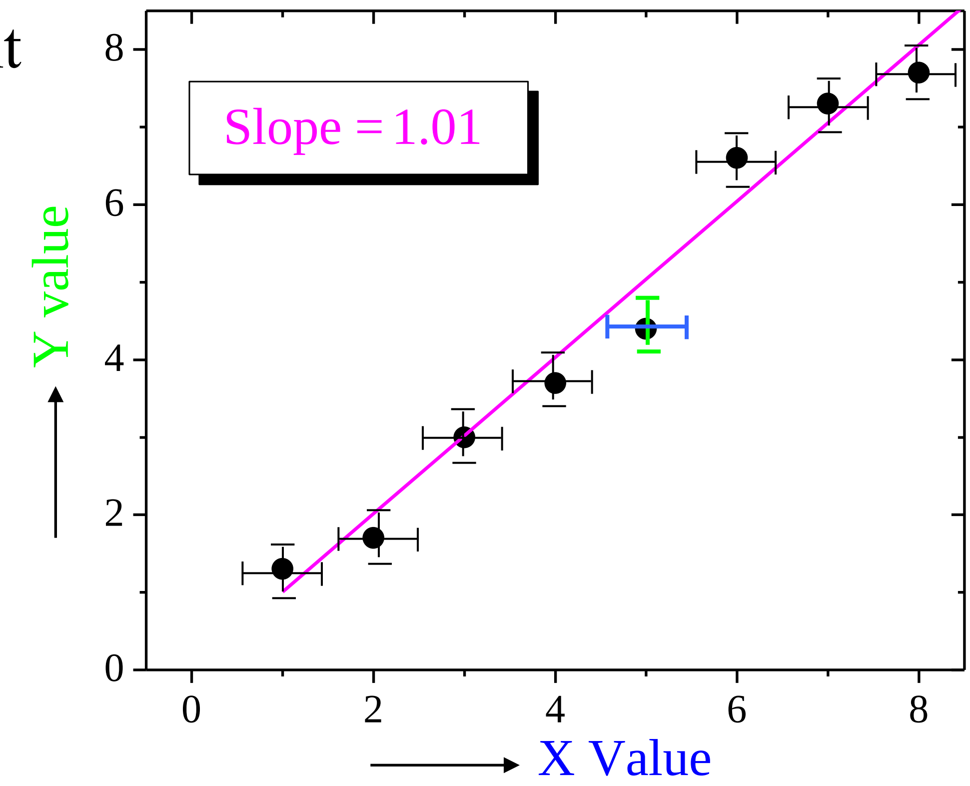
b) Best estimate is the weighted mean:

$$\bar{u} = \frac{w_A u_A + w_B u_B}{w_A + w_B} = \frac{\frac{1}{1}332 + \frac{1}{9}339}{\frac{1}{1} + \frac{1}{9}} = 332.7 \text{ m/s}$$

$$\sigma_{\bar{u}} = \frac{1}{\sqrt{1/w_A + 1/w_B}} = \frac{1}{\sqrt{1/1 + 1/9}} = 0.9 m/s$$

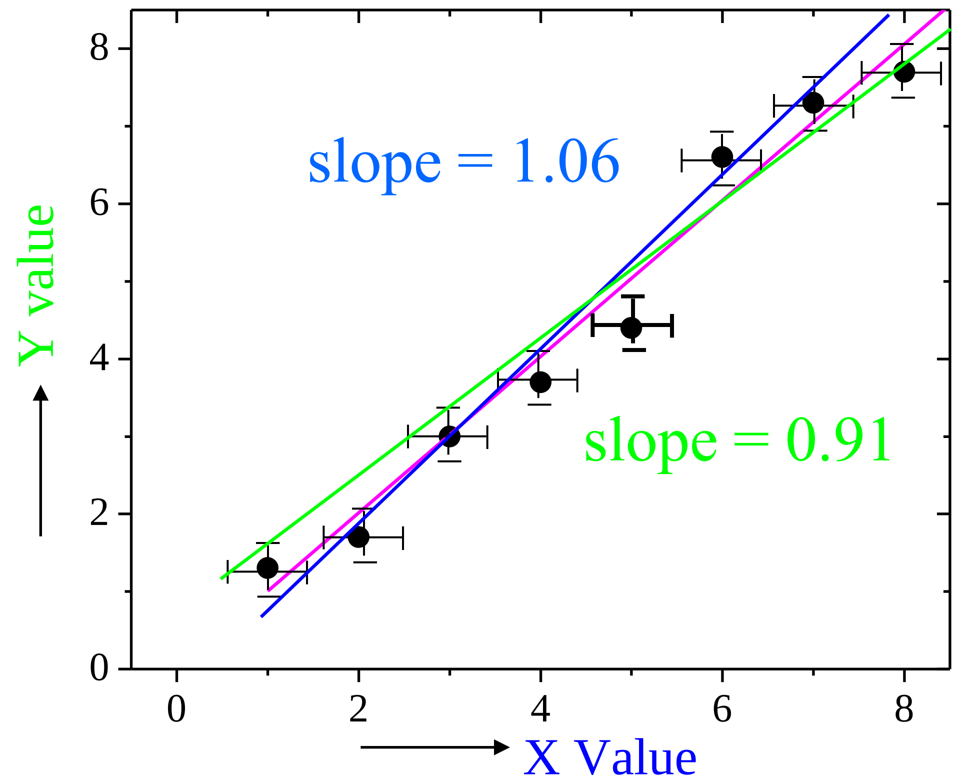
Linear Relationships: $y = A + Bx$

- Data would lie on a straight line, except for errors
- What is 'best' line through the points?
- What is uncertainty in constants?
- How well does the relationship describe the data?
- Velocity vs. time @ constant acceleration
- Ohms law



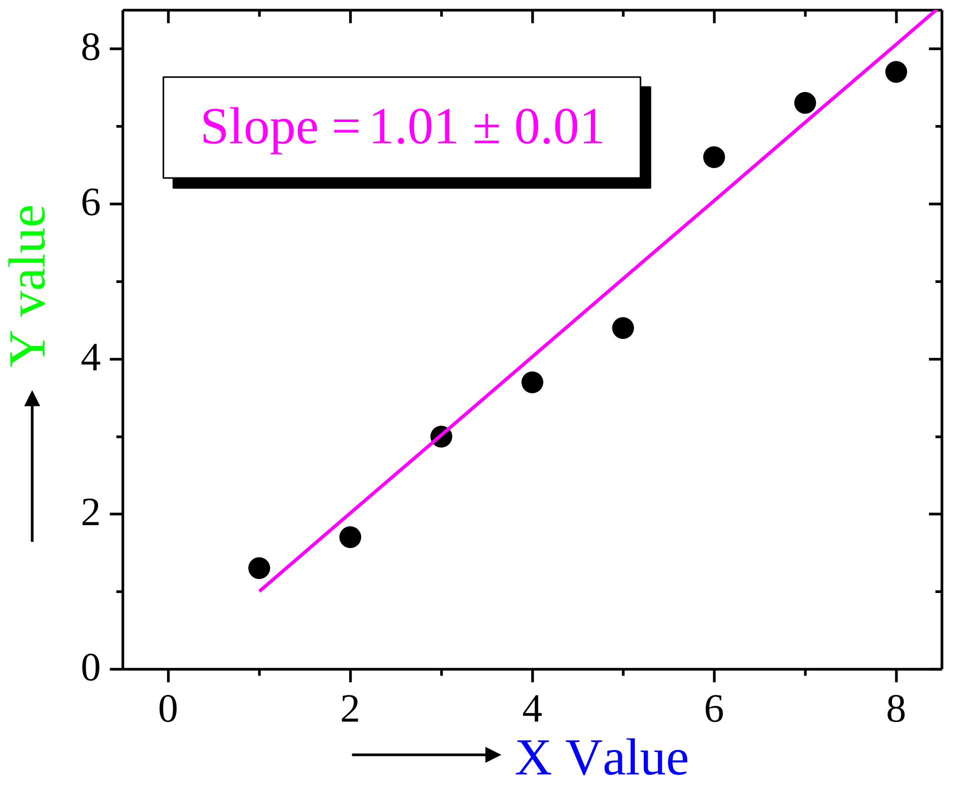
A Rough Cut

- Best means 'line close to all points'
- Draw various lines that pass through data points
- Estimate error in constants from range of values
- Good fit if points within error bars of line

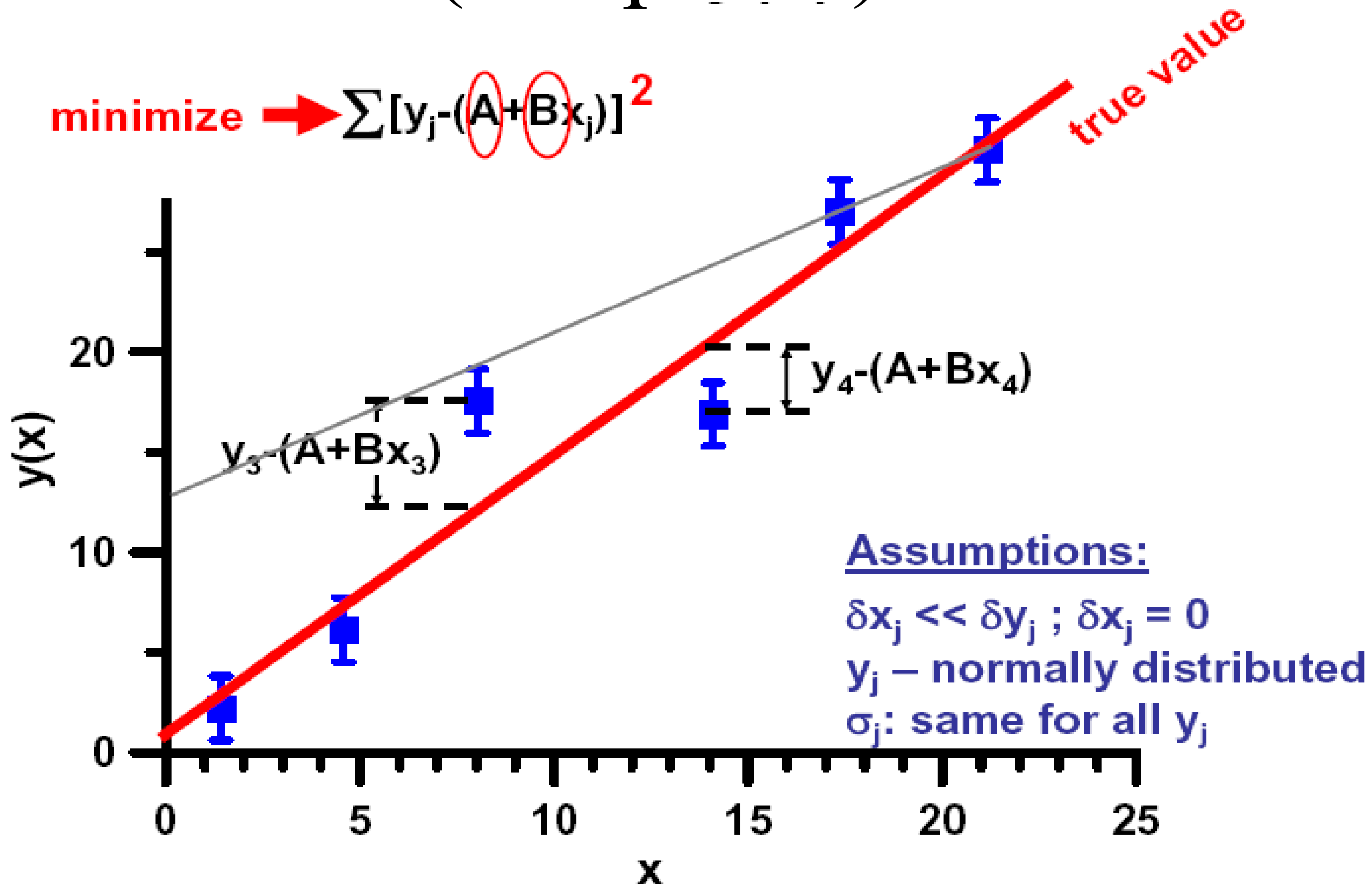


More Analytical

- Best means ‘minimize the square of the deviations between line and points’
- Can use error analysis to find constants, error

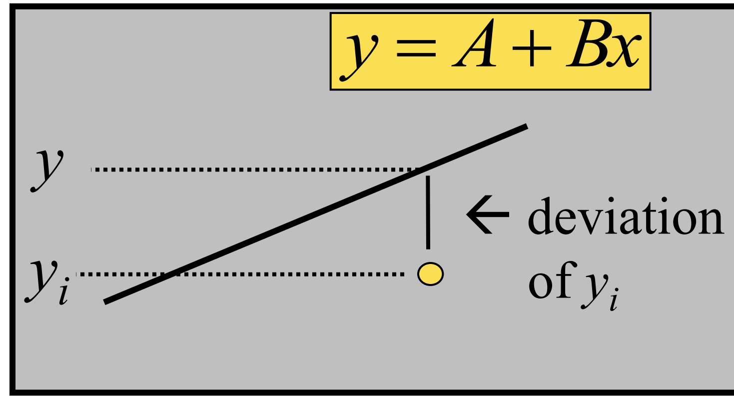


The Details of How to Do This (Chapter 8)



Finding the coefficients A and B

- Want to find *A, B* that minimize difference between data and line
- Since line above some data, below other, minimize sum of squares of deviations
- Find *A, B* that minimize this sum



$$y_i - y = y_i - A - Bx_i$$

$$\sum_{i=1}^{N} (y_i - A - Bx_i)^2$$

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^2 = 0$$

Finding *A* and *B*

- After minimization, solve equations for *A* and *B*
- Looks nasty, not so bad…
- See Taylor, example 8.1

$$\frac{\partial}{\partial A} = \sum y_i - AN - B\sum x_i = 0$$

$$\frac{\partial}{\partial B} = \sum x_i y_i - A\sum x_i + B\sum x_i^{\,2} = 0$$

$$A = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{\Delta}$$

$$B = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

$$\Delta = N\sum x_i^{\,2} - \left(\sum x_i\right)^2$$

Uncertainty in Measurements of *y*

- Before, measure several times and take standard deviation as error in *y*
- Can't now, since y_i's are different quantities
- Instead, find standard deviation of deviations

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - A - Bx_i)^2}$$

Uncertainty in *A* and *B*

- *A, B* are calculated from x_i, y_i
- Know error in x_i, y_i ; use error propagation to find error in *A, B*
- A distant extrapolation will be subject to large uncertainty

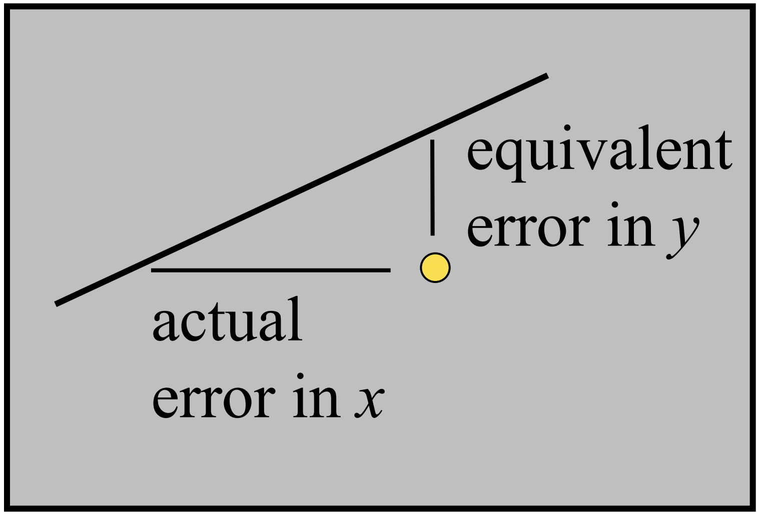
$$\sigma_A = \sigma_y \sqrt{\frac{\sum x_i^2}{\Delta}}$$

$$\sigma_B = \sigma_y \sqrt{\frac{N}{\Delta}}$$

$$\Delta = N \sum x_i^2 - \left(\sum x_i\right)^2$$

Uncertainty in x

- So far, assumed negligible uncertainty in x
- If uncertainty in x, not y, just switch them
- If uncertainty in both, convert error in x to error in y, then add errors



$$\Delta y = B\Delta x$$

$$\sigma_y(equiv) = B\sigma_x$$

$$\sigma_y(equiv) = \sqrt{\sigma_y{}^2 + (B\sigma_x)^2}$$

Other Functions

$$y = Ae^{Bx}$$

- Convert to linear
- Can now use least squares fitting to get ln *A* and *B*

$$y = Ae^{Bx}$$
$$\ln y = \ln A + Bx$$

Experiment #2
Oscillations and Damping RLC Circuit – DC response

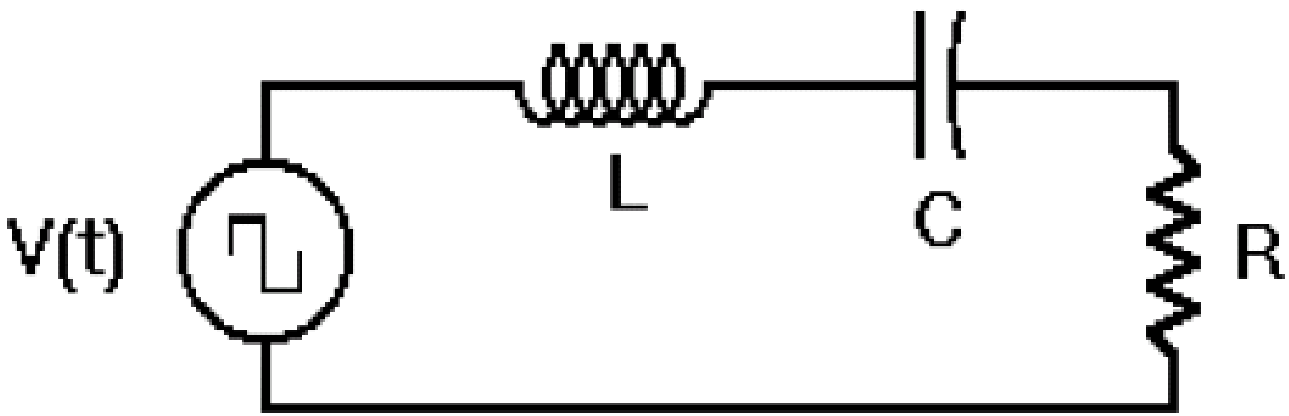


Figure 1 *LRC circuit for this experiment*

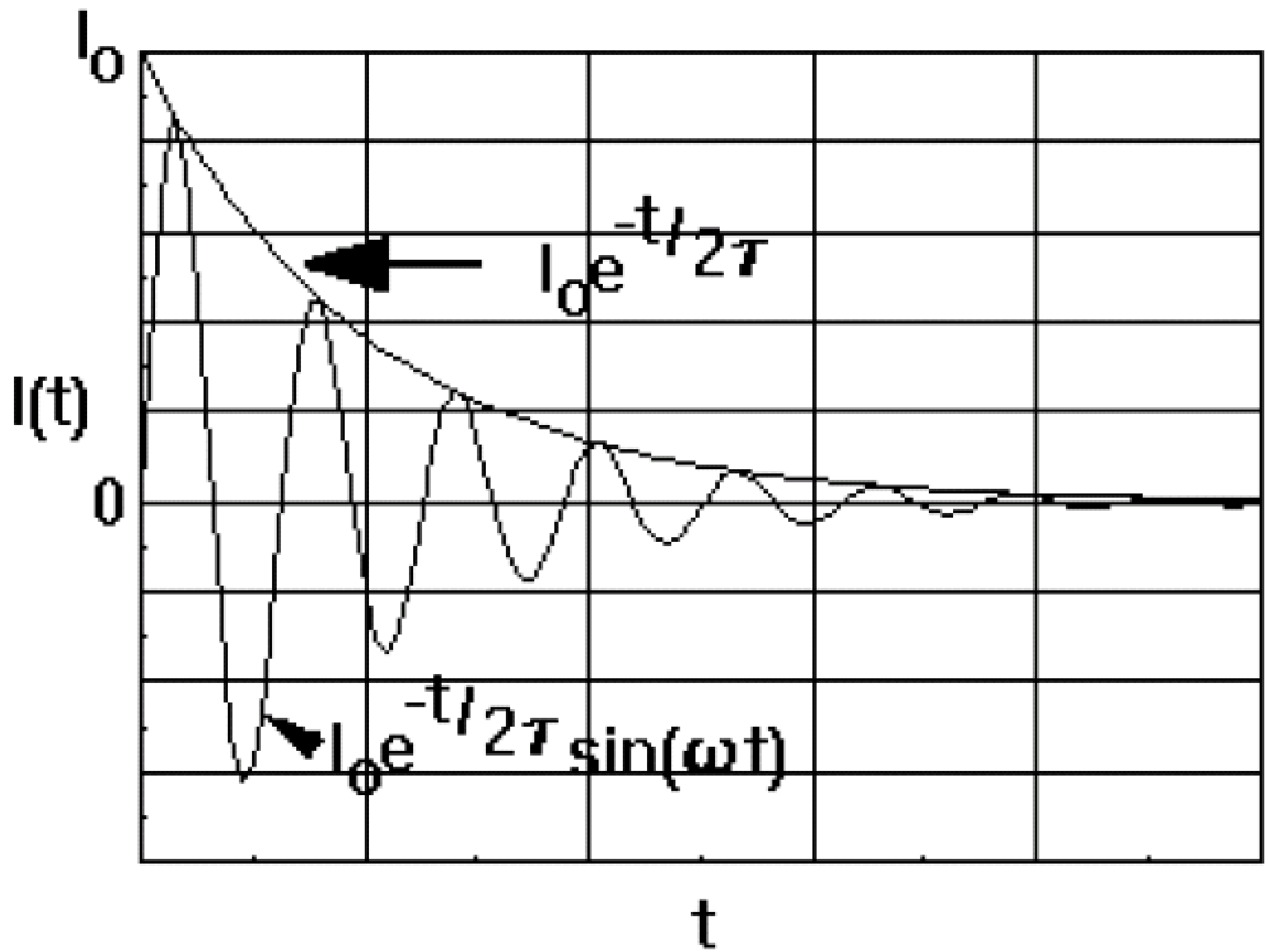
RLC Circuit Response

$$I(t) = I_0 e^{\frac{-R}{2L}t} \sin \omega t$$

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = \omega_0^2 - \frac{1}{4\tau^2}$$

$$\omega_0^2 = \frac{1}{LC} \qquad \tau = \frac{L}{R}$$

Graph of RLC Circuit Response



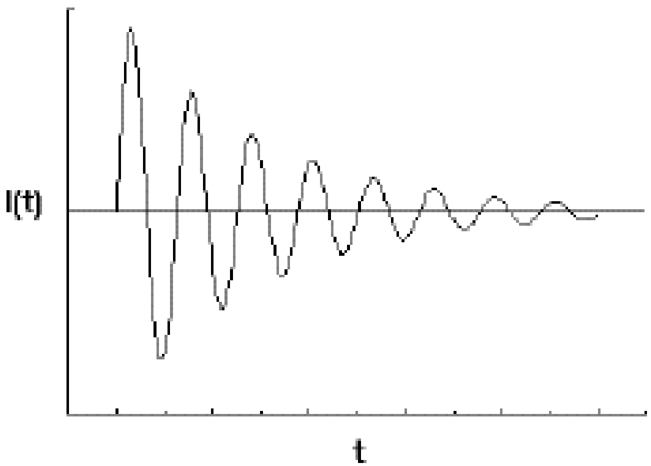
Critical Damping

Define critical damping time constant

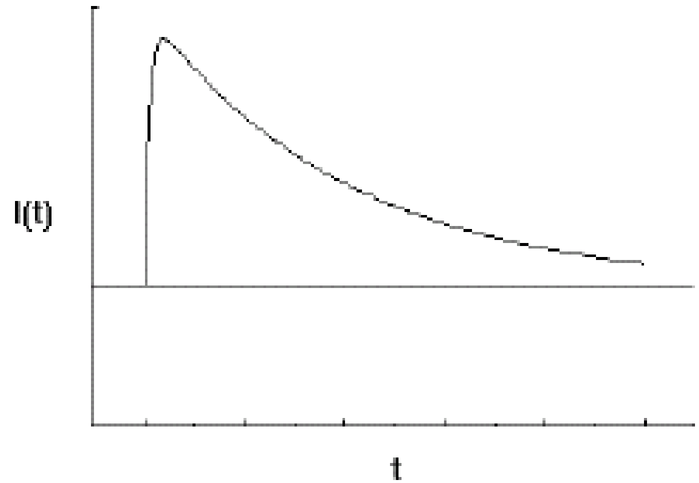
$$\tau = \tau_C = \frac{1}{2\omega_0}$$

No oscillations observed

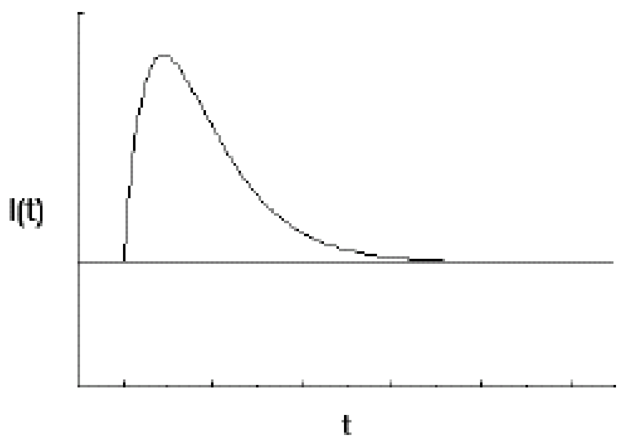
Three Regimes for Damping



Underdamped($\tau > \tau_c$)



Overdamped($\tau < \tau_c$)



Critical ($\tau = \tau_c$)

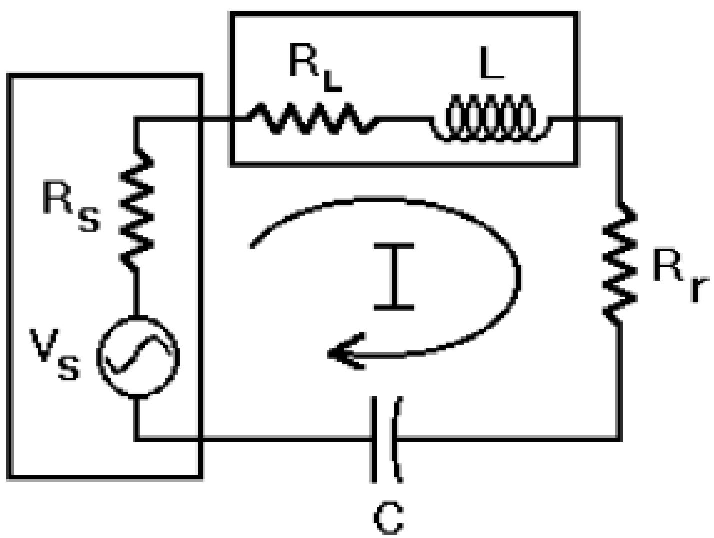
Lab Objectives

1) Determine ω and Q

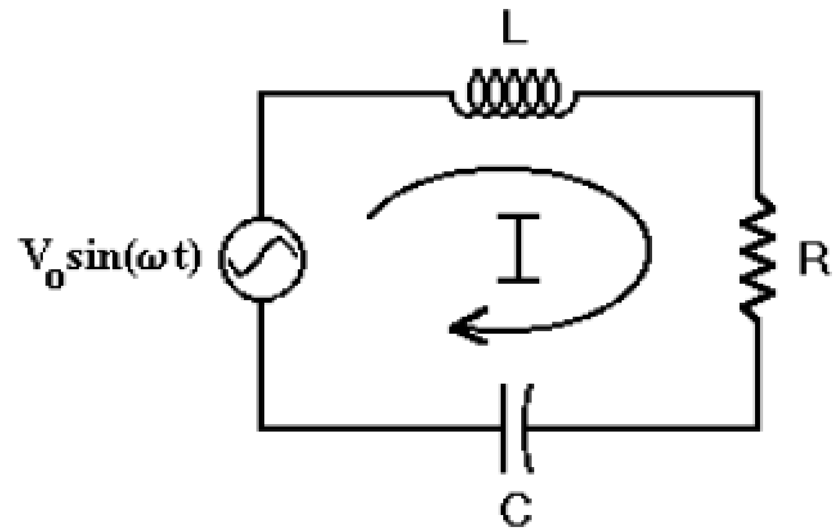
2) Achieve Critical Damping

3) Determine unknown L

Lab 3 Resonance –Sinusoidal Response

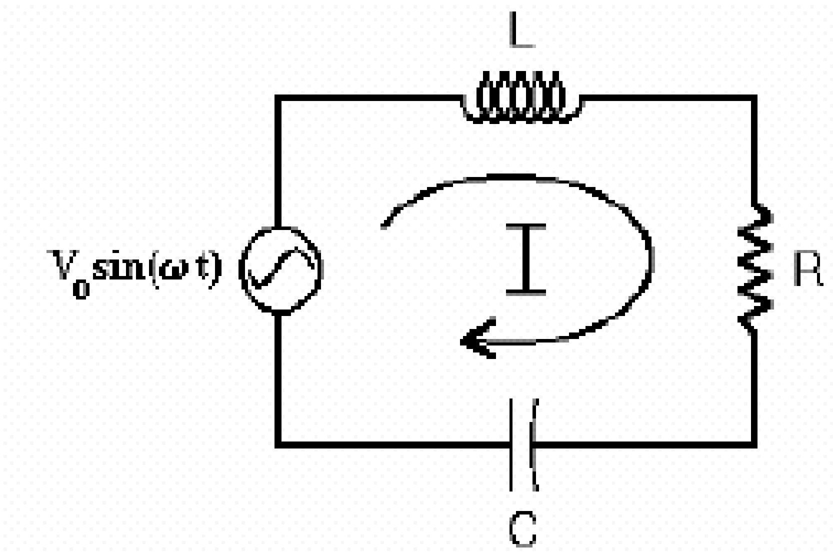


Complete circuit



Model circuit

Lab 3 Resonance



$$V_r = R_r I = V_0 \frac{R_r}{R} \frac{e^{i(\omega t + \phi)}}{\sqrt{1 + Q^2\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]^2}}$$

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

Uncertainty in Q

$$Q = \omega_0/(\omega_2 - \omega_1)$$

$$Q = \omega_0/(\Delta\omega) \quad where\ \Delta\omega = \omega_2 - \omega_1$$

$$\varepsilon(Q) = \{\varepsilon(\omega_0)^2 + \varepsilon(\Delta\omega)^2\}^{1/2}$$

$$\varepsilon(\omega_0) = \delta(\omega_0)/\omega_0$$

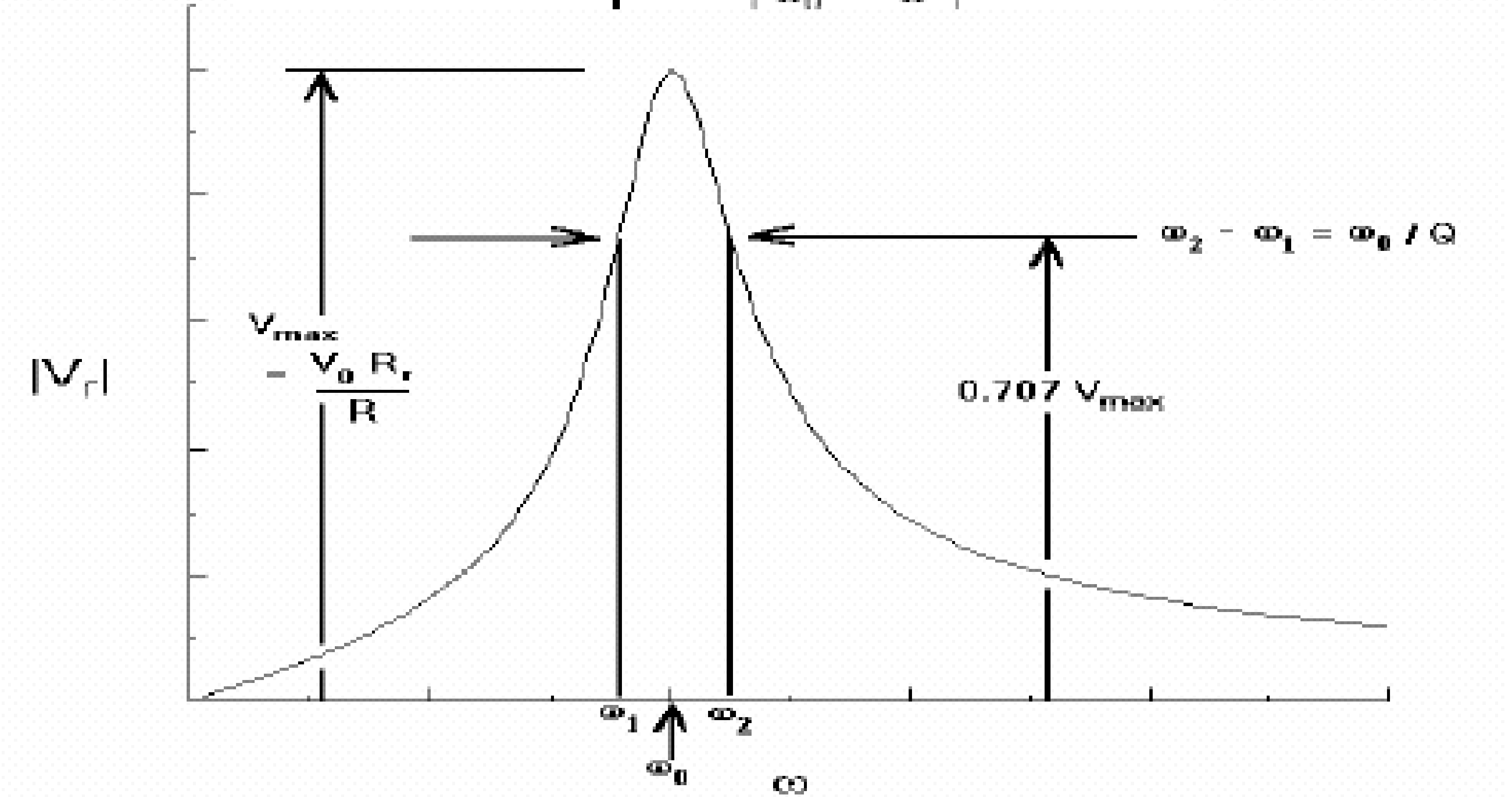
$$\varepsilon(\Delta\omega) = \delta(\Delta\omega)/\Delta\omega$$

$$\varepsilon(\omega_2 - \omega_1) = \delta(\omega_2 - \omega_1)/\ \omega_2 - \omega_1$$

$$\delta(\omega_2 - \omega_1) = \{\delta(\omega_2)^2 + \delta(\omega_1)^2\}^{1/2}$$

Voltage Response

$$|V_r| = \frac{V_0 R_r}{R} \frac{1}{\sqrt{1 + Q^2 \left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right]^2}}$$



Origin and Voltage Response

Derived Equation

$$\left|V_R\right| = \left|I\right|\left|Z_R\right| = V_0 \frac{\left|Z_R\right|}{\left|Z_{Total}\right|}$$

$$= \frac{V_0 R_R}{R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

Origin fit Equation

$$y = \frac{A}{\sqrt{1 + B^2\left(\frac{x}{C} - \frac{C}{x}\right)^2}}$$

$y =$

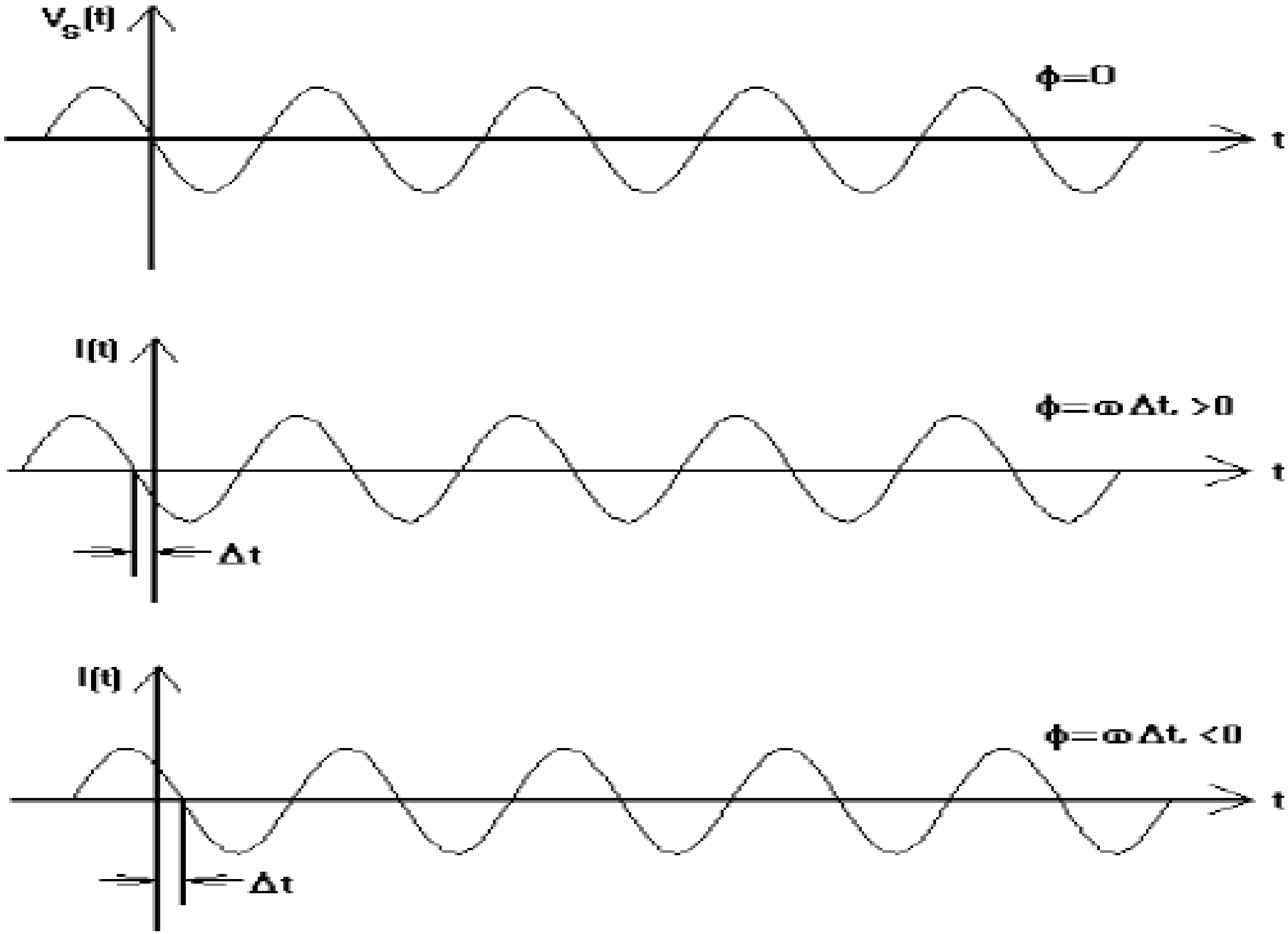
$x =$

$A =$

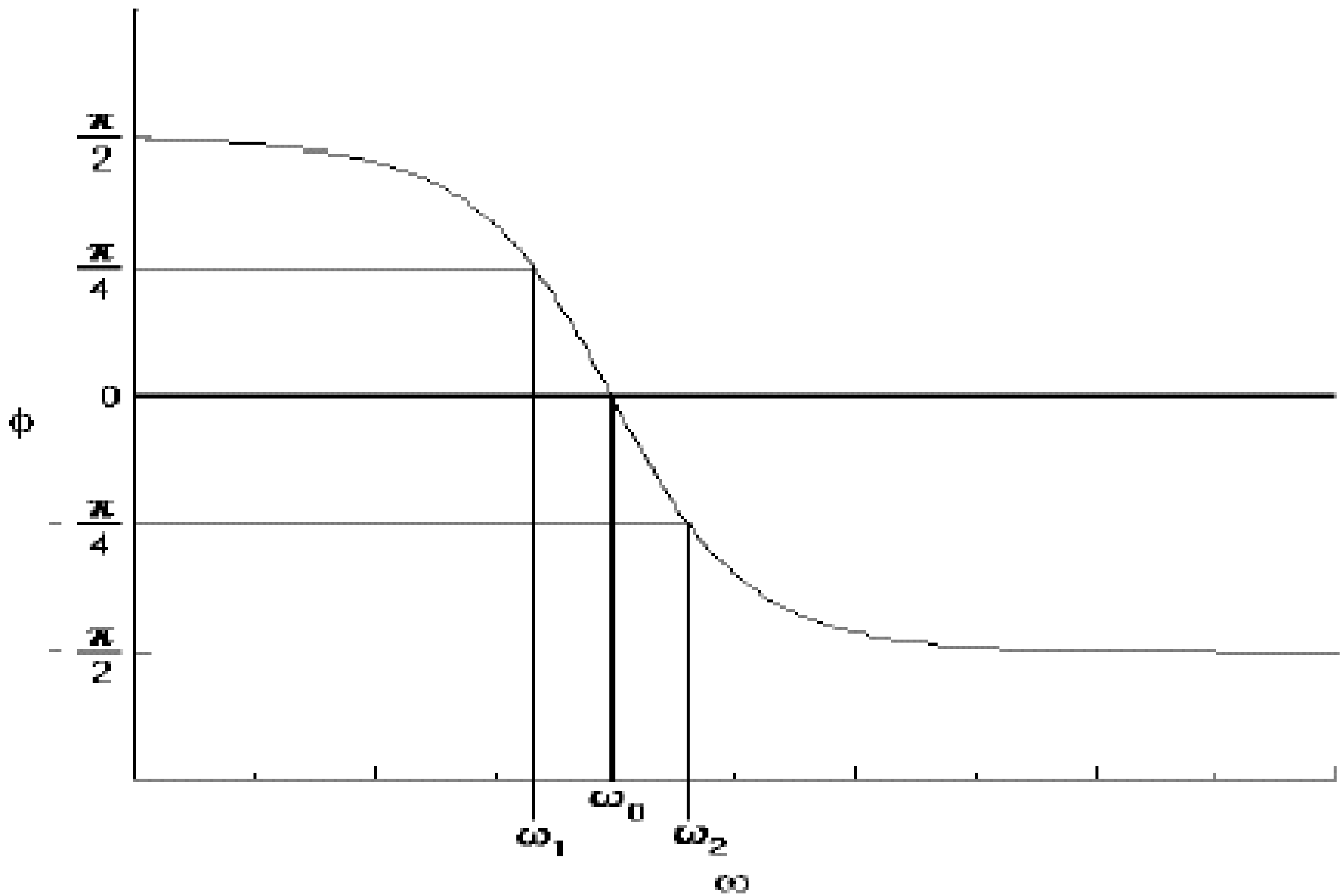
$B =$

$C =$

Phase Shifts



Phase Response



Q-Multiplier

$$\frac{V_{C0}}{V_0} = Q$$

Maximum voltage across capacitor is Q times driving voltage V_0

Outline Lab # 3

1). Preliminary calculations of ω_0 and Q
2). Measure ω_0 and Q
3). Graph Frequency Response
4). Q dependence on R
5). Q-Multiplier
6). Analysis
7). Conclusions

Remember

- CAPE evaluations
- Lab Writeup
- Read next session’s lab description, do prelab
- Homework 5 (Taylor 6.1, 6.4)
- Read Taylor through chapter 8